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Compression of Khalimsky topological spaces

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Abstract. Aiming at the study of the compression of Khalimsky topological spaces which is an interesting field in digital geometry and computer science, the present paper develops a new homotopy thinning suitable for the work. Since Khalimsky continuity of maps between Khalimsky topological spaces has some limitations of performing a discrete geometric transformation, the paper uses another continuity (see Definition 3.4) that can support the discrete geometric transformation and a homotopic thinning suitable for studying Khalimsky topological spaces. By using this homotopy, we can develop a new homotopic thinning for compressing the spaces and can write an algorithm for compressing 2D Khalimsky topological spaces.

1. Introduction

Digital geometry is an approach to understanding and compressing qualitative properties of digital images which have been studied in computer science, viewed as non-Hausdorff subspaces of $\mathbf{Z}^n$ with some particular choices of non-Hausdorff topology, $n \in \mathbf{N}$, where $\mathbf{Z}^n$ is the set of points in the Euclidean nD space with integer coordinates and $\mathbf{N}$ represents the set of natural numbers. The idea of this subject is that qualitative features of images are often in the center of interest in computer image processing, and further that they provide a useful compression of images.

In digital geometry one of the interesting areas is the Khalimsky nD space which is a locally finite space and satisfies the separation axiom T_0 instead of the Hausdorff separation axiom if $n \geq 2$. In addition, the Khalimsky 1D space satisfies the separation axiom T_1 [21]. Thus the present paper mainly studies subspaces of the Khalimsky nD space from the viewpoint of digital geometry.

In relation to the study of discrete objects in $\mathbf{Z}^n$, we have used many tools from combinatorial topology, graph theory, Khalimsky topology and so forth [7–9, 20, 21, 23, 24, 26, 30]. Motivated by Alexandroff spaces in [1], the Khalimsky nD space, denoted by $(\mathbf{Z}^n, T^n)$, was established and its study includes the papers [8, 13, 21, 28, 31]. Since the topology $(\mathbf{Z}^n, T^n)$ is established on the Euclidean nD space, it is useful to consider a subset $X \subset \mathbf{Z}^n$ to be a subspace of $(\mathbf{Z}^n, T^n)$ denoted by (X, T^n_X), $n \geq 1$ [1, 13].

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In this regard, both a Khalimsky continuous map and a Khalimsky homeomorphism have been often used in digital geometry [5, 6, 13, 18, 20–22, 24, 28, 31]. But it is well known that the graph k-connectivity of $\mathbf{Z}^n$ and every topological connectedness as well as Khalimsky connectedness are partially compatible with each other [4, 5, 25]. Besides, a Khalimsky continuous map has some limitations of performing a discrete geometric transformation such as a rotation by 90° and a parallel translation with an odd vector (see Remark 3.3). In order to overcome this difficulty, it can be helpful to take into account a reasonable k-adjacency relations of $\mathbf{Z}^n$ on (X, T^n_X). Thus, considering a Khalimsky topological space (X, T^n_X) with one of the graph k-adjacency relations of (2.1), we call it a *space* if there is no danger of ambiguity and use the notation $(X, k, T^n_X) := X_{n,k}$. The paper [13] introduced the category (briefly, *CTC*) consisting of both a set $Ob(CTC)$ of $X_{n,k}$ and a set $Mor(X_{n_0,k_0}, Y_{n_1,k_1})$ of (k_0, k_1)-continuous maps between each pair X_{n_0,k_0} and Y_{n_1,k_1} in $Ob(CTC)$ (see Definition 3.4). The present paper uses the continuity in *CTC* (see Definition 3.4) instead of both the Khalimsky continuity and the digital continuity in [3, 29] which leads to the development of a homotopy suitable for studying Khalimsky topological spaces.

The paper proposes that an approach to the study of (X, T^n_X) from the viewpoint of *CTC* can overcome the limitation of Khalimsky continuity mentioned above. This is one of the reasons why we study a Khalimsky topological space with graph k-connectivity and continuity of map in *CTC*. Finally, in relation to the compression of Khalimsky topological spaces in *CTC*, we can use a homotopy in *CTC* and develop a homotopic thinning which can substantially contribute to the compression of the spaces $X_{n,k}$.

This paper is organized as follows. Section 2 provides some basic notions. Section 3 compares a Khalimksy continuous map with continuous maps in *CTC* and further, it refers to some utilities of the category *CTC*. Section 4 develops a homotopy thinning in *CTC* and proposes a method of compressing Khalimsky topological spaces in terms of the homotopic thinning in *CTC*. In addition it suggests an algorithm for compressing spaces $X_{2,k}$. Section 5 concludes the paper with a summary and a further work.

2. Preliminaries

Let us now review some basic notions and properties of Khalimsky nD spaces. Khalimsky topology arises from the Khalimsky line. More precisely, *Khalimsky line topology* on $\mathbf{Z}$ is induced from the subbase $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ [1] (see also [21]). Namely, the family of the subset $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbf{Z}} : m, n \in \mathbf{Z}\}$ is a basis of the Khalimsky line topology on $\mathbf{Z}$ denoted by $(\mathbf{Z}, T)$. Indeed, Khalimsky line topology has useful properties. For instance, the Khalimsky line $(\mathbf{Z}, T)$ is connected and if one point is removed, then it consists of two components and is finally not connected [21], which is the similar property of the real line with the usual topology $(\mathbf{R}, U)$, where $\mathbf{R}$ means the set of real numbers. Furthermore, the usual product topology on $\mathbf{Z}^n$ induced from $(\mathbf{Z}, T)$, denoted by $(\mathbf{Z}^n, T^n)$, is called the *Khalimsky nD space*. In the present paper each space $X \subset \mathbf{Z}^n$ will be considered to be a subspace (X, T^n_X) induced from the Khalimsky nD space $(\mathbf{Z}^n, T^n)$.

Let us recall basic terminology of the structure of $(\mathbf{Z}^n, T^n)$. A point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is called *pure open* if all coordinates are odd, and *pure closed* if each of the coordinates is even [21] and the other points in $\mathbf{Z}^n$ is called *mixed* [21]. In each of the spaces of Figures 1, 2, 3, 4, 5, 6 and 7 a black big dot stands for a pure open point and the symbols ■ and • mean a pure closed point and a mixed point, respectively.

Since a Khalimsky continuous map f need not preserve the digital connectivity of $Dom(f)$, it is meaningful to study a *multi-dimensional Khalimsky topological space* (X, T_X) *with k-connectivity* denoted by $X_{n,k}$. Thus let us recall the digital k-connectivity of $\mathbf{Z}^n$, as follows.

As a generalization of the commonly used k-adjacency relations of $\mathbf{Z}^2$ and $\mathbf{Z}^3$ [26, 29], the k-adjacency relations of $\mathbf{Z}^n$ were represented in [7] (see also [9]) as follows.

For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$-(*briefly*, k-)adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In this operator $k := k(m, n)$ is the number of points q which are k-adjacent to a given point p according to the numbers m and n in $\mathbf{N}$, where " :=" means *equal* by definition. Indeed, this $k(m, n)$-adjacency is another presentation of the k-adjacency of [7, 9] (for more details, see [17]). Consequently, this operator leads to the representation of the k-adjacency relations of $\mathbf{Z}^n$ [16]:

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \tag{2.1}$$

where $C_i^n = \frac{n!}{n-i)!\,i!}$.
For instance, 8-, 32-, 64- and 80-adjacency relations of $\mathbf{Z}^4$ are considered and further, 10-, 50-, 130-, 210- and 242-adjacency relations of $\mathbf{Z}^5$ are obtained.

Owing to the *digital k-connectivity paradox* [26], a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of $\mathbf{Z}^n$ is usually considered in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$, $X \subset \mathbf{Z}^n$ is the set of points we regard as belonging to the set depicted, k represents an adjacency relation for X and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$, where $k \neq \bar{k}$ except $X \subset \mathbf{Z}$ [29]. But the paper is not concerned with the $\bar{k}$-adjacency of X. We say that the pair (X, k) is *a digital space with k-adjacency* (briefly, *(binary) digital space*) in $\mathbf{Z}^n$ and a subset (X, k) of $(\mathbf{Z}^n, k)$ is *k-connected* if it is not a union of two disjoint non-empty sets not k-adjacent to each other [26]. In other words, for a set (X, k) in $\mathbf{Z}^n$, two distinct points $x, y \in X$ are called *k-connected* if there is a k-path $f : [0, m]_{\mathbf{Z}} \to X$ whose image is a sequence $(x_0, x_1, \cdots, x_m)$ consisting of the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$. The number m is called the *length* of this k-path [26]. For a digital space (X, k) and a point $x \in X$, we say that the maximal k-connected subset of (X, k) containing the point $x \in X$ is the *k-(connected) component* of a point $x \in X$ [26]. For a digital graph connectivity k, a *simple k-path* in X is the sequence $(x_i)_{i \in [0,m]_{\mathbf{Z}}}$ such that x_i and x_j are k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [26]. Further, a simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [10], is the simple k-path $(x_i)_{i \in [0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i + 1 (mod\, l)$ or $i = j + 1 (mod\, l)$ [26].

3. Comparison of a Khalimsky continuous map and continuous maps in *CTC*

In this section by comparing a Khalimsky continuous map with a continuous map in *CTC*, we refer to some limitations of Khalimsky continuity of maps between Khalimsky topological spaces so that we can speak out merits of a Khalimsky topological space with digital k-connectivity and justify the (k_0, k_1)-continuity of Definition 3.4 which will be used in the paper. In the Khalimsky nD space $(\mathbf{Z}^n, T^n)$, as usual, consider a subset $X \subset \mathbf{Z}^n$ to be a subspace (X, T^n_X) induced from $(\mathbf{Z}^n, T^n)$, where $T^n_X = \{O \cap X | O \in T^n\}$.

In this paper we mainly study spaces (X, T^n_X) with one of the k-adjacency relations of $\mathbf{Z}^n$ that is denoted by $(X, k, T^n_X) := X_{n,k}$ [13] and is called a *space*. In relation to the establishment of various kinds of continuities of maps between spaces $X_{n,k}$ [13], in digital geometry we have often used the following *digital k-neighborhood*, denoted by $N_k(x, \varepsilon)$, [8] (see also [9]) based on the notions of both digital adjacency and a simple k-path in Section 2.

Definition 3.1. ([8]; see also [9]) Let (X, k) be a digital space, $X \subset \mathbf{Z}^n$, $x, y \in X$, and $\varepsilon \in \mathbf{N}$. By the digital k-neighborhood $N_k(x, \varepsilon)$ we denote the set

$$\{y \in X : l_k(x, y) \leq \varepsilon\} \cup \{x\},$$

where $l_k(x, y)$ is the length of a shortest simple k-path x to y in X. Besides, we put $l_k(x, y) = \infty$ if there is no k-path from x to y. Thus, if the k-component of x is the singleton $\{x\}$, then we assume that $N_k(x, \varepsilon) = \{x\}$ for any $\varepsilon \in \mathbf{N}$.

Consider digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$ and further, a map $f : (X, k_0) \to (Y, k_1)$. Then the digital continuity of f in [3] can be equivalently represented in this way [15]:
The map f is digitally (k_0, k_1)-continuous at a point $x \in X$ if and only if

$$f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1). \tag{3.1}$$

By using the digital (k_0, k_1)-continuity of (3.1), we obtain a *digital topological category*, briefly *DTC*, consisting of the following two sets [9] (see also [13]):
(1) A set of objects (X, k) in $\mathbf{Z}^n$;
(2) For every ordered pair of objects (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a set of all digitally (k_0, k_1)-continuous maps $f : (X, k_0) \to (Y, k_1)$ as morphisms.

In *DTC*, for $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$ we assume $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} | a \leq n \leq b\}$ with 2-adjacency [26].

Let us now recall the *Khalimksy topological k-neighborhood* which can be used for establishing continuity of maps between the spaces $X_{n,k}$ in *CTC* (see Definition 3.4).

Definition 3.2. ([8]; see also [13, 20]) Consider a space $X_{n,k} := X$, $x, y \in X$, and $\varepsilon \in \mathbf{N}$.

(1) A subset V of X is called a Khalimsky topological *neighborhood* of x if there exists $O_x \in T^n_X$ such that $x \in O_x \subseteq V$.

(2) If a digital k-neighborhood $N_k(x, \varepsilon)$ is a Khalimsky topological neighborhood of x in (X, T^n_X), then this set is called a Khalimsky topological k-neighborhood of x with radius ε and we use the notation $N^*_k(x, \varepsilon)$ instead of $N_k(x, \varepsilon)$.

In $A_{2,4}$ of Figure 1(a), no $N^*_4(a_i, 1)$ exists, $i \in \{0, 8\}$ because the smallest open set containing the point a_0 (resp, a_8) is the set $\{a_{11}, a_0, a_1, a_2\}$ (resp. $\{a_6, a_7, a_8, a_9\}$). In addition, we can obtain that $N^*_4(a_0, 2) = \{a_{10}, a_{11}, a_0, a_1, a_2\}$ and further, $N^*_4(a_8, 2) = \{a_6, a_7, a_8, a_9, a_{10}\}$.

Let us recall Khalimsky (briefly, K-)continuity of maps between Khalimsky topological spaces: As usual, for two Khalimsky topological spaces $(X, T^{n_0}_X) := X$ and $(Y, T^{n_1}_Y) := Y$ a map $f : X \to Y$ is called continuous at the point $x \in X$ if for any open set $O_{f(x)} \subset Y$ containing the point $f(x)$ there is an open set $O_x \subset X$ containing the point x such that $f(O_x) \subset O_{f(x)}$. In terms of the Khalimsky continuity of the map f, we obtain the *Khalimsky topological category*, briefly *KTC*, consisting of the following two sets [13]:
(1) A set of objects (X, T^n_X);
(2) For every ordered pair of objects $(X, T^{n_0}_X)$ and $(Y, T^{n_1}_Y)$ a set of all Khalimsky (briefly, K-)continuous maps $f : (X, T^{n_0}_X) \to (Y, T^{n_1}_Y)$ as morphisms.

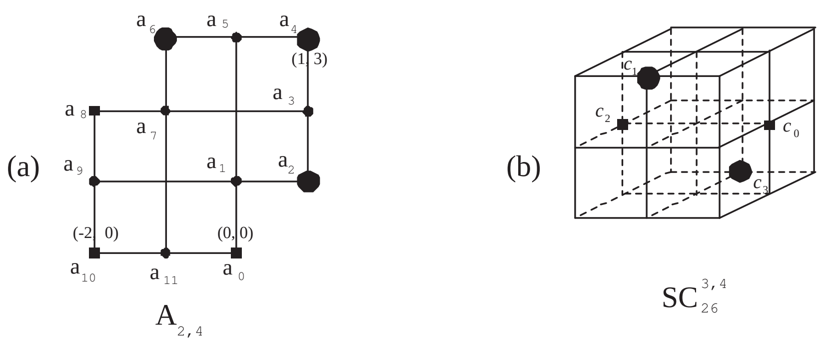

Figure 1: Existence of $N^*_k(x, \varepsilon)$.

Hereafter, $SC^{n,l}_k$ established in Section 2 is assumed to be a subspace of $(\mathbf{Z}^n, T^n)$. As already mentioned above, we can observe the following limitations of K-continuous maps in $Mor(KTC)$.

Remark 3.3. (1) Let $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ be a K-continuous map. Then f need not map a k_0-connected subset into a k_1-connected one [13] (see also the maps f and g in Figure 2(a) and (b) of the present paper). More precisely, let us consider the two maps:
$f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ in Figure 2(a) and (b), respectively. While they are K-continuous maps, f (resp. g) cannot preserve the 4-connectivity (resp. the 8-connectivity).

(2) Regard $SC_k^{n,l} := (c_i)_{i\in[0,l-1]_Z}$ as a subspace induced from the Khalimsky nD space $(\mathbf{Z}^n, T^n)$ and consider the self-map $f : SC_k^{n,l} \to SC_k^{n,l}$ given by $f(c_i) = c_{i+1(mod\, l)}$. Then f need not be a K-continuous map. More precisely, assume $SC_{26}^{3,4} := (c_i)_{i\in[0,3]_Z}$ as a subspace of $(\mathbf{Z}^3, T^3)$ (see Figure 1(b)) and the self-map $f : SC_{26}^{3,4} \to SC_{26}^{3,4}$ given by $f(c_i) = c_{i+1(mod\, 4)}$. Then we can clearly observe that f cannot be a K-continuous map because $O_{c_0} = \{c_3, c_0, c_1\}$, $O_{c_1} = \{c_1\}$, $O_{c_2} = \{c_1, c_2, c_3\}$, and $O_{c_3} = \{c_3\}$, where O_x means the smallest open set containing the point x.

(3) Let us consider the map $f : (\mathbf{Z}, T) \to (\mathbf{Z}, T)$ given by $f(t) = t + 1$ which is a parallel translation with an odd vector. Then we can clearly observe that f cannot be a K-continuous map.

In view of Remark 3.3, to study spaces $X_{n,k}$, we need to use another continuity that can support the digital geometric transformation mentioned in Remark 3.3.

Definition 3.4. ([13]) For two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$ a function $f : X \to Y$ is said to be (k_0, k_1)-continuous at a point $x \in X$ if $f(N^*_{k_0}(x, r)) \subset N^*_{k_1}(f(x), s)$, where the number r is the least element of $\mathbf{N}$ such that $N^*_{k_0}(x, r)$ contains an open set including the point x and s is the least element of $\mathbf{N}$ such that $N^*_{k_1}(f(x), s)$ contains an open set including the point $f(x)$.
Furthermore, we say that a map $f : X \to Y$ is (k_0, k_1)-continuous if the map f is (k_0, k_1)-continuous at every point $x \in X$.

In Definition 3.4 if such a neighborhood $N^*_{k_1}(f(x), \varepsilon)$ does not exist, then we clearly say that f cannot be (k_0, k_1)-continuous at the point x. Further, in Definition 3.4 if $k_0 = k_1$ and $n_0 = n_1$, then we call it a *k_0-continuous map*.

Let us now recall the *category* [13], denoted by CTC, consisting of the following two sets:
- A set of objects $X_{n,k}$ denoted by Ob(CTC);
- For every ordered pair of spaces X_{n_0,k_0} and Y_{n_1,k_1} in Ob(CTC) a set $Mor(X_{n_0,k_0}, Y_{n_1,k_1})$ of (k_0, k_1)-continuous maps as morphisms.

In CTC, for $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a, b]_{\mathbf{Z}} = \{a \le n \le b\}$ can be assumed to be a subspace of $(\mathbf{Z}, T)$ if it is related to the Khalimsky topology $(\mathbf{Z}, T)$. Then $([a, b]_{\mathbf{Z}}, T_{[a,b]_{\mathbf{Z}}})$ is called a *Khalimsky interval* [28] and is briefly denoted by $[a, b]_{\mathbf{Z}}$. For a map $f : X_{n,k} \to Y_{n,k}$ let us compare K-continuity of f in KTC with k-continuity of f in CTC. Indeed, according to the dimension n and connectedness of $X_{n,k}$ and $Y_{n,k}$, some intrinsic features appear.

Theorem 3.5. *Let $f : X_{1,2} \to Y_{1,2}$ be a map. Then K-continuity of f in KTC implies 2-continuity of f in CTC. But the converse does not hold.*

Proof. In the Khalimsky line $(\mathbf{Z}, T)$ each point $t \in \mathbf{Z}$ has $N^*_2(t, 1) = \{t - 1, t, t + 1\} \subset \mathbf{Z}$ and further, the point t is connected to the points $t - 1$ and $t + 1$. Thus a K-continuous map $f : X_{1,2} \to Y_{1,2}$ implies a 2-continuous map in CTC because a K-continuous map preserves connectedness under the Khalimsky line topology. Let us now examine if the assertion is true or not with the following four cases at every point $x \in X_{1,2}$ and its image $f(x) := y \in Y_{1,2}$.

(Case 1) Assume that both x and y are pure closed points. Then a K-continuous map $f : X_{1,2} \to Y_{1,2}$ is equivalent to a 2-continuous map in CTC because $N^*_2(x, 1) = O_x$ and $N^*_2(y, 1) = O_y$, where O_t means the smallest open set containing $t \in \mathbf{Z}$ under the subspace topologies on $X_{1,2}$ and $Y_{1,2}$.

(Case 2) Assume that both x and y are pure open points so that $O_x = \{x\}$ and $O_y = \{y\}$. Since from the hypothesis we obtain that $f(O_x) \subset O_y$, we clearly observe that $f(N^*_2(x, 1)) \subset N^*_2(y, 1)$ because $O_x \subset N^*_2(x, 1)$ and $O_y \subset N^*_2(y, 1)$ and further, if there is a point $x_1 \in N^*_2(x, 1)$ with $x_1 \neq x$ such that $f(x_1) \notin N^*_2(y, 1)$, then f cannot be a K-continuous map at the point x_1 because f does not preserve connectedness between the points x_1 and x.

(Case 3) Assume that x is a pure closed point and y is a pure open point. Owing to both the hypothesis of the K-continuity of f and the fact that $N^*_2(x, 1) = O_x$, we obtain that $f(N^*_2(x, 1)) \subset O_y = \{y\}$, which implies that $f(N^*_2(x, 1)) \subset N^*_2(y, 1)$ because $O_y \subset N^*_2(y, 1)$.

(Case 4) Assume that x is a pure open point and y is a pure closed point. Owing to the hypothesis of the K-continuity of f, we obtain that $f(O_x) \subset O_y = N_2^*(y,1)$. Further, we clearly observe that $f(N_2^*(x,1)) \subset N_2^*(y,1)$. If not, suppose that there is a point $x_1 \in N_2^*(x,1)$ with $x_1 \neq x$ such that $f(x_1) \notin N_2^*(y,1)$. Then the map f cannot be a K-continuous map at the point x_1 because f does not preserve connectedness between the points x_1 and x.

In view of the above four cases, we can conclude that a K-continuous map $f : X_{1,2} \to Y_{1,2}$ implies a 2-continuous map in *CTC*.

But the converse does not hold as demonstrated by the following example: Consider the map f in Remark 3.3(3) which is a parallel translation with an odd vector. While f is a 2-continuous map in *CTC*, as already mentioned in Remark 3.3(3), it cannot be a K-continuous map. □

Remark 3.6. Regardless of connectedness and disconnectedness of the two spaces $X_{1,2}$ and $Y_{1,2}$, the assertion of Theorem 3.2 is valid.

Unlike Theorem 3.2, if $n \geq 2$, then we obtain the following:

Theorem 3.7. *Let $f : X_{n,k} \to Y_{n,k}$ be a map, $n \geq 2$. Then none of K-continuity of f and k-continuity of f in CTC implies the other.*

Proof. Assume that $f : X_{n,k} \to Y_{n,k}$ is a K-continuous map on $X_{n,k}$, $n \geq 2$. Namely, for each point $x \in X_{n,k}$ and its image $f(x) \in Y_{n,k}$ under the map f we can obtain that for the smallest open set $O_{f(x)} \subset Y$ containing the point $f(x)$ there is the smallest open set $O_x \subset X$ containing the point x such that

$$f(O_x) \subset O_{f(x)}. \tag{3.2}$$

But the property (3.2) need not imply the property

$$f(N_k^*(x,\delta)) \subset N_k^*(f(x),\epsilon) \tag{3.3}$$

for some δ and ϵ in $\mathbf{N}$ (see Cases 1-1, 1-2 and 2 below), which means that K-continuity of f does not imply k-continuity of f in *CTC*.
Conversely, assume that $f : X_{n,k} \to Y_{n,k}$ satisfies the property (3.3). Then we can observe that the property (3.3) need not imply the property (3.2) (see Cases 1-3, 1-4 and 2 below), which means that k-continuity of f in *CTC* does not propose K-continuity of f.

(Case 1) In case $n = 2$, by using several examples described in 2D Khalimsky spaces, we can prove the assertion.

(Case 1-1) In Figure 2(a) assume the map $f : X_1 := \{x_i \mid i \in [0,3]_{\mathbf{Z}}\} \to Y_1 := \{y_i \mid i \in [0,1]_{\mathbf{Z}}\}$ defined by $f(\{x_0,x_1\}) = \{y_0\}$ and $f(\{x_2,x_3\}) = \{y_1\}$. While the map f is K-continuous on X_1, it cannot be 4-continuous in *CTC* (see the point x_2). More precisely, since the smallest open set containing x_2 is the set $\{x_2,x_3\}$, we obtain that $N_4^*(x_2,1) = \{x_1,x_2,x_3\}$ and further, $N_4^*(y_1,1) = \{y_1\}$. Since $f(N_4^*(x_2,1))$ cannot be a subset of $N_4^*(y_1,1)$, we can conclude that f cannot be 4-continuous at the point x_2.

(Case 1-2) In Figure 2(b) consider the map $g : X_2 := \{x_i \mid i \in [0,2]_{\mathbf{Z}}\} \to Y_2 := \{y_i \mid i \in [0,2]_{\mathbf{Z}}\}$ given by $g(\{x_0\}) = \{y_0\}$ and $g(\{x_1,x_2\}) = \{y_2\}$. Indeed, the map g is K-continuous on X_2 because O_{x_0} and O_{x_1} are the sets $\{x_0\}$ and $\{x_1,x_2\}$, respectively. But it cannot be 8-continuous in *CTC* (see the points x_0 and x_1) because $g(N_8^*(x_0,1))$ cannot be a subset of $N_8^*(y_0,1) = \{y_0\}$, where $N_8^*(x_0,1) = \{x_0,x_1\}$.

(Case 1-3) In Figure 2(c) assume the map $h : X_3 := \{x_i \mid i \in [0,2]_{\mathbf{Z}}\} \to Y_3 := \{y_i \mid i \in [0,1]_{\mathbf{Z}}\}$ defined by $h(\{x_0,x_1\}) = \{y_0\}$ and $h(\{x_2\}) = \{y_1\}$. While the map h is 8-continuous in *CTC*, it cannot be K-continuous on X_3 (see the point x_1) because O_{x_1} is the total set X_3 and for each $y_i \in Y_3$ we obtain that $O_{y_i} = \{y_i\}, i \in \{0,1\}$.

(Case 1-4) In Figure 2(d) consider the map $i : X_4 := \{x_j \mid j \in [0,2]_{\mathbf{Z}}\} \to Y_4 := \{y_j \mid j \in [0,1]_{\mathbf{Z}}\}$ given by $i(\{x_0,x_1\}) = \{y_1\}$ and $i(\{x_2\}) = \{y_0\}$. While the map i is 4-continuous in *CTC* because $N_4^*(y_0,1) = N_4^*(y_1,1) = Y_4$ and $N_4^*(x_2,1) = \{x_1,x_2\}$, it cannot be K-continuous on X_4 (see the point x_1) because O_{x_1} is the total set X_4 which implies that $N_4^*(x_1,1) = X_4$.

In view of the above four cases, if $n = 2$, then the proof is completed.

(Case 2) In case $n \geq 3$, by using a method similar to that used in the proof of the case that $n = 2$, we can prove the assertion. □

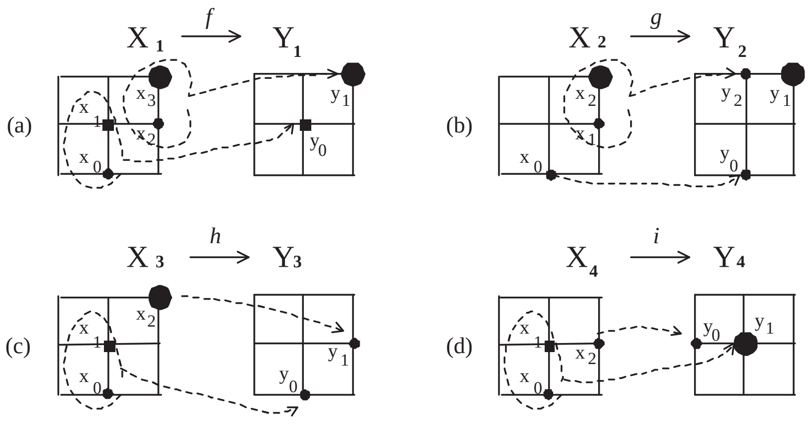

Figure 2: Comparison of K-continuity of KTC and k-continuity of CTC, $k \in \{4, 8\}$.

In relation to the proof of Theorem 3.4, we can observe that the two spaces X_2 and Y_2 in Figure 2(b) are not connected. In addition, we can also observe that Y_3 in Figure 2(c) cannot be connected either. Thus, in Theorem 3.4 if we further assume that $Dom(f)$ and $Codom(f)$ are connected, then we can obtain the result different from Theorem 3.4.

Theorem 3.8. *Let $f : X_{n,k} \to Y_{n,k}$ be a map such that $k = 3^n - 1$, $n \geq 2$, and let both $X_{n,k}$ and $Y_{n,k}$ be connected. Then K-continuity of f implies $(3^n - 1)$-continuity of f in CTC. But the converse does not hold.*

Proof. Assume that $f : X_{n,3^n-1} \to Y_{n,3^n-1}$ is a K-continuous map. For every connected space $A_{n,k} \subset X_{n,k}$, $k = 3^n - 1$ its image $f(A_{n,k})$ is connected in $Y_{n,k}$. Owing to this property, for every point $x \in X_{n,k}$ and its image $f(x) \in Y_{n,k}$ there are $O_x \subset X_{n,k}$ and $O_{f(x)} \subset Y_{n,k}$ such that $f(O_x) \subset O_{f(x)}$ and further, $N^*_{3^n-1}(x, 1) \subset X_{n,k}$ and $N^*_{3^n-1}(f(x), 1) \subset Y_{n,k}$ such that $O_x \subset N^*_{3^n-1}(x, 1)$ and $O_{f(x)} \subset N^*_{3^n-1}(f(x), 1)$. Finally, we obtain that

$$f(N^*_{3^n-1}(x, 1)) \subset N^*_{3^n-1}(f(x), 1).$$

If not, suppose that there is a point $x \in N^*_{3^n-1}(x, 1) \setminus O_x$ such that $f(x) \notin N^*_{3^n-1}(f(x), 1)$, then the map f cannot be a K-continuous map at the point x, which contradicts the hypothesis.

Let us now prove that the converse does not hold in terms of the following example. Consider the space $SC^{n,4}_{3^n-1} := (c_i)_{i\in[0,3]_Z}$ in which two points are pure open and the others are pure closed, and the self-map $f : SC^{n,4}_{3^n-1} \to SC^{n,4}_{3^n-1}$ given by $f(c_i) = c_{i+1(mod\,4)}$. Then we can observe that f is $(3^n - 1)$-continuous from the viewpoint of CTC. But it cannot be a K-continuous map. For instance, consider the space $SC^{3,4}_{26} := (c_i)_{i\in[0,3]_Z}$ in Remark 3.3(2) and the self-map $f : SC^{3,4}_{26} \to SC^{3,4}_{26}$. While f is 26-continuous in CTC, it cannot be a K-continuous map. □

In Theorem 3.5 if we replace the $(3^n - 1)$-adjacency by a k-adjacency of (2.1) with $k \neq 3^n - 1$, then the assertion of Theorem 3.5 cannot be true.

Theorem 3.9. *In Theorem 3.5, if $k \neq 3^n - 1$, then unlike the assertion of Theorem 3.5, we obtain the following: Let $f : X_{n,k} \to Y_{n,k}$ be a map with $k \neq 3^n - 1$ and $n \geq 2$ such that $X_{n,k}$ and $Y_{n,k}$ are connected. Then none of K-continuity of f and k-continuity of f in CTC implies the other.*

Proof. (Case 1) In case $n = 2$, let us prove that K-continuity of f need not imply 4-continuity of f in CTC: Assume the self-map $f : X_{2,4} \to X_{2,4}$ in Figure 3(a) given by $f(\{x_0\}) = \{x_0\}$ and $f(\{x_1, x_2\}) = \{x_2\}$. While the map f is a K-continuous map, it cannot be a 4-continuous map in CTC (see the point x_1).

Conversely, let us prove that 4-continuity in *CTC* need not imply *K*-continuity in *KTC* with the following example: Consider the self-map $f : X_{2,4} \to X_{2,4}$ in Figure 3(b) given by $f(x_i) = x_{i+1(mod\,8)}$. While f is a 4-continuous map in *CTC*, it cannot be a *K*-continuous map.

(Case 2) In case $n = 3$, let us now prove that *K*-continuity need not imply k-continuity in *CTC*, $k \neq 26$. (Case 2-1) In case $k = 18$, assume the map $f : X_{3,18} \to Y_{3,18}$ in Figure 3(c) given by $f(\{x_0, x_1\}) = \{y_0\}$ and $f(\{x_2, x_3\}) = \{y_2\}$. Then we observe that both $X_{3,18}$ and $Y_{3,18}$ are connected, and f is *K*-continuous. But it cannot be 18-continuous at the point x_2 because $f(N^*_{18}(x_2, 1))$ is not a subset of $N^*_{18}(f(x_2), 1)$, where $N^*_{18}(x_2, 1) = \{x_1, x_2, x_3\}$ and $N^*_{18}(f(x_2), 1) = \{y_1, y_2\}$ because $O_{x_2} = \{x_2, x_3\}$ and $O_{f(x_2)} = \{f(x_2)\}$ with $f(x_2) = y_2$. In view of this investigation, we can obtain that *K*-continuity of f in *KTC* cannot support 18-continuity of f in *CTC*.

Conversely, let us prove that 18-continuity in *CTC* need not imply *K*-continuity in *KTC* with the following example: Consider the self-map $f : W_{3,18} \to W_{3,18}$ in Figure 3(d) given by $f(w_i) = w_{i+1(mod\,4)}$. While f is an 18-continuous map in *CTC*, it cannot be a *K*-continuous map.

(Case 2-2) In case $k = 6$, let us prove that *K*-continuity of f in *KTC* need not imply 6-continuity of f in *CTC*: Assume the map $f : X_{3,6} \to Y_{3,6}$ in Figure 3(e) given by $f(\{x_0\}) = \{y_0\}$ and $f(\{x_1, x_2, x_3\}) = \{y_3\}$. Then we can observe that both $X_{3,6}$ and $Y_{3,6}$ are connected and further, f is *K*-continuous. But it cannot be 6-continuous at the point x_1 because $f(N^*_6(x_1, 2))$ is not a subset of $N^*_6(f(x_1), 1)$, where $N^*_6(x_1, 2)$ is the smallest 6-neighborhood of the point x_1 so that $N^*_6(x_1, 2)$ is the total set $X_{3,6}$.

Conversely, let us prove that 6-continuity of a map f in *CTC* need not imply *K*-continuity of f in *KTC* with the following example: Consider the self-map $f : Z_{3,6} \to Z_{3,6}$ in Figure 3(f) given by $f(z_i) = z_{i+1(mod\,12)}$. Since each point $z_i \in Z_{3,6}$ has $N^*_6(z_i, 1) = \{z_{i-1(mod\,12)}, z_i, z_{i+1(mod\,12)}\}$, f is a 6-continuous map in *CTC*. But f cannot be a *K*-continuous map.

In view of this investigation, in case $n = 3$ if $k \neq 26$, then it turns out that none of *K*-continuity of f in *KTC* and k-continuity of f in *CTC* implies the other.

(Case 3) In case $n \geq 4$ and $k \neq 3^n - 1$ with the hypothesis that $X_{n,k}$ and $Y_{n,k}$ are connected, by using the similar methods used for proving Cases 1 and 2 above, we can clearly prove that none of *K*-continuity of f in *KTC* and k-continuity of f in *CTC* implies the other. □

As already mentioned in Remark 3.3, while *K*-continuous maps in *KTC* have some limitations of proceeding the digital geometric transformation related to a rotation, a parallel translation and so forth, we can observe that a (k_0, k_1)-continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ in *CTC* overcomes the limitations (see Remark 3.10), which invokes strong merits of *CTC*.

Remark 3.10. (1) For a (k_0, k_1)-continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ in *CTC*, regardless of connectedness or disconnectedness of $Dom(f)$, f maps a k_0-connected subset into a k_1-connected one if $k_i = 3^{n_i} - 1, n_i \in \mathbf{N}, i \in \{0, 1\}$. More precisely, in a Khalimsky topological space (X, T^n_X) since each point $x \in X_{n_0,k_0}$ (resp. $y \in Y_{n_1,k_1}$) has $N^*_{3^{n_0}-1}(x, 1)$ (resp. $N^*_{3^{n_1}-1}(y, 1)$) which is equal to $N_{3^{n_0}-1}(x, 1)$ (resp. $N_{3^{n_1}-1}(y, 1)$), we can clearly observe that a $(3^{n_0} - 1, 3^{n_1} - 1)$-continuous map in *CTC* is equivalent to a digitally $(3^{n_0} - 1, 3^{n_1} - 1)$-continuous map in *DTC*.

(2) Assume $SC^{n,l}_k := (c_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ to be a subspace of $(\mathbf{Z}^n, T^n)$, where $k = 3^n - 1$. Consider the self-map $f : SC^{n,l}_k \to SC^{n,l}_k$ in Remark 3.3(2) given by

$$f(c_i) = c_{i+1(mod\,l)}. \tag{3.4}$$

While f need not be a *K*-continuous map, f is a k-continuous map in *CTC* because each point $c_i \in SC^{n,l}_k$ has $N^*_k(c_i, 1) \subset SC^{n,l}_k$.
If $k \neq 3^n - 1$, then the map of (3.4) need not be a k-continuous map in *CTC*.
For instance, consider the set $A_{2,4} := (a_i)_{i\in[0,11]_{\mathbf{Z}}}$ in Figure 1(a). Assume the set $A_{2,4}$ with Khalimsky topology. Then the space can be regarded as $SC^{2,12}_4$ in *KTC* or *CTC*. Let us consider the self-map $f : A_{2,4} \to A_{2,4}$ defined by $f(a_i) = a_{i+1(mod\,12)}$ from the viewpoint of *KTC* or *CTC*. Then we can observe that f is neither *K*-continuous nor 4-continuous (see the point a_0).

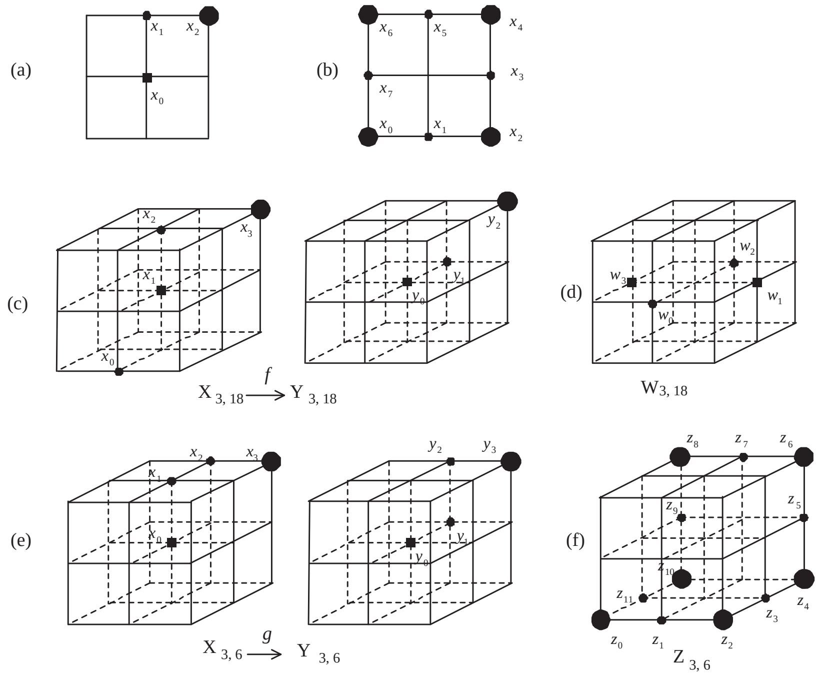

Figure 3: Comparison of K-continuity of KTC and k-continuity of CTC, $k \neq 3^n - 1$.

If for $SC_k^{n,l} := (c_i)_{i\in[0,l-1]_\mathbf{Z}}$ in CTC each point $c_i \in SC_k^{n,l} := (c_i)_{i\in[0,l-1]_\mathbf{Z}} \subset (\mathbf{Z}^n, T^n)$ has $N_k^*(c_i, 1) \subset SC_k^{n,l}$, then the self-map defined by the map of (3.4) is always a k-continuous map.

(3) Consider the map $f : (\mathbf{Z}, T) \to (\mathbf{Z}, T)$ given by $f(t) = t + (2m + 1)$ which is a parallel translation with an odd vector, $m \in \mathbf{N}$ (see Remark 3.3(3)). Then we can clearly observe that f is a 2-continuous map in CTC because each point $t \in \mathbf{Z}$ has $N_2^*(t, 1) = \{t - 1, t, t + 1\}$.

In view of Remark 3.10, by using the continuity of Definition 3.4, we can study Khalimsky topological spaces without the limitations of K-continuity of f in KTC discussed in Remark 3.3.

In relation to the classification of spaces $X_{n,k}$, we can also observe some utilities of CTC. Let us now recall a Khalimsky homeomorphism in KTC and a (k_0, k_1)-homeomorphism in CTC, as follows. In KTC we can say that for two Khalimsky spaces $(X, T_X^{n_0}) := X$ and $(Y, T_Y^{n_1}) := Y$ a map $h : X \to Y$ is a Khalimsky (briefly, K-) homeomorphism if h is a K-continuous bijection and further, $h^{-1} : Y \to X$ is K-continuous [13].

In view of Remarks 3.3 and 3.10, we need to use another homeomorphism in CTC for classifying spaces $X_{n,k}$ in CTC. In relation to the classification of the spaces $X_{n,k}$, by using the continuity of Definition 3.4, we can establish the following:

Definition 3.11. ([13]) In CTC, for two spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$ a function $f : X \to Y$ is said to be a (k_0, k_1)-homeomorphism if
(1) f is bijective, and
(2) f is a (k_0, k_1)-continuous map and further, f^{-1} is a (k_1, k_0)-continuous map.
Then we say that the space X is (k_0, k_1)-homeomorphic to Y.

In Definition 3.11 if $k_0 = k_1$ and $n_0 = n_1$, then we use the terminology *k_0-homeomorphism* instead of *(k_0, k_1)-homeomorphism*. By comparing a K-homeomorphism with a k-homeomorphism, we can observe some merits of a k-homeomorphism.

Example 3.12. (1) Let us consider the map $f : X \to Y$ in Figure 4(a) defined by $f(x_i) = y_i, i \in [0, 7]_\mathbf{Z}$. Even though the spaces X and Y have the same cardinality, the bijection f cannot be a K-homeomorphism (see the points x_3 and x_5). Meanwhile, we can observe that the map f is an 8-homeomorphism in *CTC* because each point $x \in X$ (resp. $y \in Y$) has $N_8^*(x, 1)$ (resp. $N_8^*(y, 1)$).

(2) Consider the map $g : Z := (z_i)_{i\in[0,3]_{\mathbf{Z}}} \to W := (w_i)_{i\in[0,3]_{\mathbf{Z}}}$ in Figure 4(b) given by $g(z_i) = w_i, i \in [0, 3]_{\mathbf{Z}}$. Then we can observe that the bijection g can be a K-homeomorphism. But we may undergo an eccentric situation at the points w_2 and w_3 which are the images of $g(z_2)$ and $g(z_3)$, respectively. More precisely, while $z_3 \in N_8(z_2, 1)$, we can obtain $g(z_3) \notin N_8(g(z_2), 1)$ in which we can do an unusual experience.
Meanwhile, we can observe that the map g cannot be an 8-homeomorphism in *CTC* in which a k-homoeomorphism in *CTC* is substantially helpful tool for classifying Khalimsky topological spaces.

(3) In Figure 4(c) consider the map $h : B := (b_i)_{i\in[0,5]_{\mathbf{Z}}} \to C := (c_i)_{i\in[0,5]_{\mathbf{Z}}}$ in $\mathbf{Z}^3$ given by $h(b_i) = c_i, i \in [0, 5]_{\mathbf{Z}}$. While the bijection h cannot be a K-homeomorphism, it can be an 18-homeomorphism in *CTC*. More precisely, we can observe that (B, T^3_B) has a base

$$\{\{b_0\}, \{b_1, b_2\}, \{b_3\}, \{b_4, b_5\}, \{b_2\}, \{b_4\}\}$$

and (C, T^3_C) has a base

$$\{\{c_0, c_1, c_5\}, \{c_2, c_3, c_4\}, \{c_1\}, \{c_5\}, \{c_2\}, \{c_4\}\}.$$

While h cannot be a K-homeomorphism, it is an 18-homeomorphism because each point $b_i \in B$ (resp. $c_i \in C$) has $N_{18}^*(b_i, 1)$ (resp. $N_{18}^*(c_i, 1)$).

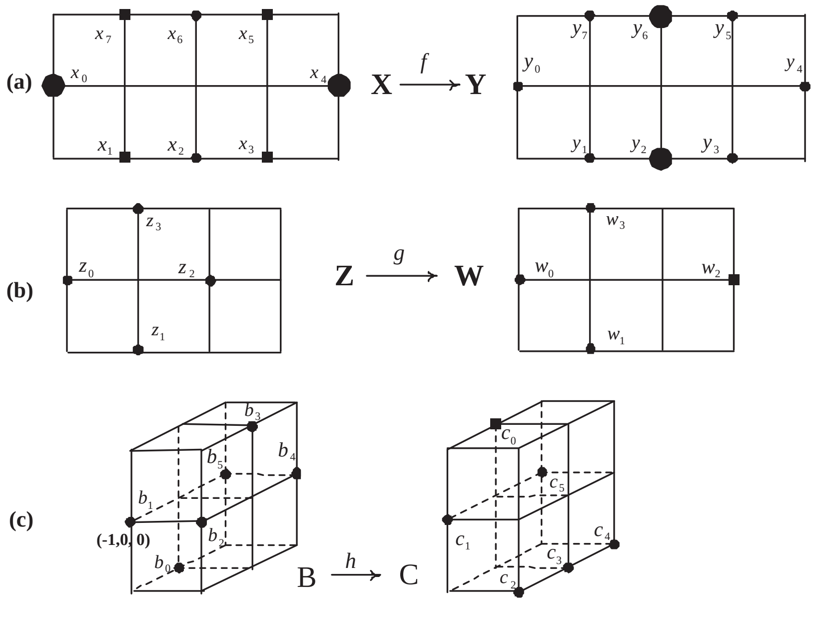

Figure 4: Comparison of two spaces with the same cardinality up to a k-homeomorphism in *CTC*.

In view of Example 3.8, by comparing a K-homeomorphism in *KTC* with a k-homeomorphism in *CTC*, we can observe that the latter has strong merits of classifying spaces $X_{n,k}$.

4. Compressing method of Khalimsky topological spaces in *CTC*

In *DTC* established in Section 3 a digital k-homotopy has contributed to the classification of digital spaces (X, k) in terms of a digital fundamental group and a discrete deck transformation group (or an automorphism group) [2, 3, 9–12, 24, 27]. Similarly, in *CTC* we can consider a homotopy for studying the spaces $X_{n,k}$. Thus, in this section we study a (k_0, k_1)-homotopy which is suitable for studying spaces $X_{n,k}$ (see

Definition 4.1). For a space X_{n_0,k_0} and its subspace A_{n_0,k_0}, consider a space pair $(X_{n_0,k_0}, A_{n_0,k_0}) := (X, A)_{n_0,k_0}$. For two space pairs $(X, A)_{n_0,k_0}$ and $(Y, B)_{n_1,k_1}$, we say that $f : (X, A)_{n_0,k_0} \to (Y, B)_{n_1,k_1}$ is (k_0, k_1)-continuous if $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is (k_0, k_1)-continuous and $f(A_{n_0,k_0}) \subset B_{n_1,k_1}$. As a Khalimsky topological analogue of the (k_0, k_1)-homotopy in *DTC* [12], in *CTC* we can establish a (k_0, k_1)-*homotopy relative to* (briefly, *rel.*)A_{n_0,k_0} in terms of the (k_0, k_1)-continuity of Definition 3.4.

Definition 4.1. ([19]) In *CTC*, for four spaces $X_{n_0,k_0} := X$ and $Y_{n_1,k_1} := Y$, a subspace $A_{n_0,k_0} := A \subset X_{n_0,k_0}$ and a Khalimsky interval $[0, m]_{\mathbf{Z}}$, let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose that there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous;
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.

Then we say that F is a (k_0, k_1)-homotopy between f and g, and f and g are (k_0, k_1)-homotopic in Y. And we use the notation $f \simeq_{(k_0,k_1)} g$.

- If, further, for all $t \in [0, m]_{\mathbf{Z}}$, then the induced map F_t on A is a constant which is the prescribed function from A to Y. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$. Then, we say that the homotopy is a (k_0, k_1)-*homotopy relative to* (briefly, rel.)A and denote it $f \simeq_{(k_0,k_1)rel.A} g$. In particular, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy.

If $X = [0, m_X]_{\mathbf{Z}}$, for all $t \in [0, m]_{\mathbf{Z}}$, we have $F(0, t) = F(0, 0)$ and $F(m_X, t) = F(m_X, 0)$, then we say that F holds the endpoints fixed.

In *CTC*, as an analogous version of the notion of *digital k-contractibility* of [3], we say that a space $X_{n,k} := X$ is *pointed k-contractible* if the identity map 1_X is pointed k-homotopic relative to $\{x_0\}$ in X to a constant map with the space consisting of some point $x_0 \in X$.

Example 4.2. Both of two spaces $X_{2,8}$ and $Y_{2,8}$ in Figure 5 are pointed 8-contractible.

Proof. First of all, let us prove the 8-contractibility of $X_{2,8}$ in Figure 5(a). Let us consider the map $H : X_{2,8} \times [0, 3]_{\mathbf{Z}} \to X_{2,8}$ given by
$H(x_i, 0) = x_i, i \in [0, 9]_{\mathbf{Z}}$;
$H(\{x_4, x_5\}, 1) = \{x_6\}, H(\{x_2, x_3\}, 1) = \{x_1\}$, and $H(x_i, 1) = x_i, i \in \{0, 1, 6, 7, 8, 9\}$;
$H(\{x_2, x_3, x_4, x_5, x_6\}, 2) = \{x_1\}, H(\{x_7, x_8\}, 2) = \{x_9\}$, and $H(x_i, 2) = x_i, i \in \{0, 1, 9\}$; and
$H(x_i, 3) = x_0, i \in [0, 9]_{\mathbf{Z}}$.
Then we can observe that this map H is an 8-homotopy which makes $X_{2,8}$ 8-contractible in *CTC*.

By the similar method used for proving the 8-contractibility of $X_{2,8}$, we can prove the 8-contractibility of $Y_{2,8}$. □

For a space pair $(X, A)_{n,k}$, consider the inclusion map $i : A_{n,k} \to X_{n,k}$. The paper [18] established the notion of k-retract in *CTC*.

Definition 4.3. ([18]) In *CTC* $A_{n,k} := A$ is called a *k-retract* of $X_{n,k}$ if there is a k-continuous map $r : X_{n,k} \to A_{n,k}$ such that $r(a) = a$ for all $a \in A_{n,k}$.

In other words, the map r satisfies the identity $r \circ i = 1_A$. As an analogy of a strong k-deformation retract in *DTC* [12], we can establish the following:

Definition 4.4. In *CTC*, for a space pair $(X, A)_{n_0,k_0} := (X_{n_0,k_0} := X, A_{n_0,k_0} := A)$, $A_{n,k}$ is said to be a strong k-deformation retract of $X_{n,k}$ if there is a k-retraction r of $X_{n,k}$ to $A_{n,k}$ such that $F : i \circ r \simeq_{k\cdot rel.A} 1_X$, *i.e.*, the strong k-deformation satisfies the condition $F(x, t) = x$ for $x \in A, t \in [0, m]_{\mathbf{Z}}$. Then the k-homotopy F is called a strong k-deformation of $X_{n,k}$ to $A_{n,k}$. Then the point $x \in X_{n,k} \setminus A_{n,k}$ is called a strongly k-deformable point.

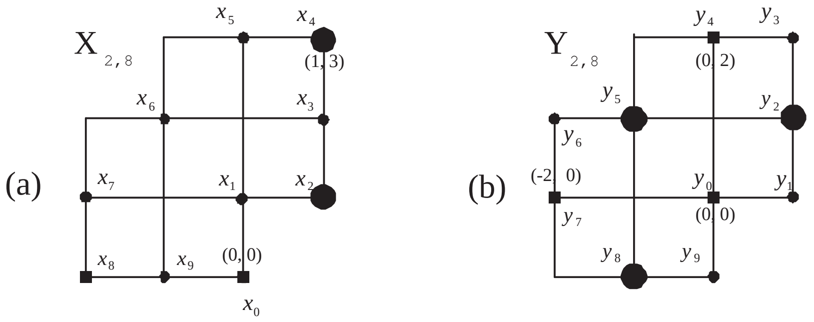

Figure 5: 8-contractibility in *CTC*.

Furthermore, in relation to the compression of spaces $X_{n,k}$ in *CTC*, we introduce the notion of k-homotopic thinning which can be used in topology and the field of applied science.

Definition 4.5. In *CTC*, for a space $X_{n,k}$ we can delete all strongly k-deformable points from $X_{n,k}$ in terms of a strong k-deformation retract. Then this processing is called a k-homotopic thinning.

Example 4.6. In *CTC*, consider the space $X = \{x_i \mid i \in [0, 12]_{\mathbf{Z}}\} := X_{3,18}$ in Figure 6. Then we can observe that the subspaces $X_1 := (X_1)_{3,18}$ and $X_2 := (X_2)_{3,18}$ in Figure 6 are two types of strong 18-deformation retracts of X.

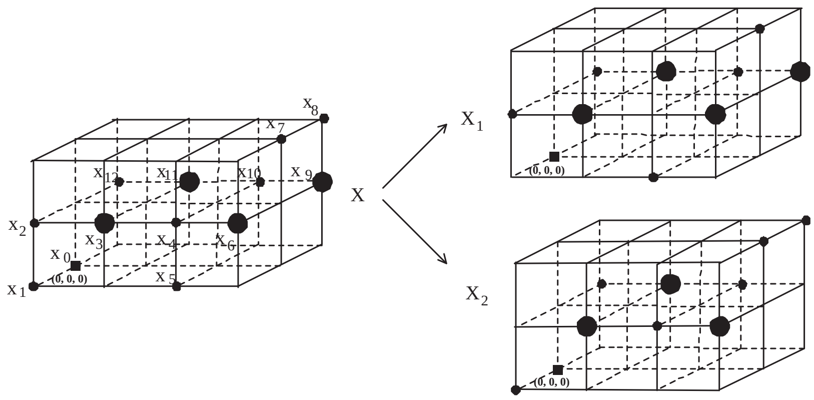

Figure 6: Different types of strong 18-deformation retracts of a given space in *CTC*.

Based on the k-homotopic thinning method in *CTC*, we can write an algorithm for compressing 2D spaces $X_{2,k}$ in *CTC*. In this section we assume that every point $x \in X_{2,k}$ has an $N_k^*(x, \epsilon) \subset X_{2,k}$ for some element ϵ of $\mathbf{N}$. If there is no hypothesis of the existence of $N_k^*(x, \epsilon) \subset X_{2,k}$, then we cannot perform the compression of a space $X_{2,k}$.

[An algorithm for compressing non-empty spaces $X_{2,k}$ in terms of the k-homotopic thinning in *CTC*]

(1) Consider a space $X_{2,k}$ in *CTC*.

(2) Scan the given space $X_{2,k}$ row by row from the top to the bottom and from the left to the right. Finally, if there is no point in the row, then restart to scan from the next row from the left to the right.

(3) According to the step(2) above, we can meet the first point $x \in X_{2,k}$ and label it as x_1. Next, keep going to the labeling on the next points to the right, e.g. $x_2, x_3, \cdots$ and so on (see the spaces $X_{2,4}$ and $Y_{2,8}$ in Figure 7 (a) and (b)).

(4) According to the step(3) above, after labeling the points in $X_{2,k}$, examine if the labeled point is a k-homotopic thinning point from the first labeled point x_1 to the next $x_2, x_3, \cdots$ and so on (see the labeled points in Figure 7 (a) and (b)). If yes, then delete it. Next, according to the order of the labeled points, keep going the k-homotopic thinning of all labeled points.

(5) In (4), when examining if the point $x \in X_{2,k}$ is a k-homotopic thinning point, if the labeled point is not a k-homotopic thinning point, then remain the point and keep going to the right and proceed the above work (4). Finally, if there is no point in the row then restart the work of (4) from the next row.

(6) After finishing the works of (1)-(5), we can finally obtain a k-homotopic thinned space from the given space $X_{2,k}$ (see the k-homotopic thinned spaces in Figure 7 (a) and (b), $k \in \{4, 8\}$).

Example 4.7. Consider the two spaces $X_{2,4} := X$ and $Y_{2,8} := Y$ in Figure 7 (a) and (b). According to the above algorithm, we can proceed the compression of X (resp. Y) in terms of a 4-homotopic thinning of X (resp. an 8-homotopic thinning of Y).

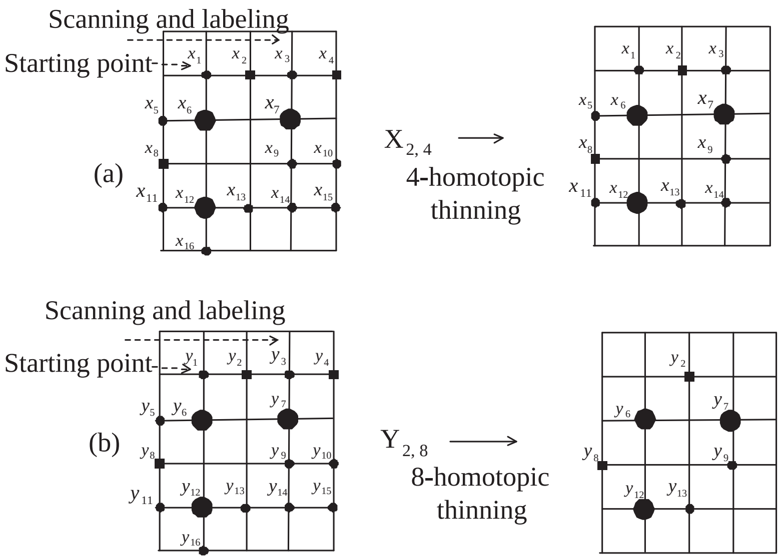

Figure 7: (a) Configuration of a 4-homotopic thinning in *CTC*; (b) Configuration of an 8-homotopic thinning in *CTC*.

5. Summary and further work

In relation to the compression of Khalimsky topological spaces, we have developed a k-homotopic thinning of spaces $X_{n,k}$ in *CTC* and further, have suggested an algorithm for it. Such a case might be made by demonstrating that the methods yield useful information about image processing algorithms, or provide useful compression, or that they in any way increase our understanding of the digital images in a concretely applicable way.

As a further work, we need to find a map between Khalimsky topological spaces expanding both a K-continuous map and a (k_0, k_1)-continuous map in *CTC*.

References

[1] P. Alexandroff, Diskrete Raume, Mat. Sb. 2 (1937) 501–518.
[2] R. Ayala, E. Domínguez, A.R. Francés, A. Quintero, Homotopy in digital spaces, Discrete Appl. Math. 125 (2003) 3–24.
[3] L. Boxer, A classical construction for the digital fundamental group, J. Math. Imaging Vision 10 (1999) 51–62.
[4] A. Bretto, Compatible topologies on graphs: An application to graph isomorphism problem complexity, Theoretical Computer Science (2006) 255–272.
[5] U. Eckhardt, L.J. Latecki, Topologies for digital spaces $\mathbf{Z}^2$ and $\mathbf{Z}^3$, Computer Vision Image Understanding 95 (2003) 261–262.
[6] N.D. Georgiou, S.E. Han, Generalized topological function space and a classification of generalized computer topological spaces, Filomat 26 (2012) 539–552.
[7] S.E. Han, Computer topology and its applications, Honam Math. J. 25 (2003) 153–162.
[8] S.E. Han, Comparison between digital continuity and computer continuity, Honam Math. J. 26 (2004) 331–339.
[9] S.E. Han, Non-product property of the digital fundamental group, Information Sci. 171 (2005) 73–91.
[10] S.E. Han, Connected sum of digital closed surfaces, Information Sci. 176 (2006) 332–348.
[11] S.E. Han, Discrete homotopy of a closed k-surface, Lecture Notes in Computer Science 1953 (2006) 214–225.
[12] S.E. Han, Strong k-deformation retract and its applications, J. Korean Math. Soc. 44 (2007) 1479–1503.
[13] S.E. Han, Continuities and homeomorphisms in computer topology and their applications, J. Korean Math. Soc. 45 (2008) 923–952.
[14] S.E. Han, Comparison among digital fundamental groups and its applications, Information Sci. 178 (2008) 2091–2104.
[15] S.E. Han, Equivalent (k_0, k_1)-covering and generalized digital lifting, Information Sci. 178 (2008) 550–561.
[16] S.E. Han, The k-homotopic thinning and a torus-like digital image in $\mathbf{Z}^n$, J. Math. Imaging Vision 31 (2008) 1–16.
[17] S.E. Han, KD-(k_0, k_1)-homotopy equivalence and its applications, J. Korean Math. Soc. 47 (2010) 1031–1054.
[18] S.E. Han, Extension problem of several continuities in computer topology, Bull. Korean Math. Soc. 47 (2010) 915–932.
[19] S.E. Han, Homotopy equivalence which is suitable for studying Khalimsky nD spaces, Topology Appl. 159 (2012) 1705–1714.
[20] S.E. Han, N.D. Georgiou, On computer topological function space, J. Korean Math. Soc. 46 (2009) 841–857.
[21] E. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology Appl. 36 (1991) 1–17.
[22] In-Soo Kim, S.E. Han, C.J. Yoo, The pasting property of digital continuity, Acta Appl. Math. 110 (2010) 399–408.
[23] O. Kiselman, Digital Jordan curve theorems, Lecture Notes in Computer Science 1953 (2000) 46–56.
[24] T.Y. Kong, A digital fundamental group, Comput. Graphics 13 (1989) 159–166.
[25] T.Y. Kong, The Khalimsky topologies are precisely those simply connected topologies on $\mathbf{Z}^n$ whose connected sets include all $2n$-connected set but no 3^{n-1}-disconnected sets, Theoretical Computer Science 305 (2003) 221–235.
[26] T.Y. Kong, A. Rosenfeld, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, 1996.
[27] R. Malgouyres, Homotopy in 2-dimensional digital images, Theoretical Computer Science 230 (2000) 221–233.
[28] E. Melin, Extension of continuous functions in digital spaces with the Khalimsky topology, Topology Appl. 153 (2005) 52–65.
[29] A. Rosenfeld, Digital topology, Amer. Math. Monthly 86 (1979) 76–87.
[30] J. Šlapal, A digital analogoue of the Jordan curve theorem, Discrete Appl. Math. 139 (2004) 231–251.
[31] J. Šlapal, Digital Jordan curves, Topology Appl. 153 (2006) 3255–3264.