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Hermite-Hadamard type inequalities for the *m*- and (α, m)-logarithmically convex functions

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Abstract. In the paper, the authors introduce concepts of m- and (α, m)-logarithmically convex functions and establish some Hermite-Hadamard type inequalities of these classes of functions.

1. Introduction

For convex functions, the following Hermite-Hadamard type inequalities were given in [8].

Theorem A ([8]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q} \tag{1.1}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}. \tag{1.2}$$

The m-convex function was defined in [12] as follows.

Definition 1.1. *A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex if*

$$f(tx + m(1-t)y) \leq t f(x) + m(1-t) f(y) \tag{1.3}$$

holds for all $x, y \in [0, b]$, $t \in [0, 1]$, and $m \in (0, 1]$.

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In [1, p. 48, Theorem 2] and [2], the following Hermite-Hadamard type inequality for m-convex functions was proved.

Theorem B. *Let $f : [0,\infty) \to \mathbb{R}$ be m-convex and $m \in (0,1]$. If $f \in L([a,b])$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \min\biggl\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\biggr\}. \tag{1.4}$$

The (α,m)-convex function was defined in [7] as follows.

Definition 1.2. *A function $f : [0,b] \to \mathbb{R}$ is said to be (α,m)-convex if*

$$f(tx+m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{1.5}$$

is valid for all $x,y \in [0,b]$, $t \in [0,1]$, and $(\alpha,m) \in (0,1]\times(0,1]$.

For (α,m)-convex functions, the following Hermite-Hadamard type inequalities appeared in [5].

Theorem C ([5, Theorem 2.2])**.** *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a,b]$ for some given numbers $m \in (0,1]$ and $q \in [1,\infty)$, then*

$$\biggl|f\Bigl(\frac{a+b}{2}\Bigr) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\biggr| \le \frac{b-a}{4}\min\biggl\{\biggl(\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\biggr)^{1/q}, \biggl(\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\biggr)^{1/q}\biggr\}. \tag{1.6}$$

Theorem D ([5, Theorem 3.1])**.** *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α,m)-convex on $[a,b]$ for $(\alpha,m) \in (0,1]\times(0,1]$ and $q \in [1,\infty)$, then*

$$\begin{aligned}\biggl|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\biggr| &\le \frac{b-a}{2}\Bigl(\frac{1}{2}\Bigr)^{1-1/q} \\ &\quad\times \min\biggl\{\biggl[v_1|f'(a)|^q + v_2 m\biggl|f'\Bigl(\frac{b}{m}\Bigr)\biggr|^q\biggr]^{1/q}, \biggl[v_2 m\biggl|f'\Bigl(\frac{a}{m}\Bigr)\biggr|^q + v_1|f'(b)|^q\biggr]^{1/q}\biggr\},\end{aligned} \tag{1.7}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\Bigl(\alpha + \frac{1}{2^\alpha}\Bigr) \quad \text{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\Bigl(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\Bigr). \tag{1.8}$$

The aim of this paper is to introduce concepts of m- and (α,m)-logarithmically convex functions, and then to present some Hermite-Hadamard type inequalities for them.

2. Definitions and lemmas

Firstly we introduce concepts of m- and (α,m)-logarithmically convex functions.

Definition 2.1. *A function $f : [0,b] \to (0,\infty)$ is said to be m-logarithmically convex if the inequality*

$$f(tx+m(1-t)y) \le [f(x)]^t[f(y)]^{m(1-t)} \tag{2.1}$$

holds for all $x,y \in [0,b]$, $m \in (0,1]$, and $t \in [0,1]$.

Obviously, if putting $m=1$ in Definition 2.1, then f is just the ordinary logarithmically convex function on $[0,b]$.

Definition 2.2. *A function $f : [0, b] \to (0, \infty)$ is said to be (α, m)-logarithmically convex if*

$$f(tx + m(1-t)y) \le [f(x)]^{t^\alpha}[f(y)]^{m(1-t^\alpha)} \tag{2.2}$$

holds for all $x, y \in [0, b]$, $(\alpha, m) \in (0, 1] \times (0, 1]$, and $t \in [0, 1]$.

Clearly, when taking $\alpha = 1$ in Definition 2.2, then f becomes the standard m-logarithmically convex function on $[0, b]$.

Secondly, we recite the following lemmas which will be used in proofs of our main results.

Lemma 2.1 ([3]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}t. \tag{2.3}$$

Lemma 2.2 ([4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then*

$$f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = (b-a)\bigg[\int_0^{1/2} tf'(ta+(1-t)b)\,\mathrm{d}t + \int_{1/2}^1 (1-t)f'(ta+(1-t)b)\,\mathrm{d}t\bigg]. \tag{2.4}$$

3. Hermite-Hadamard type inequalities

In this section, we will present several Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions.

Theorem 3.1. *Let $I \supset [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0, 1] \times (0, 1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2}\Big(\frac{1}{2}\Big)^{1-1/q}\left|f'\Big(\frac{b}{m}\Big)\right|^m [E_1(\alpha, m, q)]^{1/q} \tag{3.1}$$

is valid for $q \ge 1$, where

$$\mu = \frac{|f'(a)|}{|f'(b/m)|^m}, \quad E_1(\alpha, m, q) = \begin{cases} \dfrac{1}{2}, & \mu = 1, \\ F_1(\mu, \alpha q), & \mu < 1, \\ \mu^{(1-\alpha)q}F_1(\mu, \alpha q), & \mu > 1, \end{cases} \tag{3.2}$$

and

$$F_1(u, v) = \frac{1}{v^2 \ln^2 u}\Big[v(u^v - 1)\ln u - 2\big(u^{v/2} - 1\big)^2\Big] \tag{3.3}$$

for $u, v > 0$ and $u \ne 1$.

Proof. When $q > 1$, by Definition 2.2, Lemma 2.1, and Hölder inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &= \frac{b-a}{2}\left|\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}t\right| \\ &\le \frac{b-a}{2}\Big(\int_0^1 |1-2t|\,\mathrm{d}t\Big)^{1-1/q}\Big(\int_0^1 |1-2t||f'(ta+(1-t)b)|^q\,\mathrm{d}t\Big)^{1/q} \\ &\le \frac{b-a}{2}\Big(\frac{1}{2}\Big)^{1-1/q}\left|f'\Big(\frac{b}{m}\Big)\right|^m\Big(\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}t\Big)^{1/q}.\end{aligned}$$

For $\mu = 1$, we have

$$\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}\,t = \int_0^1 |1-2t|\,\mathrm{d}\,t = \frac{1}{2}.$$

For $\mu < 1$, we have $\mu^{qt^\alpha} \le \mu^{\alpha q t}$, thereby

$$\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}\,t \le \int_0^1 |1-2t|\mu^{\alpha q t}\,\mathrm{d}\,t = \frac{\alpha q \mu^{\alpha q}\ln\mu - \alpha q\ln\mu - 2\mu^{\alpha q} + 4\mu^{\alpha q/2} - 2}{\alpha^2 q^2 \ln^2\mu}.$$

For $\mu > 1$, we have $\mu^{qt^\alpha} \le \mu^{q(\alpha t+1-\alpha)}$, thereby

$$\int_0^1 |1-2t|\mu^{qt^\alpha}\,\mathrm{d}\,t \le \mu^{q(1-\alpha)}\int_0^1 |1-2t|\mu^{\alpha q t}\,\mathrm{d}\,t = \mu^{q(1-\alpha)}\frac{\alpha q \mu^{\alpha q}\ln\mu - \alpha q\ln\mu - 2\mu^{\alpha q} + 4\mu^{\alpha q/2} - 2}{\alpha^2 q^2 \ln^2\mu}.$$

Thus, the inequality (3.1) follows.

When $q = 1$, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &= \frac{b-a}{2}\left|\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}\,t\right| \\ &\le \frac{b-a}{2}\int_0^1 |1-2t||f'(a)|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)}\mathrm{d}\,t \le \frac{b-a}{2}\left|f'\left(\frac{b}{m}\right)\right|^m E_1(\alpha,m,q).\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Corollary 3.2. *Let $I \supset [0,\infty)$ be an open real interval and let $f: I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for some given numbers $m \in (0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left|f'\left(\frac{b}{m}\right)\right|^m [E_1(1,m,q)]^{1/q} \tag{3.4}$$

holds for $q \ge 1$, where E_1 is defined as in Theorem 3.1.

Theorem 3.3. *Let $I \supset [0,\infty)$ be an open real interval and let $f: I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α,m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha,m) \in (0,1]\times(0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-3/q}\left|f'\left(\frac{b}{m}\right)\right|^m E_2(\alpha,m,q) \tag{3.5}$$

is valid for $q \ge 1$, where μ is defined by (3.2) *and*

$$E_2(\alpha,m,q) = \begin{cases} 2\left(\frac{1}{8}\right)^{1/q}, & \mu = 1 \\ [F_2(\mu,\alpha q)]^{1/q} + [F_3(\mu,\alpha q)]^{1/q}, & 0 < \mu < 1 \\ \mu^{1-\alpha}\left\{[F_2(\mu,\alpha q)]^{1/q} + [F_3(\mu,\alpha q)]^{1/q}\right\}, & \mu > 1 \end{cases} \tag{3.6}$$

with

$$F_2(u,v) = \frac{1}{v^2\ln^2 u}\left(\frac{v}{2}u^{v/2}\ln u - u^{v/2} + 1\right) \quad \text{and} \quad F_3(u,v) = \frac{1}{v^2\ln^2 u}\left(u^v - \frac{v}{2}u^{v/2}\ln u - u^{v/2}\right) \tag{3.7}$$

for $u, v > 0$ and $u \ne 1$.

Proof. When $q > 1$, by Definition 2.2, Lemma 2.2, and Hölder inequality yield

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le (b-a)\left[\int_0^{1/2} t|f'(ta+(1-t)b)|\,\mathrm{d}t+\int_{1/2}^1(1-t)|f'(ta+(1-t)b)|\,\mathrm{d}t\right]\\ &\le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-3/q}\left\{\left[\int_0^{1/2} t|f'(a)|^{qt^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{mq(1-t^\alpha)}\mathrm{d}t\right]^{1/q}+\left[\int_{1/2}^1(1-t)|f'(a)|^{qt^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{mq(1-t^\alpha)}\mathrm{d}t\right]^{1/q}\right\}\\ &= \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-3/q}\left|f'\left(\frac{b}{m}\right)\right|^m\left\{\left(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\right)^{1/q}+\left[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

If $\mu = 1$, we have

$$\left(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\right)^{1/q}+\left[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\right]^{1/q}=2\left(\frac{1}{8}\right)^{1/q}.$$

If $\mu < 1$, we obtain

$$\begin{aligned}\left(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\right)^{1/q}+\left[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\right]^{1/q} &\le \left(\int_0^{1/2} t\mu^{\alpha qt}\,\mathrm{d}t\right)^{1/q}+\left[\int_{1/2}^1(1-t)\mu^{\alpha qt}\,\mathrm{d}t\right]^{1/q}\\ &= \left[\frac{1}{\alpha^2q^2\ln^2\mu}\left(\frac{\alpha q}{2}\mu^{\alpha q/2}\ln\mu-\mu^{\alpha q/2}+1\right)\right]^{1/q}+\left[\frac{1}{\alpha^2q^2\ln^2\mu}\left(\mu^{\alpha q}-\frac{\alpha q}{2}\mu^{\alpha q}\ln\mu-\mu^{\alpha q/2}\right)\right]^{1/q}.\end{aligned}$$

If $\mu > 1$, then

$$\begin{aligned}\left(\int_0^{1/2} t\mu^{qt^\alpha}\,\mathrm{d}t\right)^{1/q}+\left[\int_{1/2}^1(1-t)\mu^{qt^\alpha}\,\mathrm{d}t\right]^{1/q} &\le \left(\int_0^{1/2} t\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t\right)^{1/q}+\left[\int_{1/2}^1(1-t)\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t\right]^{1/q}\\ &= \mu^{1-\alpha}\left\{\left[\frac{1}{\alpha^2q^2\ln^2\mu}\left(\frac{\alpha q}{2}\mu^{\alpha q/2}\ln\mu-\mu^{\alpha q/2}+1\right)\right]^{1/q}+\left[\frac{1}{\alpha^2q^2\ln^2\mu}\left(\mu^{\alpha q}-\frac{\alpha q}{2}\mu^{\alpha q}\ln\mu-\mu^{\alpha q/2}\right)\right]^{1/q}\right\}.\end{aligned}$$

Thus, the inequality (3.5) follows.

When $q = 1$, we have

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{2}\left[\int_0^{1/2} t|f'(ta+(1-t)b)|\,\mathrm{d}t+\int_{1/2}^1(1-t)|f'(ta+(1-t)b)|\,\mathrm{d}t\right]\\ &\le (b-a)\left[\int_0^{1/2} t|f'(a)|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)}\mathrm{d}t+\int_{1/2}^1(1-t)|f'(a)|^{t^\alpha}\left|f'\left(\frac{b}{m}\right)\right|^{m(1-t^\alpha)}\mathrm{d}t\right]\\ &\le (b-a)\left|f'\left(\frac{b}{m}\right)\right|^m E_2(\alpha,m,1).\end{aligned}$$

This completes the proof of Theorem 3.3. □

Corollary 3.4. *Let $I \supset [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for some given numbers $m \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-3/q}\left|f'\left(\frac{b}{m}\right)\right|^m E_2(1,m,q) \tag{3.8}$$

holds for $q \ge 1$, where E_2 is defined as in Theorem 3.3.

Theorem 3.5. *Let $f, g : [0,\infty) \to (0,\infty)$ such that $f, g \in L([a,b])$ for $0 \le a < b < \infty$. If $f(x)$ is (α, m_1)-logarithmically convex and $g(x)$ is (α, m_2)-logarithmically convex on $[0, \frac{b}{m_i}]$ for $(\alpha, m_i) \in (0,1] \times (0,1]$ and $i = 1, 2$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le (b-a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2} E_3(\alpha), \tag{3.9}$$

where

$$\eta = f(a)g(a)\left[f\left(\frac{b}{m_1}\right)\right]^{-m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{-m_2} \quad \text{and} \quad E_3(\alpha) = \begin{cases} 1, & \eta = 1, \\ \dfrac{\eta^\alpha - 1}{\alpha \ln \eta}, & 0 < \eta < 1, \\ \dfrac{\eta^{1-\alpha}(\eta^\alpha - 1)}{\alpha \ln \eta}, & \eta > 1. \end{cases} \tag{3.10}$$

Proof. The (α, m)-logarithmic convexity of $f(x)$ and $g(x)$ yields

$$f\left(ta + m_1(1-t)\left(\frac{b}{m_1}\right)\right) \le [f(a)]^{t^\alpha}\left[f\left(\frac{b}{m_1}\right)\right]^{m_1(1-t^\alpha)} \quad \text{and} \quad g\left(ta + m_2(1-t)\left(\frac{b}{m_2}\right)\right) \le [g(a)]^{t^\alpha}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)},$$

from which it follows that

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}x &= (b-a)\int_0^1 f(ta + 1 - tb)g(ta + 1 - tb)\,\mathrm{d}t \\ &\le (b-a)\int_0^1 [f(a)]^{t^\alpha}[g(a)]^{t^\alpha}\left[f\left(\frac{b}{m_1}\right)\right]^{m_1(1-t^\alpha)}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2(1-t^\alpha)}\,\mathrm{d}t \\ &= (b-a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\int_0^1\left\{f(a)g(a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2}\right\}^{t^\alpha}\,\mathrm{d}t.\end{aligned}$$

When $\eta = 1$, we have $\int_0^1 \eta^{t^\alpha}\,\mathrm{d}t = 1$. When $\eta < 1$, we have

$$\int_0^1 \eta^{t^\alpha}\,\mathrm{d}t \le \int_0^1 \eta^{\alpha t}\,\mathrm{d}t = \frac{\eta^\alpha - 1}{\alpha \ln \eta}.$$

When $\eta > 1$, we have

$$\int_0^1 \eta^{t^\alpha}\,\mathrm{d}t \le \int_0^1 \eta^{\alpha t + 1 - \alpha}\,\mathrm{d}t = \frac{\eta^{1-\alpha}(\eta^\alpha - 1)}{\alpha \ln \eta}.$$

Theorem 3.5 is thus proved. □

Corollary 3.6. *Let $f, g : [0,\infty) \to (0,\infty)$ such that $f \cdot g \in L([a,b])$ for $0 \le a < b < \infty$. If $f(x)$ is m_1-logarithmically convex and $g(x)$ is m_2-logarithmically convex on $[0, \frac{b}{m_i}]$ for $i = 1, 2$ and some given numbers $m_1, m_2 \in (0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le (b-a)\left[f\left(\frac{b}{m_1}\right)\right]^{m_1}\left[g\left(\frac{b}{m_2}\right)\right]^{m_2} E_3(1), \tag{3.11}$$

where E_3 is defined as in Theorem 3.5.

Corollary 3.7. *Let $f, g : [0,\infty) \to (0,\infty)$ such that $f \cdot g \in L([a,b])$ for $0 \le a < b < \infty$. If $f(x)$ and $g(x)$ are (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $(\alpha, m) \in (0,1] \times (0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le (b-a)\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]^m E_3(\alpha), \tag{3.12}$$

where E_3 is defined as in Theorem 3.5.

Corollary 3.8. *Let $f, g : [0,\infty) \to (0,\infty)$ such that $f \cdot g \in L([a,b])$ for $0 \le a < b < \infty$. If $f(x)$ and $g(x)$ are m-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for some given number $m \in (0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}\,x \le (b-a)\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]^m E_3(1), \tag{3.13}$$

where E_3 is defined as in Theorem 3.5.

Remark 3.1. *In [6, 9–11, 13–16] the authors and their coauthors obtained some results on Hermite-Hadamard type inequalities for convex functions and for m- and (α, m)-geometrically convex functions.*

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