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The ditopology generated by pre-open and pre-closed sets, and submaximality in textures

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Abstract. The authors consider the ditopology generated by pre-open and pre-closed sets and investigate submaximality in ditopological texture spaces.

1. Introduction

Generalized open sets and generalized continuous functions have been firstly studied by Levine [13] who introduced semi-open sets and semi-continuity. The term "pre-open set" was introduced by Mashour, Abd El-Monsef and El-Deeb [14] in 1982 but the concept had appeared much earlier. For example Corson and Michael [8] used the term "locally dense" for pre-open sets in 1964. The notions of strongly compactness and M-precontinuity defined by using pre-open sets have been studied by Mashour, Abd El-Monsef, Hasenein and Noiri [16] and many authors [1, 12].

The notions of pre-open, pre-closed sets, strong compactness, strong cocompactness, strong stability and strong costability in ditopological texture spaces were introduced in [10, 11]. We will continue to study these concepts and the ditopology generated by pre-open and pre-closed sets in texture spaces.

Ditopological texture spaces were introduced by L.M. Brown as a point based setting for the study of fuzzy sets and provides a unified setting for the study of topology, bitopology and fuzzy topology [3–6]. However, the development of the theory has proceeded largely independently. On the one hand the notion of di-uniformity has been introduced in [17] and continued in [18–20], and a textural analogue of the notion of proximity, called a diextremity, was given in [24]. On the other hand, in [7, 10] compactness and in [11] strong compactness in ditopological texture spaces were introduced including M-prebicontinuous difunctions and preservation of strong compactness and cocompactness however early works in this area started in [3]. In the same direction the notion of real compactness in ditopological texture spaces was introduced in [21] and continued with (real) compactifications, also dicompleteness in [22, 23]. The notions of β-open, β-closed sets and β-compactness in ditopological texture spaces were introduced in [9].

The layout of the paper is as follows. In Section 2, the topology τ^p generated by pre-open sets and the co-topology κ^p generated by pre-closed sets are considered and as an important result "if $(S, \mathcal{S}, \tau^p, \kappa^p)$ is dicompact, then $(S, \mathcal{S}, \tau, \kappa)$ is strongly dicompact" has been proved. An important characterization for

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strong dicompactness was given in [11] and in this paper we prove this characterization via the ditopology generated by pre-open and pre-closed sets and bi-submaximality of the ditopological texture spaces.

In Section 3 the notions of prebicontinuity and strong prebicontinuity are defined and their properties are investigated. In addition the relationship between precontinuity (precocontinuity) and $({\tau_1}^p, \tau_2)$ continuity $(({\kappa_1}^p, \kappa_2)$ cocontinuity) is studied.

It was mentioned earlier in [11] that an arbitrary intersection of pre-closed sets is pre-closed and an arbitrary join of pre-open sets is pre-open. In the last section we deal with characterizations of pre-open (pre-closed) sets and finding under which conditions the family of pre-open (pre-closed) sets is a topology (co-topology). The notions of submaximality and co-submaximality are defined and it is shown that the ditopological texture space $(S, \mathcal{S}, \tau^p, \kappa^p)$ is bi-submaximal whether or not $(S, \mathcal{S}, \tau, \kappa)$ is submaximal.

To complete the introduction we recall some necessary concepts from [3–6].

Texture space: ([3]) Let S be a set. We work within a subset $\mathcal{S}$ of the power set $\mathcal{P}(S)$ called a *texturing*. A texturing is a point-separating, complete, completely distributive lattice with respect to inclusion, which contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $\mathcal{S}$ is a texturing of S the pair $(S, \mathcal{S})$ is called a *texture*.

For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S,\ s \notin P_u\}$$

are called respectively, the p-sets and q-sets of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts.

In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions, or equivalently if $P_s \not\subseteq Q_s$ for all $s \in S$. In this case $(S, \mathcal{S})$ is said to be *plain*.

Complementation: ([3]) A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

Example:

1. For any set X, $(X, \mathcal{P}(X), \pi_X)$ is the complemented *discrete texture* representing the usual set structure of X. Here the complementation $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complementation. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.
2. For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$.
3. The texture $(\mathbb{L}, \mathcal{L})$ is defined by $\mathbb{L} = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$. For $r \in \mathbb{L}$ $P_r = (0, r] = Q_r$

Ditopology: A *dichotomous topology* on $(S, \mathcal{S})$, or *ditopology* for short, is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

(τ_1) $S, \emptyset \in \tau$,
(τ_2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$,
(τ_3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,
(κ_1) $S, \emptyset \in \kappa$,
(κ_2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$,
(κ_3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

The elements of τ are called open and those of κ closed. We refer to τ as the topology and κ as the cotopology of (τ, κ).

If (τ, κ) is a ditopology on a complemented texture $(S, \mathcal{S}, \sigma)$, then we say that (τ, κ) *is complemented* if the equality $\kappa = \sigma[\tau]$ is satisfied. In this study, a complemented ditopological texture space is denoted by $(S, \mathcal{S}, \sigma, \tau, \kappa)$.

For $A \in \mathcal{S}$ the closure clA and interior $intA$ of A are defined by the equalities

$$clA = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and } intA = \bigvee\{G \in \tau \mid G \subseteq A\}$$

If (τ, κ) is a complemented ditopology on $(S, \mathcal{S}, \sigma)$, then we have $\sigma(clA) = int\sigma(A)$ and $\sigma(intA) = cl\sigma(A)$.

Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. In the following definition we consider $\mathcal{P}(S) \otimes \mathcal{T}$ and denote the p–sets and q–sets by $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ respectively. The notion of di-function is derived from that of direlation [5]. Now we recall the definition of difunction.

Difunction: ([5]) A *difunction* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is a direlation (f, F) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ satisfying the conditions:

(DF1) For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

(DF2) For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

If (f, F) is a difunction on $(S, \mathcal{S})$ to $(T, \mathcal{T})$, then (f, F) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists\, s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Bicontinuity: The difunction $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is called continuous if $B \in \tau_2 \Longrightarrow F^{\leftarrow}B \in \tau_1$, cocontinuous if $B \in \kappa_2 \Longrightarrow f^{\leftarrow}B \in \kappa_1$, and bicontinuous if it is both continuous and cocontinuous.

The early works on compactness in ditopological texture spaces was begun in [3] and continued in [7, 10]. Now let us recall some concepts from [7] which will be needed.

Let (τ, κ) be a ditopology on $(S, \mathcal{S})$ and take $A \in \mathcal{S}$. The family $\{G_i \mid i \in I\}$ is called open cover of A if $G_i \in \tau$ for all $i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$. A closed cocover can be defined dually, i.e. the family $\{F_i \mid i \in I\}$ is called closed cocover of A if $F_i \in \kappa$ for all $i \in I$ and $\bigcap_{i \in I} F_i \subseteq A$.

Dicompactness: ([7]) Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$ and $A \in \mathcal{S}$.

1. A is called compact if whenever $\{G_i \mid i \in I\}$ is an open cover of A, then there is a finite subset J of I with $A \subseteq \bigcup_{j \in J} G_j$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called compact if S is compact.
2. A is called cocompact if $\{F_i \mid i \in I\}$ is a closed cocover of A, then there is a finite subset J of I with $\bigcap_{j \in J} F_j \subseteq A$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called cocompact if $\emptyset$ is cocompact.
3. (τ, κ) is called stable if every $K \in \kappa$ with $K \neq S$ is compact.
4. (τ, κ) is called costable if every $G \in \tau$ with $G \neq \emptyset$ is cocompact.

A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called dicompact if it is compact, cocompact, stable and costable.

Example: Consider the texture $(\mathbb{I}, \mathcal{I})$ of Example (2) with the natural ditopology

$$\tau_{\mathbb{I}} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\} \quad \kappa_{\mathbb{I}} = \{[0, r] \mid r \in \mathbb{I}\} \cup \{\emptyset\}.$$

The ditopological texture space $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is dicompact.

2. A di-topology generated by pre-open and pre-closed sets

The notions of pre-open set and pre-closed set in ditopological texture spaces was introduced in [10, 11]. In this section we introduce a topology generated by pre-open sets and a co-topology generated by pre-closed sets. Now we recall the definitions and some properties of pre-open sets and pre-closed sets.

Definition 2.1. ([10]) Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$.

1. An element $G \in \mathcal{S}$ is *pre-open* if $G \subseteq intclG$.
2. An element $F \in \mathcal{S}$ is *pre-closed* if $clintF \subseteq F$.

As in the topological case, every τ-open set is pre-open but the converse is generally not true. Likewise every κ-closed set is pre-closed, but not conversely as in the following example.

Example 2.2. Let $\mathbb{L} = (0,1]$ and $\mathcal{L} = \{(0,t] \mid t \in (0,1]\}$. Then $(\mathbb{L},\mathcal{L})$ is a texture and take $\tau = \kappa = \{\emptyset,\mathbb{L}\}$. Consider the set $A = (0,\frac{1}{2}] \in \mathcal{L}$ the equality $clA = \mathbb{L} = intclA$ shows that A is pre-open however not open. Similarly A is pre-closed but not closed.

Clearly, there is no relation between pre-open and pre-closed sets in the general ditopological texture spaces but for the complemented ditopological texture space we have:

Proposition 2.3. ([11]) *Let $(S,\mathcal{S},\sigma,\tau,\kappa)$ be a complemented ditopological texture space and $A \in \mathcal{S}$. Then*

1. *A is pre-open iff $\sigma(A)$ is pre-closed.*
2. *A is pre-closed iff $\sigma(A)$ is pre-open.*

Example 2.4. ([11]) Let $(X,\mathcal{P}(X))$ be the discrete texture with complementation $\pi_X(Y) = X \setminus Y$, $Y \in \mathcal{P}(X)$. For a topology τ on X let $\tau^c = \{\pi_X(G) \mid G \in \tau\}$. Then $(X,\mathcal{P}(X),\pi_X,\tau,\tau^c)$ is a complemented ditopological texture space in which the pre-open sets and pre-closed sets are precisely the pre-open and pre-closed sets of (X,τ) respectively.

Note that by [11] an arbitrary intersection of pre-closed sets is pre-closed, and an arbitrary join of pre-open sets is pre-open. Indeed, let F_i, $i \in I$, be pre-closed sets. Then $\bigcap_{i\in I} F_i \subseteq F_i$, whence $clint\bigcap_{i\in I} F_i \subseteq clintF_i \subseteq F_i$ for all $i \in I$ thus $clint\bigcap_{i\in I} F_i \subseteq \bigcap_{i\in I} F_i$.

We will denote by $PO(S)$ the family of pre-open sets and by $PC(S)$ the family of pre-closed sets in $(S,\mathcal{S},\tau,\kappa)$.

In general for a topological space (X,τ) one can not expect the family $PO(X)$ to be topology since the intersection of a finite number of pre-open sets may not be pre-open. It follows that the pre-open sets in a topological space (X,τ) form a subbase for another topology which is finer than τ on X.

Similarly for a ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ since the intersection of a finite number of pre-open sets may not be pre-open, the family of $PO(S)$ may not be a topology and dually since the union of finite number of pre-closed sets may not be pre-closed, the family of $PC(S)$ may not be a co-topology but each of these classes generates a topology and a co-topology in a natural way. This gives rise to following definition:

Note that the following definition was given very briefly in [11].

Definition 2.5. Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological texture space.

1. The topology τ^p is defined by taking pre-open sets as a subbase on S.
2. The co-topology κ^p is defined by taking pre-closed sets as a subbase on S.

Hence the ditopology (τ^p,κ^p) is called the ditopology generated by pre-open and pre-closed sets with respect to the ditopology (τ,κ) and the ditopological texture space generated by a ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is denoted by $(S,\mathcal{S},\tau^p,\kappa^p)$.

It is clear that $\tau \subseteq \tau^p$ and $\kappa \subseteq \kappa^p$.

Now let us recall the definition of strong compactness, strong cocompactness, strong stability and strong costability. Let (τ,κ) be a ditopology on $(S,\mathcal{S})$ and take $A \in \mathcal{S}$. The family $\{G_j \mid j \in J\}$ is called a pre-open cover of A if $G_j \in PO(S)$ for all $j \in J$ and $A \subseteq \bigvee_{j\in J} G_j$. Pre-closed cocover can be defined dually i.e. the family $\{F_j \mid j \in J\}$ is called a pre-closed cocover of A if $F_j \in PC(S)$ for all $j \in J$ and $\bigcap_{j\in J} F_j \subseteq A$.

Definition 2.6. ([11, Definition 2.7]) Let (τ,κ) be a ditopology on the texture $(S,\mathcal{S})$ and $A \in \mathcal{S}$.

1. A is called strongly compact if every pre-open cover of A has a finite subcover. (τ,κ) is called strongly compact if S is strongly compact.
2. A is called strongly cocompact if every pre-closed cocover of A has a finite subcocover. (τ,κ) is called strongly cocompact if $\emptyset$ is strongly cocompact.

3. (τ, κ) is called strongly stable if every pre-closed set $F \in \mathcal{S} \setminus \{S\}$ is strongly compact.
4. (τ, κ) is called strongly costable if every pre-open set $G \in \mathcal{S} \setminus \{\emptyset\}$ is strongly cocompact.

Strong compactness and strong cocompactness are independent of one another, however for the complemented ditopological texture space they are equivalent. Similarly strong stability and strong costability are independent but they are equivalent for the complemented ditopological texture spaces. As a strong version of the notion of dicompactness, a ditopological texture space which has all four properties strongly compact, strongly stable, strongly cocompact and strongly costable is called strongly dicompact. It is clear that a strongly dicompact ditopological texture space is dicompact [11].

The generalization of the Alexander's subbase theorem was given in [7] to prove the Tychonoff theorem for compactness and cocompactness in ditopological texture spaces.

As an important result for topological spaces we recall that a topological space (X, τ) is strongly compact if and only if the topological space (X, τ^p) is compact [12]. For ditopological texture spaces we have the following result.

Theorem 2.7. *Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$.*

(1) The following are equivalent:
   - (a) *(τ, κ) is strongly compact.*
   - (b) *(τ^p, κ) is compact.*
   - (c) *(τ^p, κ^p) is compact.*

(2) The following are equivalent:
   - (a) *(τ, κ) is strongly cocompact.*
   - (b) *(τ, κ^p) is cocompact.*
   - (c) *(τ^p, κ^p) is cocompact.*

Proof. (1) $(a) \Rightarrow (b)$ Let C be a cover of S by members of τ^p. Since $(S, \mathcal{S}, \tau, \kappa)$ is strongly compact C has a finite subcover. By [7, Theorem 2.14] we obtain (τ^p, κ) is compact.

$(b) \Rightarrow (c)$ Since compactness depends on the topology not cotopology it is clear.

$(c) \Rightarrow (a)$ Let C be a cover of S consisting of pre-open sets of τ. Then $C \subseteq PO(S)$. Since (τ^p, κ^p) is compact C has a finite subcover by [7, Theorem 2.14].

Without any proof the equivalence of $(a) \Longleftrightarrow (b)$ has been given in [11]. Note here that (2) is dual to (1) and the proof is clear. □

Theorem 2.8. *Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$. Then*

(1) If (τ^p, κ^p) is stable, then (τ, κ) is strongly stable.
(2) If (τ^p, κ^p) is costable, then (τ, κ) is strongly costable.

Proof. (1) Let (τ^p, κ^p) be a stable ditopology and $F \in \mathcal{S} \setminus \{S\}$ be a pre-closed set. We want to show that F is strongly compact. Now let C be a pre-open cover of F. Since the ditopology (τ^p, κ^p) is stable, $F \in \kappa^p$ is compact. Thus there is a finite subset J such that $F \subseteq \bigcup_{j \in J} G_j$. That is, we find a finite subcover $C' \subseteq C \subseteq \tau^p$ such that $F \subseteq \bigcup C'$ and so it is clear that F is strongly compact.

(2) Suppose that (τ^p, κ^p) is a co-stable ditopology and $G \in \mathcal{S} \setminus \{\emptyset\}$ is a pre-open set. Let C be a pre-closed co-cover of G, that is, $\bigcap C = \bigcap_{i \in I} F_i \subseteq G$. Since (τ^p, κ^p) is co-stable, the set G which is pre-open is co-compact. Thus there is a finite subset J such that $\bigcap_{j \in J} F_j \subseteq G$. Consequently we find a finite subcover $C' \subseteq C \subseteq \kappa^p$ such that $\bigcap C' \subseteq G$ so G is strongly co-compact. □

Now we may state the following result.

Corollary 2.9. *For a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ if $(S, \mathcal{S}, \tau^p, \kappa^p)$ is dicompact, then $(S, \mathcal{S}, \tau, \kappa)$ is strongly dicompact.*

Proof. It is straightforward by Theorems 2.7 and 2.8. □

These results enable us to produce some results on strong compactness and strong cocompactness similar to results on compactness and cocompactness. For example, the following characterizations of strong dicompactness which is analogous to those for dicompactness was proved in [11]. Now we will give a new proof via the ditopology (τ^p, κ^p) generated by pre-open and pre-closed sets.

Theorem 2.10. ([11, Theorem 4.7]) *The following are equivalent for a ditopological texture space* $(S, \mathcal{S}, \tau, \kappa)$.

(1) $(S, \mathcal{S}, \tau, \kappa)$ *is strongly dicompact.*

(2) Every pre-closed, co-pre-open difamily with the finite exclusion property is bound.

(3) Every pre-open, co-pre-closed dicover has a subdicover which is finite and co-finite.

Note that in this theorem the implication (2) $\Longrightarrow$ (1) can be proved via the ditopology (τ^p, κ^p) generated by pre-open and pre-closed sets. Indeed, suppose that every pre-closed, co-pre-open difamily with the finite exclusion property is bound and the space $(S, \mathcal{S}, \tau, \kappa)$ is not strongly dicompact. In this case, by Corollary 2.9, the space $(S, \mathcal{S}, \tau^p, \kappa^p)$ is not dicompact, and so by [3, Theorem 3.5], there is a closed co-open difamily $\mathcal{D}$ with the finite exclusion property which is not bound. Since closed sets are pre-closed and open sets are pre-open, $\mathcal{D}$ is a pre-closed, co-pre-open difamily which is not bound with finite exclusion property and this is a contradiction.

3. Strongly prebicontinuous difunctions

It is known that strong compactness of topological spaces is preserved under M-precontinuity [15, 16]. In [11] the authors investigated the preservation of strong compactness, strong cocompactness, strong stability and strong costability under surjective difunctions. First of all we begin by recalling the definition of M-prebicontinuity for difunctions.

Definition 3.1. ([11]) Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a difunction.

1. (f, F) is called *M-precontinuous* if for every pre-open set $G \in \mathcal{S}_2$ the set $F^{\leftarrow} G \in \mathcal{S}_1$ is pre-open.
2. (f, F) is called *M-precocontinuous* if for every pre-closed set $K \in \mathcal{S}_2$ the set $f^{\leftarrow} K \in \mathcal{S}_1$ is pre-closed.
3. (f, F) is called *M-prebicontinuous* if it is M-precontinuous and M-precocontinuous.

On the other hand, the notion of precontinuity for functions have been defined in [12]. Now let us generalize this notion and "strong" forms of it to difunctions and we have the dual notions as expected.

Definition 3.2. Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$, $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be a difunction.

1. (f, F) is called *precontinuous* (*strongly precontinuous*) if for every open (pre-open) set $G \in \mathcal{S}_2$ the set $F^{\leftarrow} G \in \mathcal{S}_1$ is pre-open (open).
2. (f, F) is called *precocontinuous* (*strongly precocontinuous*) if for every closed (pre-closed) set $K \in \mathcal{S}_2$ the set $f^{\leftarrow} K \in \mathcal{S}_1$ is pre-closed (closed).
3. (f, F) is called *prebicontinuous* (*strongly prebicontinuous*) if it is precontinuous (*strongly precontinuous*) and precocontinuous (*strongly precocontinuous*).

Now by the above definitions we have:

Theorem 3.3. *A difunction* $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ *is*

Strongly-prebicontinuous $\Longrightarrow$ *bicontinuous* $\Longrightarrow$ *prebicontinuous.*

Proof. Let (f,F) be a strongly-precontinuous difunction from $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ to $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$. Take a set $G \in \tau_2$; then $G \in \mathcal{S}_2$ is a pre-open set and since (f,F) is strongly-precontinuous $F^{\leftarrow}G \in \tau_1$. For the second implication take a set $G \in \tau_2$. Since (f,F) be a bicontinuous difunction $F^{\leftarrow}G \in \tau_1$, then $F^{\leftarrow}G$ is pre-open.

The proof of the implications strongly-precocontinuous $\Longrightarrow$ cocontinuous $\Longrightarrow$ precocontinuous is dual and omitted.

□

Theorem 3.4. *A difunction $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is*

Strongly-prebicontinuous $\Longrightarrow$ M-prebicontinuous $\Longrightarrow$ prebicontinuous.

Proof. For the first implication, let (f,F) be a strongly-precontinuous difunction from $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ to $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$. Take a pre-open set $G \in \mathcal{S}_2$. Since (f,F) is strongly-precontinuous $F^{\leftarrow}G \in \mathcal{S}_1$ is open and then is pre-open. For the second implication, let (f,F) be a M-precontinuous difunction from $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ to $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$. Take an open set $G \in \mathcal{S}_2$. Since every open set is pre-open and (f,F) is M-precontinuous $F^{\leftarrow}G \in \mathcal{S}_1$ is pre-open.

For the difunction (f,F) the proof of the implications

strongly-precocontinuous $\Longrightarrow$ M-precocontinuous $\Longrightarrow$ precocontinuous

is dual of the above and is omitted. □

In the following we state the conditions under which the converse of Theorem 3.4 holds.

Proposition 3.5. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$, $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces and $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be a difunction.*

(1) If $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is bi-T_2 and strongly costable, then (f,F) is precontinuous $\Longrightarrow$ M−precontinuous,
(2) If $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is bi-T_2 and strongly costable, then (f,F) is M-precontinuous $\Longrightarrow$ strongly precontinuous,
(3) If $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is bi-T_2 and strongly stable, then (f,F) is precocontinuous $\Longrightarrow$ M−precocontinuous,
(4) If $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is bi-T_2 and strongly stable, then (f,F) is M-precocontinuous $\Longrightarrow$ strongly precocontinuous.

Proof. (1) Let $G \in \mathcal{S}_2$ be a pre-open set. Since $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is strongly costable G is open by [11, Corollary 3.10] and $F^{\leftarrow}G \in \mathcal{S}_1$ is pre-open as (f,F) is precontinuous hence (f,F) is M-precontinuous.

(2) Let $G \in \mathcal{S}_2$ be a pre-open set. Since (f,F) is M-precontinuous and $(S_1,\mathcal{S}_1)$ is strongly costable $F^{\leftarrow}G \in \mathcal{S}_1$ is pre-open hence it is an open set by [11, Corollary 3.10]. Thus (f,F) is strongly precontinuous.

(3) and (4) are dual to (1) and (2), respectively from [11, Corollary 3.10]. □

The relationships between M-precontinuity and $(\tau_1)^p-(\tau_2)^p$-continuity and also between M-precocontinuity and $(\kappa_1)^p-(\kappa_2)^p$-cocontinuity were investigated in [11]. Now we have the following:

Proposition 3.6. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$, $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces and $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be a difunction.*

(1) If $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is precontinuous, then $(f,F):(S_1,\mathcal{S}_1,{\tau_1}^p,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is continuous.
(2) If $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is precocontinuous, then $(f,F):(S_1,\mathcal{S}_1,\tau_1,{\kappa_1}^p)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is cocontinuous.
(3) If $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is prebicontinuous, then $(f,F):(S_1,\mathcal{S}_1,{\tau_1}^p,{\kappa_1}^p)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is bicontinuous.

Proof. (1) Take $G \in \tau_2$. Since (f,F) is precontinuous, $F^{\leftarrow}G \in \mathcal{S}_1$ is pre-open and we obtain $F^{\leftarrow}G \in {\tau_1}^p$ by the definition of τ^p.

(2) Take $K \in \kappa_2$. Since (f,F) is precocontinuous, $f^{\leftarrow}K \in \mathcal{S}_1$ is pre-closed and we obtain $f^{\leftarrow}K \in {\kappa_1}^p$ by the definition of κ^p.

(3) It is clear by (1) and (2). □

Proposition 3.7. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$, $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces and $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be a difunction. Then*

(1) $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is strongly precontinuous if and only if $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,{\tau_2}^p,\kappa_2)$ is continuous.

(2) $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is strongly precocontinuous if and only if $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,{\kappa_2}^p)$ is cocontinuous.

(3) $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is strongly prebicontinuous if and only if $(f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,{\tau_2}^p,{\kappa_2}^p)$ is bicontinuous.

Proof. (1) (⇒) Let (f,F) be strongly precontinuous and $G \in {\tau_2}^p$. Then $G=\bigvee_{i\in I}\bigcap_{j=1}^{n}{C_i}^j$ with ${C_i}^j$ are pre-open sets in $(S_2,\mathcal{S}_2)$.

Since (f,F) is strongly-precontinuous, $F^{\leftarrow}{C_i}^j \in \mathcal{S}_1$ is open for every $i \in I$. So $F^{\leftarrow}G = F^{\leftarrow}(\bigvee_{i\in I}\bigcap_{j=1}^{n}{C_i}^j) = \bigvee F^{\leftarrow}(\bigcap {C_i}^j) = \bigvee_{i\in I}\bigcap_{j=1}^{n}F^{\leftarrow}({C_i}^j)$ is open in $(S_1,\mathcal{S}_1)$. Hence (f,F) is $\tau_1-{\tau_2}^p$ continuous.

(⇐) Let (f,F) be $\tau_1-{\tau_2}^p$ continuous and $G \in \mathcal{S}_2$ be pre-open set. Therefore because of $F^{\leftarrow}G \in \tau_1$, (f,F) is strongly precontinuous.

(2) (⇒) Let (f,F) be strongly-precocontinuous and $K \in {\kappa_2}^p$. Then $K=\bigcap_{i\in I}\bigvee_{j=1}^{n}{D_i}^j$ where ${D_i}^j$ are pre-closed sets in $(S_2,\mathcal{S}_2)$.

Since (f,F) is strongly precocontinuous $f^{\leftarrow}{D_i}^j \in \mathcal{S}_1$ is closed for every $i \in I$. So $f^{\leftarrow}K = f^{\leftarrow}(\bigcap_{i\in I}\bigvee_{j=1}^{n}{D_i}^j) = \bigcap f^{\leftarrow}(\bigvee {D_i}^j) = \bigcap_{i\in I}\bigvee_{j=1}^{n}f^{\leftarrow}({D_i}^j)$ is closed in $(S_1,\mathcal{S}_1)$. Hence (f,F) is $\kappa_1-{\kappa_2}^p$ continuous.

(⇐) Let (f,F) be $\kappa_1-{\kappa_2}^p$ co-continuous and $K \in \mathcal{S}_2$ be pre-closed set. Therefore because of $f^{\leftarrow}K \in \kappa_1$, (f,F) is strongly pre-cocontinuous.

(3) immediate from (1) and (2). □

Now, the following theorem will be clear.

Theorem 3.8. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$, $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces, $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ be a difunction and $B \in \mathcal{S}_1$. In this case*

(1) If B is (τ_1,κ_1)-strongly compact and (f,F) is a precontinuous difunction, then $f^{\rightarrow}B \in \mathcal{S}_2$ is (τ_2,κ_2)-compact.

(2) If B is (τ_1,κ_1)-strongly cocompact and (f,F) is a precocontinuous difunction, then $F^{\rightarrow}B \in \mathcal{S}_2$ is (τ_2,κ_2)-cocompact.

Proof. It is obvious in view of Proposition 3.6 and [7, Theorems 2.5, 2.8] □

Corollary 3.9. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$, $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ be a surjective difunction. Then:*

(1) If $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is strongly compact and (f,F) is precontinuous, then $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is compact.

(2) If $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is strongly cocompact and (f,F) is precocontinuous, then $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is cocompact.

Proof. Straightforward by Theorem 3.8 for $B=S_1$. □

Corollary 3.10. *Let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ and $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ be ditopological texture spaces, and let $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ be bi-T_2.*

(1) If $(S_1,\mathcal{S}_1,\tau_1,\kappa_1)$ is strongly stable and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ is a prebicontinuous surjective difunction, then $(S_2,\mathcal{S}_2,\tau_2,\kappa_2)$ is stable.

(2) *If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is strongly costable and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ is a prebicontinuous surjective difunction, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is costable.*

Proof. (1) Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ be strongly stable. Since $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is bi-T_2 and by [11, Corollary 3.10] the space $(S_1, \mathcal{S}_1, {\tau_1}^p, {\kappa_1}^p)$ is stable. On the other hand, the difunction (f, F) is $({\tau_1}^p, {\kappa_1}^p) - (\tau_2, \kappa_2)$ bicontinuous by Proposition 3.6. Consequently $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is stable by [7, Theorem 3.17].
(2) Dual to (1). □

Finally, from Corollaries 3.9 and 3.10, we have the following:

Corollary 3.11. *If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is a strongly dicompact bi-T_2 space and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ is a surjective prebicontinuous difunction, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is dicompact.*

4. Submaximality in ditopological texture spaces

To prove that the family $PO(S)$ ($PC(S)$) is a topology (co-topology) it suffices to show that the intersection (union) of any two pre-open (pre-closed) sets is pre-open (pre-closed). We conclude this paper by investigating characterizations of pre-open (pre-closed) sets and by finding the conditions such that every pre-open (pre-closed) set is open (closed).

Definition 4.1. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $A \in \mathcal{S}$.

(1) If $clA = S$, then A is called a dense set in (τ, κ).
(2) If $intA = \emptyset$, then A is called a codense set in (τ, κ).
(3) If $A \subseteq S$ is dense and codense it is called bi-dense.

Note that for a complemented ditopological texture space $(S, \mathcal{S}, \sigma, \tau, \kappa)$ and for any set $A \in \mathcal{S}$ it is clear that A is dense if and only if $\sigma(A)$ is codense.
This definition justifies the following:

Proposition 4.2. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. We have the following statements:*

(1) Every dense set is pre-open,
(2) Every codense set is pre-closed,
(3) Every bidense set is pre-open and pre-closed.

Proof. (1) Let D be a dense set, that is $clD = S$. Then $intclD = intS = S \implies D \subseteq intclD = S \implies D$ is pre-open.
(2) Let B be a codense set, that is $intB = \emptyset$. Then $clintB = cl\emptyset = \emptyset \implies \emptyset = clintB \subseteq B \implies B$ is pre-closed.
(3) Immediate from (1) and (2). □

With regard to the above considerations we introduce the concept of "submaximality" in ditopological texture spaces.

Definition 4.3. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.
(i) $(S, \mathcal{S}, \tau, \kappa)$ is called submaximal if every dense subset in (τ, κ) is an element of τ.
(ii) $(S, \mathcal{S}, \tau, \kappa)$ is called co-submaximal if every codense subset in (τ, κ) is an element of κ.
(iii) A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called bi-submaximal if $(S, \mathcal{S}, \tau, \kappa)$ is submaximal and co-submaximal.

More exactly the above considerations yield a condition that a ditopological texture space to be bisubmaximal.

Theorem 4.4. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. If $PO(S) = \tau$ and $PC(S) = \kappa$, then $(S, \mathcal{S}, \tau, \kappa)$ is bi-submaximal.*

Proof. Let B be a dense set in (τ, κ). By Proposition 4.2 (1), B is pre-open and because of the equality $PO(S) = \tau$, B is an element of τ. Hence the space $(S, \mathcal{S}, \tau, \kappa)$ is submaximal.

Likewise let C be a codense set in (τ, κ). By Proposition 4.2 (2), C is pre-closed and because of $PC(S) = \kappa$, C is an element of κ. Hence the space $(S, \mathcal{S}, \tau, \kappa)$ is co-submaximal.

Consequently, the space $(S, \mathcal{S}, \tau, \kappa)$ is bi-submaximal. □

Theorem 4.5. *The ditopological texture space $(S, \mathcal{S}, \tau^p, \kappa^p)$ which is generated by a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is bi-submaximal.*

Proof. In general we have $\tau \subseteq \tau^p$ and $\kappa \subseteq \kappa^p$. Thus, if we take a dense set D and a codense set C in (τ^p, κ^p), then D is dense and C is codense in (τ, κ). In this case, D is pre-open and C is pre-closed by Proposition 4.2(2).

Consequently, $D \in \tau^p$ and $C \in \kappa^p$, that is, D is open and C is closed with respect to (τ^p, κ^p). Thus the ditopological texture space $(S, \mathcal{S}, \tau^p, \kappa^p)$ is submaximal, co-submaximal and so it is bi-submaximal. □

In topological spaces, a pre-open set can be written as an intersection of an open set and a dense set. The following example shows that this is not true for ditopological texture spaces.

Example 4.6. Consider the texture $(\mathbb{L}, \mathcal{L})$ with $\mathbb{L} = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$. Let $\tau = \kappa = \{\emptyset, (0, \frac{1}{2}], \mathbb{L}\}$. Take a set $A = (0, \frac{1}{4}]$; then $clA = (0, \frac{1}{2}]$ and $intclA = (0, \frac{1}{2}]$. Since $A \subseteq intclA$ then A is pre-open. However A can not be written as an intersection of an open set and a dense set.

However by restricting our attention to the discrete texture $(S, \mathcal{P}(S))$, we will obtain some useful characterizations for pre-open and pre-closed sets in the ditopology (τ, κ). For, if we recall $\pi_S(Y) = S \setminus Y$, $Y \subseteq \mathcal{P}(S)$ then we can define $\tau^c = \{\pi_S(G) \mid G \in \tau\}$ for a topology τ on S. Hence, $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ is a complemented ditopological texture space for which the pre-open sets and pre-closed sets are precisely same with the pre-open and pre-closed sets of (S, τ), respectively.

Corson and Michael [8] used the term "locally dense" for pre-open sets precisely, because any pre-open set in a topological space can be written as the intersection of an open set and a dense set. We now generalize this fact to the ditopological setting. In the complemented discrete ditopological texture space $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$, we will show that a pre-open set can be written as an intersection of an τ-open set and a dense set in (τ, κ) and dually, a pre-closed set can be written as an union of a κ-closed set and a codense set in (τ, κ) as following:

Lemma 4.7. a) *Let $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ be a complemented discrete ditopological texture space and $A \subseteq S$. Then the following are equivalent:*

i) $A \in PO(S)$

ii) *A is the intersection of a set in τ and a dense set in (τ, κ).*

b) *Let $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ be a complemented discrete ditopological texture space and $K \subseteq S$. Then the following are equivalent:*

i) $K \in PC(S)$

ii) *K is the union of a set in κ and a codense set in (τ, κ).*

Proof. a) (i) ⇒ (ii) : Let $A \in PO(S)$. In this case, if take the set $B = A \cup (S \setminus intclA)$ it is easy to verify that $clB = S$ and thus B is dense. Also note that the set $intclA$ is open and we have the equality $A = intclA \cap B$.

(ii)⇒ (i): Suppose that the set A is the intersection of a set $G \in \tau$ and a dense set D. Then $clA = clG$, and by $A \subseteq G \subseteq clG = clA$, we have $A \subseteq intclA$ and A is a pre-open set.

b) (i) $\Rightarrow$ (ii) : Let $K \in PC(S)$. In this case, if we take the set $C = K \cap (S \setminus clintK)$, it is easy to verify that $intC = \emptyset$ and thus C is codense. Also note that the set $clintK$ is the element of κ and we have the equality $K = clintK \cup C$.

(ii)$\Rightarrow$ (i): Suppose that the set K is the union of a set Z in κ and a codense set C. Then $intK = intZ$, and by $intK = intZ \subseteq Z \subseteq K$, we have $clintK \subseteq K$ and K is a pre-closed set. □

Note that an arbitrary intersection of pre-closed sets is pre-closed and an arbitrary join of pre-open sets is pre-open [11] in ditopological texture spaces. Hence the family $PO(S)(PC(S))$ is a topology (co-topology) if and only if the intersection (union) of any two pre-open (pre-closed) sets is pre-open (pre-closed). In order to give the required conditions such that $PO(S)(PC(S))$ is a topology (co-topology). First we require the following proposition.

Proposition 4.8. *Let $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ be a complemented discrete ditopological texture space.*

1. *If U is an open set and $A \subseteq S$, then $U \cap clA \subseteq cl(U \cap A)$.*
2. *If K is a closed set and $B \subseteq S$, then $int(K \cup B) \subseteq K \cup intB$.*

Proof. (1) Suppose that $U \cap clA \not\subseteq cl(U \cap A)$. Then there exists an $s \in S$ such that $U \cap clA \not\subseteq Q_s$ and $P_s \not\subseteq cl(U \cap A)$. In this case $U \not\subseteq Q_s$ and $clA \not\subseteq Q_s$ that is $s \in U$ and because of $clA = \bigcap\{K \in \kappa \mid A \subseteq K\} \not\subseteq Q_s$, for all $K \in \kappa$ such that $A \subseteq K$, $s \in K$.

On the other hand, since $P_s \not\subseteq cl(U \cap A) = \bigcap\{K \in \kappa \mid U \cap A \subseteq K\}$ there exists $K_1 \in \kappa$ such that $s \notin K_1$ and $U \cap A \subseteq K_1$. Clearly $A \subseteq K_1 \cup S/U$ because of $U \cap A \subseteq K_1$. Now note that $S/U \in \tau^c$ and moreover $S/U \in \kappa = \pi_S(\tau)$. Hence A is contained in the closed set $K_1 \cup S/U \in \tau^c$ and so $s \in K_1 \cup S/U$ but this is a contradiction. Consequently $U \cap clA \subseteq cl(U \cap A)$.

(2) Dual to (1) and hence omitted. □

By virtue of Proposition 4.8 we may state the following briefly.

Proposition 4.9. *Let $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ be a complemented discrete ditopological texture space and $A, B \subseteq S$*

1. *If A is open and B is pre-open, then $A \cap B$ is pre-open.*
2. *If A is closed and B is pre-closed, then $A \cup B$ is pre-closed.*

Proof. (1) Since A is open and B is pre-open set we have the statement

$$A \cap B \subseteq A \cap intclB = intA \cap intclB = int(A \cap clB) \subseteq intcl(A \cap B).$$

by Proposition 4.8(1). Hence it is clear that $A \cap B$ is pre-open.

(2) It is proved as dual to (1) by Proposition 4.8(2). □

In topological spaces the family of pre-open sets is a topology if and only if the intersection of any two dense sets is pre-open. Now we will give an analogue of this fact and the dual for the pre-closed sets in the ditopological context as shown in the following.

Theorem 4.10. *Let $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ be a complemented discrete ditopological texture space.*

a) *The family $PO(S)$ is a topology if and only if the intersection of any two dense sets is pre-open.*
b) *The family $PC(S)$ is a co-topology if and only if the union of any two codense sets is pre-closed.*

Proof. (a) ($\Rightarrow$) Suppose that $PO(S)$ is a topology and let A and B be dense sets. Since dense sets are pre-open by Proposition 4.2(1), A and B are pre-open sets. In this case by the fact that the family $PO(S)$ is a topology the intersection of pre-open sets A and B is pre-open.

($\Leftarrow$) Clearly $\emptyset, S \in PO(S)$ and an arbitrary join of pre-open sets is pre-open. We will show that the intersection of two pre-open sets is pre-open. Let A and B be pre-open sets. By Lemma 4.7(a) there exist an open set C and a dense set D such that $A = C \cap D$. Similarly for an open set E and a dense set F we have $B = E \cap F$. So

$A \cap B = C \cap D \cap E \cap F = (C \cap E) \cap (D \cap F)$. On the other hand, by hypothesis the intersection of two dense sets D, F is pre-open. Hence $A \cap B$ is pre-open by Proposition 4.9(1).

(b) ($\Rightarrow$) Let A and B be codense sets. Thus they are pre-closed by Proposition 4.2(2) and since the family $PC(S)$ is a co-topology, the union of A and B is pre-closed.

($\Leftarrow$) It is clear that $\emptyset, S \in PC(S)$ and an arbitrary intersection of pre-closed sets is pre-closed. Now let A and B be pre-closed sets. By Lemma 4.7(b) there exist a closed set C and a codense set D such that $A = C \cup D$. Similarly for a closed set E and a codense set F we have $B = E \cup F$. Then $A \cup B = C \cup D \cup E \cup F = (C \cup E) \cap (D \cup F)$. On the other hand, the union of two codense sets D, F is pre-closed by hypothesis. Hence $A \cup B$ is pre-closed by Proposition 4.9(2). □

Now we can give a condition that implies a pre-open set is open and a pre-closed set is closed.

Theorem 4.11. *If a complemented discrete ditopological texture space $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ is bi-submaximal, then $PO(S) = \tau$ and $PC(S) = \kappa$.*

Proof. Since every open set is pre-open, it is clear that $\tau \subseteq PO(S)$. In order to show the converse, take a set $A \in PO(S)$. Then A can be written as an intersection of an open set G in τ and a dense set D in (τ, κ) by Lemma 4.7 (a). On the other hand, $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ is submaximal and so every dense set in (τ, κ) is open, that is the set D is open. Therefore, the set $A = G \cap D$ is open, that is $A \in \tau$ and so $PO(S) \subseteq \tau$.

Now let us prove the equality $PC(S) = \kappa$. Firstly we have $\kappa \subseteq PC(S)$ since every closed set is pre-closed. In order to show the converse, take a set $A \in PC(S)$. In this case, A can be written as an union of a set K in κ and a codense set C in (τ, κ) by the Lemma 4.7(b). On the other hand, since $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ is co-submaximal and thus every codense set in (τ, κ) is closed, C is closed. Therefore, the set $A = K \cup C$ is closed, that is $PC(S) \subseteq \kappa$. □

Finally we have the following:

Corollary 4.12. *If a complemented discrete ditopological texture space $(S, \mathcal{P}(S), \pi_S, \tau, \tau^c)$ is bi-submaximal, then $\tau^p = \tau$ and $\kappa^p = \kappa$.*

Proof. It is clear that $\tau \subseteq \tau^p$ and $\kappa \subseteq \kappa^p$ for a general ditopological texture space. For a complemented discrete ditopological texture space, we have $\tau^p \subseteq \tau$ and $\kappa^p \subseteq \kappa$ by Theorem 4.11. □

As mentioned in the introduction, strong dicompactness is a strong version of the notion of dicompactness since strong dicompactness implies dicompactness clearly. On the other hand, in [11] the authors have shown that for bi-T_2 spaces strong dicompactness coincides with dicompactness. However without any ditopological separation axiom we have shown that if the complemented discrete ditopological texture space is bi-submaximal, then dicompactness implies strong dicompactness. It remains an open problem to determine conditions for general ditopological texture spaces.

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