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A problem related to Bárány–Grünbaum conjecture

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Abstract. In this paper we prove that for any absolute continuous Borel probability measure μ on the sphere S^2 and any $t \in [0, \frac{1}{4}]$ there exist four great semi-circles $\ell_1, \ldots, \ell_4$ emanating from a point $x \in S^2$ that partition sphere S^2 into four angular sectors $\sigma_1, \ldots, \sigma_4$, counter clockwise oriented, such that $\mu(\sigma_1) = \mu(\sigma_4) = t$, $\mu(\sigma_2) = \mu(\sigma_3) = \frac{1}{4} - t$, and $\text{area}(\sigma_1) = \text{area}(\sigma_4)$, $\text{area}(\sigma_2) = \text{area}(\sigma_3)$.

1. Introduction and statement of results

Let P be a convex body in the plane. It is know that there exist two orthogonal lines that part P into four pieces of equal area. The following natural question was asked jointly by Imre Bárány and Branko Grünbaum.

Conjecture 1.1 (Bárány–Grünbaum conjecture in $\mathbb{R}^2$). *Let P be a convex body in the plane of area 1 and $t \in [0, \frac{1}{4}]$. There exist two orthogonal lines that partition P into four pieces of area t, t, $\frac{1}{2} - t$ and $\frac{1}{2} - t$ in counter clockwise order.*

In the case when the diameter of P is at least $\sqrt{37}$ times the minimum width, the conjecture is settled by Arocha, Jernimo-Castro, Montejano, and Roldn-Pensado in [1].
The conjecture can be naturally generalized to the following question:

Let μ be an absolutely continuous, Borel, probablity measure on $\mathbb{R}^2$ and $t \in [0, \frac{1}{4}]$. There exist two orthogonal lines that partition $\mathbb{R}^2$ into four pieces that contain t, t, $\frac{1}{2} - t$ and $\frac{1}{2} - t$ amount of measure μ, in counter clockwise order.

Moving the problem from the plane to the sphere S^2 we get a similar, not equivalent, but equally resistant conjecture.

Conjecture 1.2 (Bárány–Grünbaum conjecture on S^2). *Let μ be an absolutely continuous, Borel, probablity measure on the sphere S^2 and $t \in [0, \frac{1}{4}]$. There exist two great circles ℓ_1 and ℓ_2 that are mutually orthogonal and partition sphere S^2 into four angular sectors $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, counter clockwise oriented, with the property that*

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_3) = \mu(\sigma_4).$$

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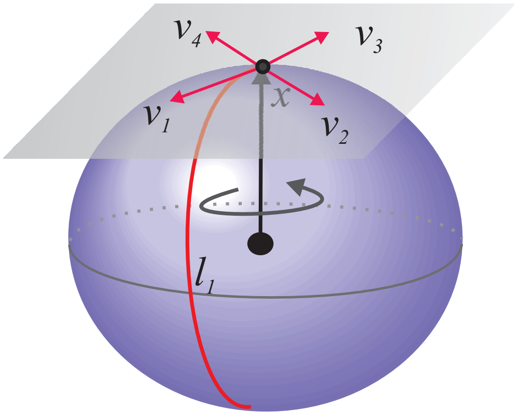

Figure 1: An example of a 4-fan

The great circles ℓ_1 and ℓ_2 on the sphere S^2 are mutually orthogonal if the unit tangent vectors v_1 and v_2 to ℓ_1 and ℓ_2 at one of two antipodal intersection points are orthogonal, i.e., $v_1 \perp v_2$.

In this paper we prove the following weaker version of the Bárány–Grünbaum conjecture on S^2.

Theorem 1.3. *Let μ be an absolutely continuous, Borel, probability measure on the sphere S^2 and $t \in [0, \frac{1}{4}]$. There exist four great semi-circles $\ell_1, \dots, \ell_4$ emanating from a point $x \in S^2$ that partition sphere S^2 into four angular sectors counter clockwise oriented:*

$$\sigma_1 = (\ell_1, \ell_2), \sigma_2 = (\ell_2, \ell_3), \sigma_3 = (\ell_3, \ell_4), \sigma_4 = (\ell_4, \ell_1),$$

having property that

$$\mu(\sigma_1) = \mu(\sigma_4) = t, \ \ \mu(\sigma_2) = \mu(\sigma_3) = \tfrac{1}{4} - t \ \ \textit{and} \ \ \operatorname{area}(\sigma_1) = \operatorname{area}(\sigma_4), \ \ \operatorname{area}(\sigma_2) = \operatorname{area}(\sigma_3).$$

The condition about the areas can be reformulated in terms of angles that determine angular sectors. Let us denote by

$$\alpha_1 = \sphericalangle(\ell_1, \ell_2), \alpha_2 = \sphericalangle(\ell_2, \ell_3), \alpha_3 = \sphericalangle(\ell_3, \ell_4), \alpha_4 = \sphericalangle(\ell_4, \ell_1).$$

Then $\operatorname{area}(\sigma_i) = \frac{\alpha_i}{2\pi} \operatorname{area}(S^2)$ for each $i \in \{1, 2, 3, 4\}$. Therefore, condition on areas read off in terms of angles as $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$, which implies that $\ell_1 \cup \ell_3$ is a great circle, or $\ell_1 = -\ell_3$.

2. From geometric problem to an equivariant problem

The proof of Theorem 1.3 is obtained via the configuration test map method. In this section we relate the claim of the theorem with the non-existence of a $\mathbb{Z}/2$-equivariant map from the Stiefel manifold $V_2(\mathbb{R}^3)$ into the sphere S^1.

A 4-*fan* $(x; \ell_1, \dots, \ell_4)$ on the sphere S^2 consists of a point $x \in S^2$ on the sphere and four pairwise different great semi-circles $\ell_1, \dots, \ell_4$ emanating from x oriented in the counter clockwise order. For a 4-fan $(x; \ell_1, \dots, \ell_4)$ we can also use notation

1. $(x; \sigma_1, \dots, \sigma_4)$ where σ_i denotes the open angular sector between ℓ_i and ℓ_{i+1}, $i \in \{1, \dots, 4\}$, $\ell_5 \equiv \ell_1$; or
2. $(x; v_1, \dots, v_4)$ where $v_i \in T_x S^2$ denotes the unite tangent vector determined by the great semicircle curve ℓ_i, $i \in \{1, \dots, 4\}$. Observe that $v_1, \dots, v_4 \in \operatorname{span}(x)^{\perp}$.

Further on, $\mathcal{F}_4$ denotes the space of all 4-fans on S^2.

Let μ be an absolutely continuous, Borel, probablity measure on the sphere S^2 and $t \in [0, \frac{1}{4}]$. Consider the following configuration space determined by μ and give t:

$$X_{\mu,t} := \{(x; \sigma_1, \dots, \sigma_4) \in \mathcal{F}_4 : \mu(\sigma_1) = \mu(\sigma_4) = t, \ \mu(\sigma_2) = \mu(\sigma_3) = \tfrac{1}{4} - t\}.$$

Every point $(x; v_1, \dots, v_4) = (x; \ell_1, \dots, \ell_4)$ in the space $X_{\mu,t}$ is completely determined by the point x and the first tangent vector v_1 to ℓ_1. After fixing $(x, v_1) \in V_2(\mathbb{R}^3)$ the remaining tangent vectors v_2, v_3, v_4 can be obtain from the condition $\mu(\sigma_1) = \mu(\sigma_4) = t$, $\mu(\sigma_2) = \mu(\sigma_3) = \frac{1}{4} - t$. Thus, $X_{\mu,t} \approx V_2(\mathbb{R}^3)$. Moreover, the configuration space $X_{\mu,t}$ has a natural free $\mathbb{Z}/2 = \langle \varepsilon \rangle$ action given by

$$\varepsilon \cdot (x; \ell_1, \ell_2, \ell_3, \ell_4) = (-x; \ell_1, \ell_4, \ell_3, \ell_2) \quad \text{or} \quad \varepsilon \cdot (x; \sigma_1, \sigma_2, \sigma_3, \sigma_4) = (-x; \sigma_4, \sigma_3, \sigma_2, \sigma_1).$$

Let us define a continuous map $f_{\mu,t} \colon X \to \mathbb{R}^2$ by

$$(x; \sigma_1, \sigma_2, \sigma_3, \sigma_4) \longmapsto (\text{area}(\sigma_1) - \text{area}(\sigma_4), \text{area}(\sigma_2) - \text{area}(\sigma_3)).$$

If we introduce the antipodal $\mathbb{Z}/2$-action on $\mathbb{R}^2$ it is not hard to see that the map τ is a $\mathbb{Z}/2$-equivariant map. Indeed, the following diagram commutes

$$\begin{array}{ccc} (x; \sigma_1, \sigma_2, \sigma_3, \sigma_4) & \xrightarrow{f_{\mu,t}} & (\text{area}(\sigma_1) - \text{area}(\sigma_4), \text{area}(\sigma_2) - \text{area}(\sigma_3)) \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ (-x; \sigma_4, \sigma_3, \sigma_2, \sigma_1) & \xrightarrow{f_{\mu,t}} & (\text{area}(\sigma_4) - \text{area}(\sigma_1), \text{area}(\sigma_3) - \text{area}(\sigma_2)). \end{array}$$

The main properties of this map, coming from its construction, are summarised in the following proposition.

Proposition 2.1. (i) *If for the given μ and t the claim of Theorem 1.3 holds, then $0 \in \text{im}\, f_{\mu,t} \subset \mathbb{R}^2$.*

(ii) *If for the given t the claim of Theorem 1.3 does not hold, then there exists a measure μ such that $0 \notin \text{im}\, f_{\mu,t} \subset \mathbb{R}^2$. Consequently, the map $f_{\mu,t}$ factors in the following way*

$$\begin{array}{ccc} V_2(\mathbb{R}^3) \approx X & \xrightarrow{\quad f_{\mu,t} \quad} & \mathbb{R}^2 \\ & {\scriptstyle g_{\mu,t}} \searrow \quad \nearrow & \\ & \mathbb{R}^2 \backslash \{0\} & \end{array}$$

where $g_{\mu,t} : V_2(\mathbb{R}^3) \to \mathbb{R}^2 \backslash \{0\}$ is a $\mathbb{Z}/2$-equivariant map.

(iii) *If for the given t the claim of Theorem 1.3 does not hold, then there exists a $\mathbb{Z}/2$-equivariant map $V_2(\mathbb{R}^3) \to S^1$. The action on S^1 is assumed to be antipodal.*

(iv) *If there is no $\mathbb{Z}/2$-equivariant map $V_2(\mathbb{R}^3) \to S^1$, then the claim of Theorem 1.3 holds for every t and any μ.*

Thus, if we prove that there is no $\mathbb{Z}/2$-equivariant map $V_2(\mathbb{R}^3) \to S^1$ we have concluded the proof of Theorem 1.3.

3. Non existence of an equivaraint map

The non-existence of a $\mathbb{Z}/2$-equivariant map $V_2(\mathbb{R}^3) \to S^1$ will be proved via the Fadell–Husseini ideal valued index theory applied to the group $\mathbb{Z}/2$ and integer coefficients. We make a quick review of basic properties of the Fadell–Husseini index theory. For further details consult the original paper of Fadell and Husseini [3] and for use of integer coefficients [2].

Consider a finite group G. Let X be a G-space, R a commutative ring with unit and $p_X \colon X \to \text{pt}$ the G-equivariant projection. The Fadell–Husseini index of the G-space X is the kernel ideal of the induced map $p_X^* \colon H_G^*(\text{pt}; R) \to H_G^*(X; R)$ in the equivariant cohomology, i.e.,

$$\text{Index}_G(X; R) := \ker\left(p_X^* \colon H_G^*(\text{pt}; R) \to H_G^*(X; R)\right).$$

Here $H_G^*(X; R) := H^*(\text{E}G \times_G X; R)$ and therefore $H_G^*(\text{pt}; R) \cong H^*(G; R)$.

Let X and Y be G-spaces and $f\colon X \to Y$ a G-equivariant map. There are following commutative triangles of G-spaces and related cohomology groups:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle p_Y} & \\ & \mathrm{pt} & \end{array} \qquad\qquad \begin{array}{ccc} H^*_G(X;R) & \xleftarrow{\ f^*\ } & H^*_G(Y;R) \\ & {\scriptstyle p_X^*}\nwarrow \quad \nearrow{\scriptstyle p_Y^*} & \\ & H^*_G(\mathrm{pt};R). & \end{array}$$

Consequently, $\ker p_Y^* \subseteq \ker p_X^*$, i.e.,

$$\mathrm{Index}_G(Y;R) \subseteq \mathrm{Index}_G(X;R) \subseteq H^*_G(\mathrm{pt};R) = H^*(G;R). \tag{1}$$

Let us recall that the cohomology ring of the group $\mathbb{Z}/2$, or $\mathbb{Z}/2$-equivariant cohomology of the point, with integer coefficients $\mathbb{Z}$ can be presented by

$$H^*(\mathbb{Z}/2;\mathbb{Z}) = \mathbb{Z}[T]/\langle 2T\rangle$$

where $\deg(T) = 2$.

In computation of indexes that follows we use the Serre spectral sequence of the Borel construction fibrations:

$$S^1 \longrightarrow \mathrm{E}\mathbb{Z}/2 \times_{\mathbb{Z}/2} S^1 \longrightarrow \mathrm{B}\mathbb{Z}/2 \quad \text{and} \quad V_2(\mathbb{R}^3) \longrightarrow \mathrm{E}\mathbb{Z}/2 \times_{\mathbb{Z}/2} V_2(\mathbb{R}^3) \longrightarrow \mathrm{B}\mathbb{Z}/2.$$

The E_2-terms, with $\mathbb{Z}$ coefficients, of these spectral sequences $E_t^{r,s}(S^1)$ and $E_t^{r,s}(V_2(\mathbb{R}^3))$ are given by

$$E_2^{r,s}(S^1) = H^r(\mathbb{Z}/2;H^s(S^1;\mathbb{Z})) \quad \textit{and} \quad E_2^{r,s}(V_2(\mathbb{R}^3)) = H^r(\mathbb{Z}/2;H^s(V_2(\mathbb{R}^3);\mathbb{Z})).$$

3.1. $\mathrm{Index}_{\mathbb{Z}/2}(S^1;\mathbb{Z})$

The group $\mathbb{Z}/2$ acts on the sphere S^1 antipodally and therefore orientation preserving. Consequently, $H^s(V_2(\mathbb{R}^3);\mathbb{Z})$ is a trivial $\mathbb{Z}/2$-module and so the E_2-term of the $E_t^{r,s}(S^1)$ spectral sequence transforms into the tensor product

$$E_2^{r,s}(S^1) = H^r(\mathbb{Z}/2;H^s(S^1;\mathbb{Z})) \cong H^r(\mathbb{Z}/2;\mathbb{Z}) \otimes H^s(S^1;\mathbb{Z}) \cong \begin{cases} H^r(\mathbb{Z}/2;\mathbb{Z}), & \text{for } s = 0,1 \\ 0, & \text{otherwise.} \end{cases}$$

Since, S^1 is a free $\mathbb{Z}/2$ space we have that $\mathrm{E}\mathbb{Z}/2 \times_{\mathbb{Z}/2} S^1 \simeq S^1/(\mathbb{Z}/2)$. The spectral sequence $E_t^{r,s}(S^1)$ converges to $H^*(\mathrm{E}\mathbb{Z}/2 \times_{\mathbb{Z}/2} S^1;\mathbb{Z}) \cong H^*(S^1/(\mathbb{Z}/2);\mathbb{Z})$. Therefore, $E_3^{r,s}(S^1) \cong E_\infty^{r,s}(S^1) = 0$ for all $r+s \geq 2$ and so

$$\mathrm{Index}_{\mathbb{Z}/2}(S^1;\mathbb{Z}) = \langle T\rangle. \tag{2}$$

3.2. $\mathrm{Index}_{\mathbb{Z}/2}(V_2(\mathbb{R}^3);\mathbb{Z})$

In order to describe the spectral sequence $E_t^{r,s}(V_2(\mathbb{R}^3))$ recall that the cohomology of the Stiefel manifold is given by

$$H^s(V_2(\mathbb{R}^3);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & s = 0,3, \\ \mathbb{Z}/2, & s = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The E_2-term of the spectral sequence $E_t^{r,s}(V_2(\mathbb{R}^3))$ can be now determined in more details

$$E_2^{r,s}(V_2(\mathbb{R}^3)) = H^r(\mathbb{Z}/2;H^s(V_2(\mathbb{R}^3);\mathbb{Z})) = \begin{cases} H^r(\mathbb{Z}/2;\mathbb{Z}), & s = 0,3, \\ H^r(\mathbb{Z}/2;\mathbb{Z}/2), & s = 2, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the 1-row of the E_2 term vanishes. Therefore, $0 \neq T \in E_2^{2,0}(V_2(\mathbb{R}^3)) \cong E_\infty^{2,0}(V_2(\mathbb{R}^3))$ and consequently

$$T \notin \mathrm{Index}_{\mathbb{Z}/2}(V_2(\mathbb{R}^3); \mathbb{Z}). \tag{3}$$

The relation (3) will be enough to conclude the proof of Theorem 1.3. Nevertheless, let us point out that the spectral sequence $E_t^{r,s}(V_2(\mathbb{R}^3))$ can be computed in all details and it can be proved that

$$\mathrm{Index}_{\mathbb{Z}/2}(V_2(\mathbb{R}^3); \mathbb{Z}) = \langle T^2 \rangle.$$

4. Proof of Theorem 1.3

According to Proposition 2.1 in order to prove Theorem 1.3 we need to prove the non-existence of a $\mathbb{Z}/2$-equivariant map $V_2(\mathbb{R}^3) \to S^1$. The $\mathbb{Z}/2$-actions on $V_2(\mathbb{R}^3)$ and S^1 are assumed to be as introduced in Section 2.

Assume the opposite, let $f \colon V_2(\mathbb{R}^3) \to S^1$ be a $\mathbb{Z}/2$-equivariant map. The basic property of the Fadell–Husseini index theory (1) implies that

$$\mathrm{Index}_{\mathbb{Z}/2}(S^1; \mathbb{Z}) \subseteq \mathrm{Index}_{\mathbb{Z}/2}(V_2(\mathbb{R}^3); \mathbb{Z}).$$

This contradicts the fact that $\mathrm{Index}_{\mathbb{Z}/2}(S^1; \mathbb{Z}) = \langle T \rangle$, (2), and $T \notin \mathrm{Index}_{\mathbb{Z}/2}(V_2(\mathbb{R}^3); \mathbb{Z})$, (3). Thus, there can not be $\mathbb{Z}/2$-equivariant map $V_2(\mathbb{R}^3) \to S^1$ and consequently Theorem 1.3 holds.

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