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The perturbation of the group inverse under the stable perturbation in a unital ring

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Abstract. Let $\Re$ be a ring with unit 1 and $a \in \Re$, $\bar{a} = a + \delta a \in \Re$ such that $a^\#$ exists. In this paper, we mainly investigate the perturbation of the group inverse $a^\#$ on $\Re$. Under the stable perturbation, we obtain the explicit expressions of $\bar{a}^\#$. The results extend the main results in [19, 20] and some related results in [18]. As an application, we give the representation of the group inverse of the matrix $\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}$ on the ring $\Re$ for certain $d, b, c \in \Re$.

1. Introduction

Let $\Re$ be a ring with unit and $a \in \Re$. we consider an element $b \in \Re$ and the following equations:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ a^k ba = a^k, \quad (4)\ ab = ba.$$

If b satisfies (1), then b is called a pseudo–inverse or 1–inverse of a. In this case, a is called regular. The set of all 1–inverse of a is denoted by $a^{\{1\}}$; If b satisfies (2), then b is called a 2–inverse of a, and a is called anti–regular. The set of all 2–inverse of a is denoted by $a^{\{2\}}$; If b satisfies (1) and (2), then b is called the generalized inverse of a, denoted by a^+; If b satisfies (2), (3) and (4), then b is called the Drazin inverse of a, denoted by a^D. The smallest integer k is called the index of a, denoted by $ind(a)$. If $ind(a) = 1$, we say a is group invertible and b is the group inverse of a, denoted by $a^\#$.

The notation so–called stable perturbation of an operator on Hilbert spaces and Banach spaces is introduced by G. Chen and Y. Xue in [4, 6]. Later the notation is generalized to Banach Algebra by Y. Xue in [19] and to Hilbert C^*–modules by Xu, Wei and Gu in [17]. The stable perturbation of linear operator was widely investigated by many authors. For examples, in [5], G. Chen and Y. Xue study the perturbation for Moore–Penrose inverse of an operator on Hilbert spaces; Q. Xu, C. Song and Y. Wei studied the stable perturbation of the Drazin inverse of the square matrices when $I - A^\pi - B^\pi$ is nonsingular in [16] and Q. Huang and W. Zhai worked over the perturbation of closed operators in [12, 13], etc..

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The Drazin inverse has many applications in matrix theory, difference equations, differential equations and iterative methods. In 1979, Campbell and Meyer proposed an open problem: how to find an explicit expression for the Drazin inverse of the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in terms of its sub-block in [1]? The representation of the Drazin inverse of a triangular matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ has been given in [3, 9, 11]. In [8], Deng and Wei studied the Drazin inverse of the anti-triangular matrix $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ and given its representation under some conditions.

In this paper, we investigate the stable perturbation for the group inverse of an element in a ring. Assume that $1 - a^\pi - \bar{a}^\pi$ is invertible, we present the expression of $\bar{a}^\#$ and $\bar{a}^D$. This extends the related results in [18, 20]. As an applications, we study the representation for the group inverse of the anti–triangular matrix $\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}$ on the ring.

2. Some Lemmas

Throughout the paper, $\Re$ is always a ring with the unit 1. In this section, we give some lemmas:

Lemma 2.1. *Let $a, b \in \Re$. Then $1 + ab$ is invertible if and only if $1 + ba$ is invertible. In this case, $(1 + ab)^{-1} = 1 - a(1 + ba)^{-1}b$ and*

$$(1 + ab)^{-1}a = a(1 + ba)^{-1}, \ b(1 + ab)^{-1} = (1 + ba)^{-1}b.$$

Lemma 2.2. *Let $a, b \in \Re$. If $1 + ab$ is left invertible, then so is $1 + ba$.*

Proof. Let $c \in \Re$ such that $c(1 + ab) = 1$. Then

$$1 + ba = 1 + bc(1 + ab)a = 1 + bca(1 + ba).$$

Therefore, $(1 - bca)(1 + ba) = 1$. □

Lemma 2.3. *Let a be a nonzero element in $\Re$ such that a^+ exists. If $s = a^+a + aa^+ - 1$ is invertible in $\Re$, then $a^\#$ exists and $a^\# = a^+s^{-1} + (1 - a^+a)s^{-1}a^+s^{-1}$.*

Proof. According to [14] or [18, Theorem 4.5.9], $a^\#$ exists. We now give the expression of $a^\#$ as follows.
Put $p = a^+a$, $q = aa^+$. Then we have

$$ps = pq = sq, \ qs = qp = sp, \ sa = a^+a^2. \tag{2.1}$$

Set $y = a^+s^{-1}$. Then by (2.1),

$$\begin{aligned} yp &= a^+s^{-1}a^+a = a^+aa^+s^{-1} = y = py, \\ pay &= a^+aaa^+s^{-1} = pqs^{-1} = p, \\ ypa &= a^+s^{-1}a^+aa = a^+a = p. \end{aligned}$$

Put $a_1 = pap = pa$, $a_2 = (1 - p)ap = (1 - p)a$. Then $a = a_1 + a_2$ and it is easy to check that $a^\# = y + a_2y^2$. Using (2.1), we can get that $a^\# = a^+s^{-1} + (1 - a^+a)a(a^+s^{-1})^2 = a^+s^{-1} + (1 - a^+a)s^{-1}a^+s^{-1}$. □

Let $M_2(\Re)$ denote the matrix ring of all 2×2 matrices over $\Re$ and let 1_2 denote the unit of $M_2(\Re)$.

Corollary 2.4. *Let $b, c \in \Re$ have group inverse $b^\#$ and $c^\#$ respectively. Assume that $k = b^\#b + c^\#c - 1$ is invertible in $\Re$. Then $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}^\#$ exists with $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}^\# = \begin{bmatrix} 0 & k^{-1}c^\#k^{-1} \\ k^{-1}b^\#k^{-1} & 0 \end{bmatrix}$.*

In particular, when $b^\#bc^\#c = b^\#b$ and $c^\#cb^\#b = c^\#c$, $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}^\# = \begin{bmatrix} 0 & b^\#bc^\# \\ c^\#cb^\# & 0 \end{bmatrix}$.

Proof. Set $a = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. Then $a^+ = \begin{bmatrix} 0 & c^\# \\ b^\# & 0 \end{bmatrix}$ and

$$a^+a + aa^+ - 1_2 = \begin{bmatrix} b^\#b + c^\#c - 1 & 0 \\ 0 & b^\#b + c^\#c - 1 \end{bmatrix} = \begin{bmatrix} k & \\ & k \end{bmatrix}$$

is invertible in $M_2(\mathfrak{R})$. Noting that $bb^\#k^{-1} = k^{-1}cc^\#$. Thus, by Lemma 2.3,

$$\begin{aligned} a^\# &= a^+ \begin{bmatrix} k^{-1} & \\ & k^{-1} \end{bmatrix} + (1_2 - a^+a) \begin{bmatrix} k^{-1} & \\ & k^{-1} \end{bmatrix} a^+ \begin{bmatrix} k^{-1} & \\ & k^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & c^\#k^{-1} + (1 - c^\#c)k^{-1}c^\#k^{-1} \\ b^\#k^{-1} + (1 - b^\#b)k^{-1}b^\#k^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & k^{-1}c^\#k^{-1} \\ k^{-1}b^\#k^{-1} & 0 \end{bmatrix}. \end{aligned}$$

When $b^\#bc^\#c = b^\#b$ and $c^\#cb^\#b = c^\#c$, $k^{-1} = k$. In this case, $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}^\# = \begin{bmatrix} 0 & b^\#bc^\# \\ c^\#cb^\# & 0 \end{bmatrix}$. □

Lemma 2.5. *Let $a, b \in \mathfrak{R}$ and p be a non–trivial idempotent element in $\mathfrak{R}$, i.e., $p \neq 0, 1$. Put $x = pap + pb(1 - p)$.*

(1) *If pap is group invertible and $(pap)(pap)^\#b(1 - p) = pb(1 - p)$, then x is group invertible too and $x^\# = (pap)^\# + [(pap)^\#]^2pb(1 - p)$.*
(2) *If x is group invertible, then so is the pap.*

Proof. (1) It is easy to check that $p(pap)^\# = (pap)^\#p = (pap)^\#$. Put $y = (pap)^\# + [(pap)^\#]^2pb(1 - p)$. Then $xyx = x$, $yxy = y$ and $xy = yx$, i.e., $y = x^\#$.

(2) Set $y_1 = px^\#p$, $y_2 = px^\#(1 - p)$, $y_3 = (1 - p)x^\#p$ and $y_4 = (1 - p)x^\#(1 - p)$. Then $x^\# = y_1 + y_2 + y_3 + y_4$. From $xx^\#x = x$, $x^\#xx^\# = x^\#$ and $xx^\# = x^\#x$, we can obtain that $y_3 = y_4 = 0$ and

$$(pxp)y_1(pxp) = pxp, \quad y_1(pxp)y_1 = y_1, \quad y_1(pxp) = (pxp)y_1,$$

that is, $(pxp)^\# = y_1$. □

At the end of this section, we will introduce the notation of stable perturbation of an element in a ring.

Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$ such that a^+ exists. Let $\bar{a} = a + \delta a \in \mathcal{A}$. Recall from [19] that $\bar{a}$ is a stable perturbation of a if $\bar{a}\mathcal{A} \cap (1 - aa^+)\mathcal{A} = \{0\}$. This notation can be easily extended to the case of ring as follows.

Definition 2.6. *Let $a \in \mathfrak{R}$ such that a^+ exists and let $\bar{a} = a + \delta a \in R$. We say $\bar{a}$ is a stable perturbation of a if $\bar{a}\mathfrak{R} \cap (1 - aa^+)\mathfrak{R} = \{0\}$.*

Using the same methods as appeared in the proofs of [19, Proposition 2.2] and [18, Theorem 2.4.7], we can obtain:

Proposition 2.7. *Let $a \in \mathfrak{R}$ and $\bar{a} = a + \delta a \in \mathfrak{R}$ such that a^+ exists and $1 + a^+\delta a$ is invertible in $\mathfrak{R}$. Then the following statements are equivalent:*

(1) *$\bar{a}^+$ exists and $\bar{a}^+ = (1 + a^+\delta a)^{-1}a^+$.*
(2) *$\bar{a}\mathfrak{R} \cap (1 - aa^+)\mathfrak{R} = \{0\}$ (that is, $\bar{a}$ is a stable perturbation if a).*
(3) *$\bar{a}(1 + a^+\delta a)^{-1}(1 - a^+a) = 0$.*
(4) *$(1 - aa^+)(1 + \delta aa^+)^{-1}\bar{a} = 0$.*
(5) *$(1 - aa^+)\delta a(1 - a^+a) = (1 - aa^+)\delta a(1 + a^+\delta a)^{-1}a^+\delta a(1 - a^+a)$.*
(6) *$\mathfrak{R}\bar{a} \cap \mathfrak{R}(1 - a^+a) = \{0\}$.*

3. Main results

In this section, we investigate the stable perturbation for group inverse and Drazin inverse of an element a in $\mathfrak{R}$.

Let $a \in \mathfrak{R}$ and $\bar{a} = a + \delta a \in \mathfrak{R}$ such that $a^{\#}$ exists and $1 + a^{\#}\delta a$ is invertible in $\mathfrak{R}$. Put $a^{\pi} = (1 - a^{\#}a)$, $\Phi(a) = 1 + \delta a a^{\pi}\delta a[(1 + a^{\#}\delta a)^{-1}a^{\#}]^2$ and $B = \Phi(a)(1 + \delta a a^{\#})$, $C(a) = a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}$. These symbols will be used frequently in this section.

Lemma 3.1. *Let $a \in \mathfrak{R}$ and $\bar{a} = a + \delta a \in \mathfrak{R}$ such that $a^{\#}$ exists and $1 + a^{\#}\delta a$ is invertible in $\mathfrak{R}$. Suppose that $\Phi(a)$ is invertible, then $(Ba)^{\#} = Baa^{\#}B^{-1}a^{\#}B^{-1}$.*

Proof. Put $P = aa^{\#}$. Noting that $\Phi(a)(1 - P) = 1 - P$, we have $P\Phi(a)P = P\Phi(a)$, $\Phi^{-1}(a)(1 - P) = (1 - P)$ and $PBP = PB$, $B^{-1}(1 - P) = (1 + \delta a a^{\#})^{-1}(1 - P)$, $a^{\#}B^{-1}(1 - P) = 0$, i.e., $a^{\#}B^{-1} = a^{\#}B^{-1}P$. Thus, $BPB^{-1}Ba = Ba$ and

$$\begin{aligned}(Ba)(Baa^{\#}B^{-1}a^{\#}B^{-1}) &= BPB^{-1} = (Baa^{\#}B^{-1}a^{\#}B^{-1})(Ba),\\ (Baa^{\#}B^{-1}a^{\#}B^{-1})(BPB^{-1}) &= Baa^{\#}B^{-1}a^{\#}B^{-1}.\end{aligned}$$

These indicate $(Ba)^{\#} = Baa^{\#}B^{-1}a^{\#}B^{-1}$. □

Theorem 3.2. *Let $a \in \mathfrak{R}$ such that $a^{\#}$ exists. Let $\bar{a} = a + \delta a \in \mathfrak{R}$ with $1 + a^{\#}\delta a$ invertible in $\mathfrak{R}$. Suppose that $\Phi(a)$ is invertible and $\bar{a}\mathfrak{R} \cap (1 - aa^{\#})\mathfrak{R} = \{0\}$. Put $D(a) = (1 + a^{\#}\delta a)^{-1}a^{\#}\Phi^{-1}(a)$. Then $\bar{a}^{\#}$ exists with*

$$\bar{a}^{\#} = (1 + C(a))(D(a) + D^2(a)\delta a a^{\pi})(1 - C(a)).$$

Proof. Put $P = aa^{\#}$. By Proposition 2.7 (3), we have $a^{\pi}(1 + \delta a a^{\#})^{-1}\bar{a} = 0$ and

$$P\bar{a}(1 + a^{\#}\delta a)^{-1} = a(aa^{\#} + a^{\#}\delta a)(1 + a^{\#}\delta a)^{-1} = a(1 + a^{\#}\delta a - a^{\pi})(1 + a^{\#}\delta a)^{-1} = a.$$

Thus, we have

$$\begin{aligned}
&(1 - C(a))\bar{a}(1 + C(a))\\
&= [P + a^{\pi}(1 + \delta a a^{\#})^{-1}]\bar{a}[1 + a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}]\\
&= P\bar{a}[1 + a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}]\\
&= P\bar{a} + P\bar{a}a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}\\
&= P\bar{a} + P\delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}\\
&= P\delta a + P[a + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}]\\
&= P\delta a(1 - P) + P\delta a P + P[a + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}]\\
&= P\delta a(1 - P) + P[\delta a + a + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}]P\\
&= P\delta a(1 - P) + P[\bar{a} + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}]P\\
&= P\delta a(1 - P) + P[\bar{a}(1 + a^{\#}\delta a)^{-1} + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}(1 + a^{\#}\delta a)^{-1}](1 + a^{\#}\delta a)P\\
&= P\delta a(1 - P) + P[a + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}(1 + a^{\#}\delta a)^{-1}](1 + a^{\#}\delta a)P\\
&= P\delta a(1 - P) + P[a + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}(1 + a^{\#}\delta a)^{-1}]a^{\#}(1 + \delta a a^{\#})a\\
&= P\delta a(1 - P) + P[aa^{\#} + \delta a a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}(1 + a^{\#}\delta a)^{-1}a^{\#}](1 + \delta a a^{\#})a\\
&= P\delta a(1 - P) + P[1 + \delta a a^{\pi}\delta a((1 + a^{\#}\delta a)^{-1}a^{\#})^2](1 + \delta a a^{\#})a\\
&= P\delta a(1 - P) + P\Phi(a)(1 + \delta a a^{\#})aP.
\end{aligned}$$

By Lemma 3.1, we have

$$P(Ba)^{\#}P = PBaa^{\#}B^{-1}a^{\#}B^{-1}P = PBPB^{-1}a^{\#}B^{-1}P = a^{\#}B^{-1} = P(Ba)^{\#}$$

and $PBa(Ba)^{\#}\delta a = P\delta a$. So $P(Ba)^{\#}P(Ba)P = P(Ba)^{\#}(Ba)$ and

$$\begin{aligned} P(Ba)PP(Ba)^{\#}P &= P(Ba)(Ba)^{\#}P = P(Ba)^{\#}(Ba)P = P(Ba)^{\#}P(Ba)P\\ P(Ba)^{\#}P(Ba)P(Ba)^{\#}P &= P(Ba)^{\#}(Ba)(Ba)^{\#}P = P(Ba)^{\#}P,\\ P(Ba)P(Ba)^{\#}P(Ba)P &= P(Ba)P(Ba)^{\#}(Ba)P = P(Ba)P, \end{aligned}$$

i.e., $(P(Ba)P)^{\#} = P(Ba)^{\#} = a^{\#}B^{-1}$. So $P(Ba)P(P(Ba)P)^{\#} = P$ and hence, we have by Lemma 2.5 (1),

$$[(1 - C(a))\bar{a}(1 + C(a))]^{\#} = a^{\#}B^{-1} + [a^{\#}B^{-1}]^2\delta a(1 - P).$$

Therefore,

$$\begin{aligned} \bar{a}^{\#} &= (1 + C(a))[(1 - C(a))\bar{a}(1 + C(a))]^{\#}(1 - C(a))\\ &= (1 + C(a))(D(a) + D^2(a)\delta a a^{\pi})(1 - C(a))\\ &= (1 + a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#})(1 + a^{\#}\delta a)^{-1}a^{\#}\Big[1 + \delta a a^{\pi}\delta a[(1 + a^{\#}\delta a)^{-1}a^{\#}]^2\Big]^{-1}\\ &\quad\times\Big[1 + (1 + a^{\#}\delta a)^{-1}a^{\#}\Big[1 + \delta a a^{\pi}\delta a[(1 + a^{\#}\delta a)^{-1}a^{\#}]^2\Big]^{-1}\delta a a^{\pi}\Big](1 - a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#}). \quad \square \end{aligned}$$

Now we consider the case when $a \in \mathfrak{R}$ and $\bar{a} = a + \delta a \in \mathfrak{R}$ such that $a^{\#}$, $\bar{a}^{\#}$ exist. Firstly, we have

Proposition 3.3. *Let $a \in \mathfrak{R}$, $\bar{a} = a + \delta a \in \mathfrak{R}$ such that $a^{\#}$, $\bar{a}^{\#}$ exist. Then the following statements are equivalent:*

(1) $\mathfrak{R} = \bar{a}\mathfrak{R} \dotplus (1 - aa^{\#})\mathfrak{R} = a\mathfrak{R} \dotplus (1 - \bar{a}\bar{a}^{\#})\mathfrak{R} = \mathfrak{R}\bar{a} \dotplus \mathfrak{R}(1 - aa^{\#}) = \mathfrak{R}a \dotplus \mathfrak{R}(1 - \bar{a}\bar{a}^{\#})$.
(2) $K = K(a, \bar{a}) = \bar{a}\bar{a}^{\#} + aa^{\#} - 1$ *is invertible.*
(3) $\bar{a}\mathfrak{R} \cap (1 - aa^{\#})\mathfrak{R} = \{0\}$, $\mathfrak{R}\bar{a} \cap \mathfrak{R}(1 - aa^{\#}) = \{0\}$ *and* $1 + \delta a a^{\#}$ *is invertible.*

Proof. (1) ⇒ (2) : Since $\mathfrak{R} = \bar{a}\mathfrak{R} \dotplus (1 - aa^{\#})\mathfrak{R} = a\mathfrak{R} \dotplus (1 - \bar{a}\bar{a}^{\#})\mathfrak{R}$, we have for any $y \in \mathfrak{R}$, there are $y_1 \in \mathfrak{R}$, $y_2 \in \mathfrak{R}$ such that

$$(1 - \bar{a}\bar{a}^{\#})y = (1 - \bar{a}\bar{a}^{\#})(1 - aa^{\#})y_1,\ \bar{a}\bar{a}^{\#}y = \bar{a}\bar{a}^{\#}aa^{\#}y_2.$$

Put $z = aa^{\#}y_2 - (1 - aa^{\#})y_1$. Then

$$K(a, \bar{a})z = (\bar{a}\bar{a}^{\#} + aa^{\#} - 1)(aa^{\#}y_2 - (1 - aa^{\#})y_1) = y.$$

Since $\mathfrak{R} = \mathfrak{R}\bar{a} \dotplus \mathfrak{R}(1 - aa^{\#}) = \mathfrak{R}a \dotplus \mathfrak{R}(1 - \bar{a}\bar{a}^{\#})$, we have for any $y \in \mathfrak{R}$, there are $y_1, y_2 \in \mathfrak{R}$ such that

$$y(1 - \bar{a}\bar{a}^{\#}) = y_1(1 - aa^{\#})(1 - \bar{a}\bar{a}^{\#}), y\bar{a}\bar{a}^{\#} = y_2aa^{\#}\bar{a}\bar{a}^{\#}.$$

Put $z = y_2aa^{\#} - y_1(1 - aa^{\#})$. Then

$$zK(a, \bar{a}) = (y_2aa^{\#} - y_1(1 - aa^{\#}))(\bar{a}\bar{a}^{\#} + aa^{\#} - 1) = y.$$

The above indicates $K(a, \bar{a})$ is invertible when we take $y = 1$.

(2) ⇒ (3) : Let $y \in \bar{a}\mathfrak{R} \cap (1 - aa^{\#})\mathfrak{R}$. Then $\bar{a}\bar{a}^{\#}y = y$, $a^{\#}y = 0$. Thus $K(a, \bar{a})y = 0$ and hence $y = 0$, that is, $\bar{a}\mathfrak{R} \cap (1 - aa^{\#})\mathfrak{R} = \{0\}$. Similarly, we have $\mathfrak{R}\bar{a} \cap \mathfrak{R}(1 - aa^{\#}) = \{0\}$.

Let $T = aK^{-1}\bar{a}^{\#} - a^{\pi}$. Since $\bar{a}a^{\#}aK^{-1} = \bar{a}$, we have $(1 + \delta aa^{\#})T = K$, that is, $(1 + \delta aa^{\#})$ has right inverse TK^{-1}.

Since $K^{-1}aa^{\#}\bar{a} = \bar{a}$, we have $(\bar{a}^{\#}K^{-1}a - a^{\pi})(1 + a^{\#}\delta a) = K$, that is, $1 + a^{\#}\delta a$ has left inverse $K^{-1}(\bar{a}^{\#}K^{-1}a - a^{\pi})$. This indicates that $1 + \delta aa^{\#}$ has left inverse $1 - \delta aK^{-1}(\bar{a}^{\#}K^{-1}a - a^{\pi})a^{\#}$ by Lemma 2.2. Finally, $1 + \delta aa^{\#}$ is invertible.

(3) ⇒ (1) : By Lemma 2.1, $1 + a^{\#}\delta a$ is also invertible. So from

$$1 + \delta aa^{\#} = \bar{a}a^{\#} + (1 - aa^{\#}),\ 1 + a^{\#}\delta a = a^{\#}\bar{a} + (1 - aa^{\#})$$

and Lemma 2.7, we get that

$$\mathfrak{R} = \bar{a}\mathfrak{R} \dotplus (1 - aa^{\#})\mathfrak{R} = \mathfrak{R}\bar{a} \dotplus \mathfrak{R}(1 - aa^{\#}).$$

We now prove that

$$\mathfrak{R} = a\mathfrak{R} + (1-\bar{a}\bar{a}^{\#})\mathfrak{R} = \mathfrak{R}a + \mathfrak{R}(1-\bar{a}\bar{a}^{\#}),\ a\mathfrak{R} \cap (1-\bar{a}\bar{a}^{\#})\mathfrak{R} = \mathfrak{R}a \cap \mathfrak{R}(1-\bar{a}\bar{a}^{\#}) = \{0\}.$$

For any $y \in a\mathfrak{R} \cap (1-\bar{a}\bar{a}^{\#})\mathfrak{R}$, we have $aa^{\#}y = y$, $\bar{a}y = 0$. So $(1 + a^{\#}\delta a)y = (1 - a^{\#}a)y = 0$ and hence $y = 0$. Similarly, we have $\mathfrak{R}a \cap \mathfrak{R}(1-\bar{a}\bar{a}^{\#}) = \{0\}$.

By Lemma 2.7, $\bar{a}^{+} = (1+a^{\#}\delta a)^{-1}a^{\#}$ and $\bar{a}^{+}\bar{a} = (1+a^{\#}\delta a)^{-1}a^{\#}a(1+a^{\#}\delta a)$. So $(1-\bar{a}^{+}\bar{a})\mathfrak{R} = (1+a^{\#}\delta a)^{-1}(1-a^{\#}a)\mathfrak{R}$. From $(1-\bar{a}^{\#}\bar{a})(1-\bar{a}^{+}\bar{a}) = 1-\bar{a}^{+}\bar{a}$, we get that $(1-\bar{a}^{+}\bar{a})\mathfrak{R} \subset (1-\bar{a}^{\#}\bar{a})\mathfrak{R}$. Note that $a\mathfrak{R} = a^{\#}\mathfrak{R}$ and $(1 + a^{\#}\delta a)a\mathfrak{R} = a^{\#}(1+\delta aa^{\#})\mathfrak{R} = a^{\#}a\mathfrak{R}$. So

$$\mathfrak{R} \supset a\mathfrak{R} + (1-\bar{a}^{\#}\bar{a})\mathfrak{R} \supset (1+a^{\#}\delta a)^{-1}a^{\#}a\mathfrak{R} + (1+a^{\#}\delta a)^{-1}(1-a^{\#}a)\mathfrak{R} = \mathfrak{R}.$$

Similarly, we can get $\mathfrak{R}a + \mathfrak{R}(1-\bar{a}^{\#}\bar{a}) = \mathfrak{R}$. □

Now we present a theorem which can be viewed as the inverse of Theorem 3.2 as follows:

Theorem 3.4. *Let $a \in \mathfrak{R}$ and $\bar{a} = a + \delta a \in \mathfrak{R}$ such that $a^{\#}$, $\bar{a}^{\#}$ exist. If $K(a,\bar{a})$ is invertible, then $\Phi(a)$ is invertible.*

Proof. Since $K(a,\bar{a})$ is invertible, we have $\bar{a}\mathfrak{R} \cap (1-aa^{\#})\mathfrak{R} = \{0\}$ and $1 + \delta aa^{\#}$ is invertible in $\mathfrak{R}$ by Propositon 3.3. Thus, from the proof of Theorem 3.2, we have

$$\begin{aligned}(1-C(a))\bar{a}(1+C(a)) &= P\delta a(1-P) + P\Phi(a)(1+\delta aa^{\#})aP\\ &= PBaP + P\delta a(1-P).\end{aligned}$$

Since $(1-C(a))\bar{a}(1+C(a))$ is group invertible, it follows from Lemma 2.5 (2) that $PBaP$ is group invertible. Hence $PBa(PBa)^{\#}\delta a(I-P) = P\delta a(I-P)$. Consequently,

$$\begin{aligned}[(1-C(a))\bar{a}(1+C(a))]^{\#} &= (1-C(a))\bar{a}^{\#}(1+C(a))\\ &= (PBa)^{\#}P + ((PBa)^{\#})^{2}\delta a(1-P).\end{aligned}$$

Thus,

$$\begin{aligned}(1-C(a))\bar{a}\bar{a}^{\#}(1+C(a)) &= [PBaP + P\delta a(1-P)][(PBa)^{\#}P + ((PBa)^{\#})^{2}\delta a(1-P)]\\ &= PBa(PBa)^{\#}P + (PBa)^{\#}\delta a(1-P)\\ (1-C(a))K(a,\bar{a})(1+C(a)) &= (1-C(a))\bar{a}\bar{a}^{\#}(1+C(a)) - (1-C(a))a^{\pi}(1+C(a))\\ &= (1-C(a))\bar{a}\bar{a}^{\#}(1+C(a)) - a^{\pi}(1+C(a))\\ &= PBa(PBa)^{\#}P + (PBa)^{\#}\delta a(1-P) - (1-P)C(a)P - (1-P)\end{aligned}$$

Since $(1-C(a))K(a,\bar{a})(1+C(a))$ is invertible, we get that

$$\rho(a) = (PBa)^{\#}PBa - (PBa)^{\#}\delta aC(a) = PBa[(PBa)^{\#}]^{2}(PBa - \delta aC(a))$$

is invertible in $P\mathfrak{R}P$. So we have $P = PBa[(PBa)^{\#}]^{2}(PBa - \delta aC(a))\rho^{-1}(a)$ and that $\Phi(a)$ has right inverse.

Set $E(a) = a^{\#}(1+\delta aa^{\#})^{-1}\delta aa^{\pi}$. Then $1 - E(a) = P + (1+a^{\#}\delta a)^{-1}a^{\pi}$ and $(1-E(a))^{-1} = 1 + E(a)$. From Lemma 2.7, we have $\bar{a}(1+a^{\#}\delta a)^{-1}a^{\pi} = 0$ and

$$\begin{aligned}a^{\#}(1+\delta aa^{\#})^{-1}\bar{a} &= (1+a^{\#}\delta a)^{-1}a^{\#}\bar{a} = (1+a^{\#}\delta a)^{-1}(1+a^{\#}\delta a - a^{\pi})\\ &= 1 - (1+a^{\#}\delta a)^{-1}a^{\pi}.\end{aligned}$$

Put $\psi(a) = 1 + [(1 + a^{\#}\delta a)^{-1}a^{\#}]^2\delta a a^{\pi}\delta a$ and $R = (1 + a^{\#}\delta a)\psi(a)$. Then

$$\begin{aligned}
&(1 + E(a))\bar{a}(1 - E(a))\\
&= [1 + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}]\bar{a}[P + (1 + a^{\#}\delta a)^{-1}a^{\pi}]\\
&= [1 + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}]\bar{a}P\\
&= \bar{a}P + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\bar{a}P\\
&= \bar{a}P + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\delta a P\\
&= aP + \delta a P + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\delta a P\\
&= (1 - P)\delta a P + P\delta a P + aP + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\delta a P\\
&= (1 - P)\delta a P + P[\bar{a} + a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\delta a]P\\
&= (1 - P)\delta a P + P(1 + \delta a a^{\#})[(1 + \delta a a^{\#})^{-1}\bar{a} + (1 + \delta a a^{\#})^{-1}a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\delta a]P\\
&= (1 - P)\delta a P + Pa(1 + a^{\#}\delta a)[a^{\#}(1 + \delta a a^{\#})^{-1}\bar{a} + a^{\#}(1 + \delta a a^{\#})^{-1}a^{\#}(1 + \delta a a^{\#})^{-1}\delta a a^{\pi}\delta a]P\\
&= (1 - P)\delta a P + Pa(1 + a^{\#}\delta a)[a^{\#}(1 + \delta a a^{\#})^{-1}\bar{a} + [(1 + a^{\#}\delta a)^{-1}a^{\#}]^2\delta a a^{\pi}\delta a]P\\
&= (1 - P)\delta a P + Pa(1 + a^{\#}\delta a)[1 + [(1 + a^{\#}\delta a)^{-1}a^{\#}]^2\delta a a^{\pi}\delta a]P\\
&= (1 - P)\delta a P + PaRP.
\end{aligned}$$

Since $(1 + E(a))\bar{a}(1 - E(a))$ is group invertible, we can deduce that aR is group invertible and

$$\begin{aligned}
[(1 + E(a))\bar{a}(1 - E(a))]^{\#} &= (1 + E(a))\bar{a}^{\#}(1 - E(a))\\
&= P(aRP)^{\#} + (1 - P)\delta a((aRP)^{\#})^2
\end{aligned}$$

and

$$\begin{aligned}
(1 + E(a))\bar{a}\bar{a}^{\#}(1 - E(a)) &= [(1 - P)\delta a P + PaRP][P(aRP)^{\#} + (1 - P)\delta a((aRP)^{\#})^2]\\
&= PaR(aRP)^{\#} + (1 - P)\delta a(aRP)^{\#}.
\end{aligned}$$

Thus, from the invertibility of $K(a, \bar{a})$, we get that

$$\begin{aligned}
&(1 + E(a))K(a, \bar{a})(1 - E(a))\\
&\quad = (1 + E(a))\bar{a}\bar{a}^{\#}(1 - E(a)) - (1 + E(a))a^{\pi}(1 - E(a))\\
&\quad = PaR(aRP)^{\#} + (1 - P)\delta a(aRP)^{\#} - (1 + E(a))a^{\pi}\\
&\quad = PaR(aRP)^{\#} + (1 - P)\delta a(aRP)^{\#} - PE(a)(1 - P) - (1 - P)
\end{aligned}$$

is invertible in $\mathfrak{R}$ and hence

$$\begin{aligned}
\eta(a) &= aRP(aRP)^{\#} - E(a)\delta a(aRP)^{\#} = [aRP - E(a)\delta a][(aRP)^{\#}]^2 aRP\\
&= [aRP - E(a)\delta a][(aRP)^{\#}]^2 a(1 + a^{\#}\delta a)\psi(a)P
\end{aligned}$$

is invertible in $P\mathfrak{R}P$. So $\psi(a)$ is left invertible and hence $\Phi(a)$ is left invertible by Lemma 2.2. Therefore, $\Phi(a)$ is invertible. □

Let $a \in \mathfrak{R}$ such that a^D exists and $ind(a) = s$. As we know if a^D exists, then a^l has group inverse $(a^l)^{\#}$ and $a^D = (a^l)^{\#}a^{l-1}$ for any $l \geq s$.

From Theorem 3.2 and Theorem 3.4, we have the following corollary:

Corollary 3.5. *Let a and b be nonzero elements in R such that a^D and b^D exist. Put $s = ind(a)$ and $t = ind(b)$. Suppose that $K(a, b) = bb^D + aa^D - 1$ is invertible in $\mathfrak{R}$. Then for any $l \geq s$ and $k \geq t$, we have*

(1) *$1 + (a^D)^l(b^k - a^l)$ is invertible in $\mathfrak{R}$ and $b^k\mathfrak{R} \cap (1 - a^D a)\mathfrak{R} = \{0\}$.*

(2) $W_{k,l} = 1+E_{k,l}Z_{k,l}(1+(a^D)^l E_{k,l})^{-1}(a^D)^l$ *is invertible in* $\mathfrak{R}$, *here* $E_{k,l} = b^k - a^l$ *and* $Z_{k,l} = a^\pi E_{k,l}(a^D)^l(1+E_{k,l}(a^D)^l)^{-1}$.
(3) $b^D = (1 + Z_{k,l})[H_{k,l} + H_{k,l}^2 E_{k,l}a^\pi](1 - Z_{k,l})b^{k-1}$, *where* $H_{k,l} = (1 + (a^D)^l E_{k,l})^{-1}(a^D)^l W_{k,l}^{-1}$.

Proof. Noting that $(a^D)^l = (a^l)^\#,\ aa^D = a^l(a^l)^\#,\ bb^D = b^k(b^k)^\#,\ l \ge s,\ k \ge t$, we have

$$K(a, b) = b^k(b^k)^\# + a^l(a^l)^\# - 1, \quad 1 + (a^D)^l(b^k - a^l) = 1 + (a^l)^\#(b^k - a^l).$$

Applying Theorem 3.2 and Theorem 3.4 to b^k and a^l, we can get the assertions. □

4. The representation of the group inverse of certain matrix on $\mathfrak{R}$

As an application of Theorem 3.2 and Theorem 3.4, we study the representation of the group inverse of $\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}$ on the ring $\mathfrak{R}$.

Proposition 4.1. *Let* $b, c, d \in \mathfrak{R}$. *Suppose that* $b^\#$ *and* $c^\#$ *exist and* $k = b^\# b + c^\# c - 1$ *is invertible. If* $b^\pi d = 0$ *or* $dc^\pi = 0$, *then* $\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}^\#$ *exists and*

$$\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}^\# = \begin{bmatrix} -k^{-1}c^\# k^{-1}b^\# k^{-1}dc^\pi k^{-1} & k^{-1}c^\# k^{-1} \\ k^{-1}b^\# k^{-1}(1 + dk^{-1}c^\# k^{-1}b^\# k^{-1}dc^\pi k^{-1}) & -k^{-1}b^\# k^{-1}dk^{-1}c^\# k^{-1} \end{bmatrix}$$

if $b^\pi d = 0$. *When* $dc^\pi = 0$, *we have*

$$\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}^\# = \begin{bmatrix} -k^{-1}b^\pi dk^{-1}c^\# k^{-1}b^\# k^{-1} & (1 + k^{-1}b^\pi dk^{-1}c^\# k^{-1}b^\# d)k^{-1}c^\# k^{-1} \\ k^{-1}b^\# k^{-1} & -k^{-1}b^\# k^{-1}dk^{-1}c^\# k^{-1} \end{bmatrix}.$$

Proof. Set $a = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, $\delta a = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}$ and $\bar{a} = \begin{bmatrix} d & b \\ c & 0 \end{bmatrix}$. Since $b^\# bk = kc^\# c$, $c^\# ck = kb^\# b$, it follows from Corollary 2.4 that

$$\begin{aligned}
1_2 + a^\#\delta a &= 1_2 + \begin{bmatrix} 0 & b^\# bk^{-1}c^\# k^{-1} \\ c^\# ck^{-1}b^\# k^{-1} & 0 \end{bmatrix}\begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k^{-1}b^\# k^{-1}d & 1 \end{bmatrix} \\
(1_2 + a^\#\delta a)^{-1}a^\# &= \begin{bmatrix} 1 & 0 \\ -k^{-1}b^\# k^{-1}d & 1 \end{bmatrix}\begin{bmatrix} 0 & k^{-1}c^\# k^{-1} \\ k^{-1}b^\# k^{-1} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & k^{-1}c^\# k^{-1} \\ k^{-1}b^\# k^{-1} & -k^{-1}b^\# k^{-1}dk^{-1}c^\# k^{-1} \end{bmatrix} \\
aa^\# &= \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & b^\# bk^{-1}c^\# k^{-1} \\ c^\# ck^{-1}b^\# k^{-1} & 0 \end{bmatrix} \\
&= \begin{bmatrix} bc^\# ck^{-1}b^\# k^{-1} & 0 \\ 0 & cb^\# bk^{-1}c^\# k^{-1} \end{bmatrix} \\
&= \begin{bmatrix} bb^\# k^{-1} & 0 \\ 0 & cc^\# k^{-1} \end{bmatrix} \\
a^\pi &= 1 - aa^\# = \begin{bmatrix} -c^\pi k^{-1} & 0 \\ 0 & -b^\pi k^{-1} \end{bmatrix} \\
\bar{a}(1_2 + a^\#\delta a)^{-1}a^\pi &= \begin{bmatrix} d & b \\ c & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -c^\# ck^{-1}b^\# k^{-1}d & 1 \end{bmatrix}\begin{bmatrix} -c^\pi k^{-1} & 0 \\ 0 & -b^\pi k^{-1} \end{bmatrix} \\
&= \begin{bmatrix} d - bc^\# ck^{-1}b^\# k^{-1}d & b \\ c & 0 \end{bmatrix}\begin{bmatrix} -c^\pi k^{-1} & 0 \\ 0 & -b^\pi k^{-1} \end{bmatrix} \\
&= \begin{bmatrix} -c^\pi k^{-1}d & b \\ c & 0 \end{bmatrix}\begin{bmatrix} -c^\pi k^{-1} & 0 \\ 0 & -b^\pi k^{-1} \end{bmatrix} = \begin{bmatrix} k^{-1}b^\pi dc^\pi k^{-1} & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

and

$$\delta a a^{\pi}\delta a = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -c^{\pi}k^{-1} & 0 \\ 0 & -b^{\pi}k^{-1} \end{bmatrix}\begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -dc^{\pi}k^{-1}d & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^{\pi}\delta a = \begin{bmatrix} -k^{-1}b^{\pi}d & 0 \\ 0 & 0 \end{bmatrix}, \quad \delta a a^{\pi} = \begin{bmatrix} -dc^{\pi}k^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

If $b^{\pi}d = 0$ or $dc^{\pi} = 0$, then $\bar{a}(1+a^{\#}\delta a)^{-1}a^{\pi} = 0$ and $\delta a a^{\pi}\delta a = 0$. Thus, $\Phi(a) = 1_2$ and $D(a) = (1+a^{\#}\delta a)^{-1}a^{\#}\Phi^{-1}(a) = (1 + a^{\#}\delta a)^{-1}a^{\#}$.

When $b^{\pi}d = 0$, $C(a) = a^{\pi}\delta a(1 + a^{\#}\delta a)^{-1}a^{\#} = 0$ and

$$D(a)\delta a a^{\pi} = \begin{bmatrix} 0 & k^{-1}c^{\#}k^{-1} \\ k^{-1}b^{\#}k^{-1} & -k^{-1}b^{\#}k^{-1}dk^{-1}c^{\#}k^{-1} \end{bmatrix}\begin{bmatrix} -dc^{\pi}k^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -k^{-1}b^{\#}k^{-1}dc^{\pi}k^{-1} & 0 \end{bmatrix}.$$

By Theorem 3.2, we have

$$\begin{aligned}
\bar{a}^{\#} &= (1_2 + C(a))(D(a) + D^2(a)\delta a a^{\pi})(1_2 - C(a)) \\
&= D(a)(1_2 + D(a)\delta a a^{\pi}) \\
&= \begin{bmatrix} 0 & k^{-1}c^{\#}k^{-1} \\ k^{-1}b^{\#}k^{-1} & -k^{-1}b^{\#}k^{-1}dk^{-1}c^{\#}k^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -k^{-1}b^{\#}k^{-1}dc^{\pi}k^{-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & k^{-1}c^{\#}k^{-1} \\ a_2 & -k^{-1}b^{\#}k^{-1}dk^{-1}c^{\#}k^{-1} \end{bmatrix},
\end{aligned}$$

where $a_1 = -k^{-1}c^{\#}k^{-2}b^{\#}k^{-1}dc^{\pi}k^{-1}$, $a_2 = k^{-1}b^{\#}k^{-1} + k^{-1}b^{\#}k^{-1}dk^{-1}c^{\#}k^{-2}b^{\#}k^{-1}dc^{\pi}k^{-1}$. Since $cc^{\#}b^{\#} = kb^{\#}$, it follows that

$$c^{\#}k^{-2}b^{\#} = c^{\#}(c^{\#}c)k^{-1}k^{-1}b^{\#} = c^{\#}k^{-1}k^{-1}c^{\#}cb^{\#} = c^{\#}k^{-1}b^{\#}.$$

So $a_1 = -k^{-1}c^{\#}k^{-1}b^{\#}k^{-1}dc^{\pi}k^{-1}$, $a_2 = k^{-1}b^{\#}k^{-1}(1 + dk^{-1}c^{\#}k^{-1}b^{\#}k^{-1}dc^{\pi}k^{-1})$.

When $dc^{\pi} = 0$, we have by Theorem 3.2,

$$\begin{aligned}
\bar{a}^{\#} &= (1_2 + C(a))D(a)(1_2 - C(a)) = (1_2 + C(a))D(a) \\
&= \begin{bmatrix} 1 & -k^{-1}b^{\pi}dk^{-1}c^{\#}k^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & k^{-1}c^{\#}k^{-1} \\ k^{-1}b^{\#}k^{-1} & -k^{-1}b^{\#}k^{-1}dk^{-1}c^{\#}k^{-1} \end{bmatrix} \\
&= \begin{bmatrix} -k^{-1}b^{\pi}dk^{-1}c^{\#}k^{-1}b^{\#}k^{-1} & (1 + k^{-1}b^{\pi}dk^{-1}c^{\#}k^{-1}b^{\#}k^{-1}d)k^{-1}c^{\#}k^{-1} \\ k^{-1}b^{\#}k^{-1} & -k^{-1}b^{\#}k^{-1}dk^{-1}c^{\#}k^{-1} \end{bmatrix}. \quad \square
\end{aligned}$$

Combining Proposition 4.1 with Corollary 2.4, we have

Corollary 4.2. *Let $b, c, d \in \mathfrak{R}$. Assume that $b^{\#}$ and $c^{\#}$ exist and satisfy conditions: $b^{\#}bc^{\#}c = b^{\#}b$, $c^{\#}cb^{\#}b = c^{\#}c$.*

(1) *If $b^{\pi}d = 0$, then* $\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}^{\#} = \begin{bmatrix} b^{\#}bc^{\#}b^{\#}db^{\pi} & b^{\#}bc^{\#} \\ c^{\#}cb^{\#}(1 - db^{\#}bc^{\#}b^{\#}db^{\pi}) & -c^{\#}cb^{\#}db^{\#}bc^{\#} \end{bmatrix}.$

(2) *If $dc^{\pi} = 0$, then* $\begin{bmatrix} d & b \\ c & 0 \end{bmatrix}^{\#} = \begin{bmatrix} b^{\pi}db^{\#}bc^{\#}b^{\#} & (1 - b^{\pi}db^{\#}bc^{\#}b^{\#}d)b^{\#}bc^{\#} \\ c^{\#}cb^{\#} & -c^{\#}cb^{\#}db^{\#}bc^{\#} \end{bmatrix}.$

Recall from [10] that an involution * on $\mathfrak{R}$ is an involutory anti–automorphism, that is,

$$(a^*)^* = a,\ (a + b)^* = a^* + b^*,\ (ab)^* = b^*a^*,\ a^* = 0 \text{ if and only if } a = 0$$

and $\mathfrak{R}$ is called the $*$–ring if $\mathfrak{R}$ has an involution.

Corollary 4.3. *Let $\mathfrak{R}$ be a $*$–ring with unit* 1 *and let p be a nonzero idempotent element in $\mathfrak{R}$. Then*

$$\begin{bmatrix} pp^* & p \\ p & 0 \end{bmatrix}^{\#} = \begin{bmatrix} pp^*(1-p) & p \\ p-(pp^*)^2(1-p) & -pp^*p \end{bmatrix}, \quad \begin{bmatrix} p^*p & p \\ p & 0 \end{bmatrix}^{\#} = \begin{bmatrix} (1-p)p^*p & p-(1-p)(p^*p)^2 \\ p & -pp^*p \end{bmatrix}.$$

Proof. Since $p^{\#} = p$, we can get the assertions easily by using Corollary 4.2. □

Remark 4.4. (1) If $\mathfrak{R}$ is a skew field and $b = c$ in Proposition 4.1, the conclusion of Proposition 4.1 is contained in [22].

(2) Let p be an idempotent matrix. The group inverse of $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ is given in [2] for some $a, b, c \in \{pp^*, p, p^*\}$. The group inverse of this type of matrices is also discussed in [7].

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