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Weighted Approximation by New Bernstein-Chlodowsky-Gadjiev Operators

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Abstract. In the present paper, we introduce Bernstein-Chlodowsky-Gadjiev operators taking into consideration the polynomials introduced by Gadjiev and Ghorbanalizadeh [2]. The interval of convergence of the operators is a moved interval as polynomials given in [2] but grows as $n \to \infty$ as in the classical Bernstein-Chlodowsky polynomials. Also their knots are shifted and depend on x.

We firstly study weighted approximation properties of these operators and show that these operators are more efficient in weighted approximating to function having polynomial growth since these operators contain a factor b_n tending to infinity. Secondly we calculate derivative of new Bernstein-Chlodowsky-Gadjiev operators and give a weighted approximation theorem in Lipchitz space for the derivatives of these operators.

1. Introduction

Due to the polynomials have significant applications in a lot of area such as mathematics and physics, nowadays a variety of their generalizations have been studied increasingly. Recently, in [2] Gadjiev and Ghorbanalizadeh constructed a new generalization of Bernstein-Stancu type polynomials given by

$$S_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right)\binom{n}{r}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-r}, \tag{1}$$

where $\frac{\alpha_2}{n+\beta_2} \le x \le \frac{n+\alpha_2}{n+\beta_2}$ and α_k, β_k, $k = 1,2$ are positive real numbers satisfying $0 \le \alpha_2 \le \alpha_1 \le \beta_2 \le \beta_1$. They studied convergence properties of these operators, showed that the new polynomials are sequences of linear positive operators in the space of continuous functions and the interval of convergence of these polynomials is a moved interval and grows to $[0,\ 1]$.

Following polynomials were introduced by I. Chlodowsky [1] in 1937 as a generalization of the Bernstein polynomials. The classical Bernstein-Chlodowsky operators are

$$C_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r},$$

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where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0,b]\subset[0,\infty)$ and $(b_n)_{n\geq 1}$ is a positive increasing sequence with the properties

$$b_n \to \infty \text{ and } \frac{b_n}{n} \to 0 \text{ as } n \to \infty. \tag{2}$$

In [8], weighted approximation properties of Bernstein-Chlodowsky operators were investigated and some generalization of Bernstein-Chlodowsky operators were given in [4] and [7].

In the present paper, using above ideas, we introduce a new construction of Bernstein-Chlodowsky-Gadjiev type operators as following:

$$T_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\alpha_3 x + \beta_3 \frac{r+\alpha_1}{n+\beta_1} b_n\right)\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} \tag{3}$$

where $\frac{\alpha_2}{n+\beta_2}b_n \leq x \leq \frac{n+\alpha_2}{n+\beta_2}b_n$, b_n satisfies (2) and $\alpha_k, \beta_k, k = 1,2,3$ are positive real numbers satisfying $\alpha_3+\beta_3 = 1$ and $0 \leq \alpha_2 \leq \alpha_1 \leq \beta_2 \leq \beta_1$. If we chose

1. $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$ with $b_n = 1$, then we obtain the classical Bernstein polynomials,
2. $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = 0$, then we obtain the classical Bernstein-Chlodowsky polynomials,
3. $\alpha_2 = \alpha_3 = \beta_2 = 0$ with $b_n = 1$, then we obtain the classical Bernstein-Stancu polynomials given in [9],
4. $\alpha_3 = 0$ with $b_n = 1$, then we obtain the Bernstein-Stancu polynomials defined by (1).

The new Bernstein-Chlodowsky-Gadjiev operators, based on functions defined on $[0,\infty)$, which are bounded on every $[0,b_n]\subset[0,\infty)$ with (2), become an approximation process in approximating unbounded functions on the unbounded infinite interval $[0,\infty)$. Also as known that, since an immediate analog of the Bohman-Korovkin theorem does not hold in the unbounded interval, some restrictions are needed. Now we give these restrictions and notations will be used throughout the paper.

Let $B_2[0,\infty)$ be the space of all functions f defined on the semi-axis $[0,\infty)$ satisfying the inequality

$$|f(x)| \leq M_f(1+x^2),$$

where M_f is a positive constant only depending on function f. Introduce

$$C_2[0,\infty) = B_2[0,\infty) \cap C[0,\infty)$$

and

$$C_2^*[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{x\to\infty} \frac{\left|f(x)\right|}{1+x^2} = K_f < \infty \right\}.$$

These spaces endowed with the norm

$$\left\|f\right\|_2 = \sup_{x\geq 0} \frac{\left|f(x)\right|}{1+x^2}. \tag{4}$$

As it follows from the Gadjiev papers [5] and [6], the Korovkin-type theorems for positive linear operators does not hold in the space $C_2[0,\infty)$ but holds in the space of $C_2^*[0,\infty)$ in the norm of $B_2[0,\infty)$ and has the following forms:

Theorem 1.1. *If the sequence of positive linear operators L_n from $C_2[0,\infty)$ to $B_2[0,\infty)$ satisfies conditions*

$$\lim_{n\to\infty} \|L_n(t^\nu;x) - x^\nu\|_2 = 0,\ \nu = 0,1,2.$$

then for any function $f \in C_2^*[0,\infty)$

$$\lim_{n\to\infty} \left\|L_n f - f\right\|_2 = 0.$$

Theorem 1.2. *For any sequence of linear positive operators $(L_n)_{n\geq 1}$ satisfying the conditions of Theorem 1.1, there exists a function $f^* \in C_2[0,\infty)$, for which*

$$\lim_{n\to\infty} \left\|L_n f^* - f^*\right\|_2 \neq 0.$$

2. Weighted Approximation

In this section we study approximation properties of $T_{n,\alpha,\beta}(f)$ using the Theorem 1.1.

Theorem 2.1. *We have*

$$\lim_{n\to\infty} \sup_{\frac{\alpha_2}{n+\beta_2}b_n \leq x \leq \frac{n+\alpha_2}{n+\beta_2}b_n} \frac{\left|T_{n,\alpha,\beta}(f;x) - f(x)\right|}{1+x^2} = 0$$

for any function $f \in C_2^*[0,\infty)$.

Proof. We use the method given in [8]. For simplification, we shall use following definition:

$$T^*_{n,\alpha,\beta}(f;x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^n f\left(\frac{r}{n}\right)\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r}. \tag{5}$$

By the binomial expansion and (3) it is obvious that

$$\begin{aligned} T_{n,\alpha,\beta}(1;x) &= T^*_{n,\alpha,\beta}(1;x) \\ &= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^n \binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} \\ &= \left(\frac{n+\beta_2}{n}\right)^n \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2} + \frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^n = 1. \end{aligned} \tag{6}$$

It follows that

$$\lim_{n\to\infty} \sup_{\frac{\alpha_2}{n+\beta_2}b_n \leq x \leq \frac{n+\alpha_2}{n+\beta_2}b_n} \frac{\left|T_{n,\alpha,\beta}(1;x) - 1\right|}{1+x^2} = 0. \tag{7}$$

From (5), we have

$$\begin{aligned} T^*_{n,\alpha,\beta}(t;x) &= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^n \frac{r}{n}\binom{n}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r} \\ &= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n-1} \binom{n-1}{r}\left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right)^{r+1} \left(\frac{n+\alpha_2}{n+\beta_2} - \frac{x}{b_n}\right)^{n-r-1} \\ &= \left(\frac{n+\beta_2}{n}\right)^n \left(\frac{n}{n+\beta_2}\right)^{n-1} \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right) \\ &= \left(\frac{n+\beta_2}{n}\right) \left(\frac{x}{b_n} - \frac{\alpha_2}{n+\beta_2}\right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} T^{*}_{n,\alpha,\beta}\left(t^{2};x\right) &= \left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=0}^{n}\frac{r^2}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &= \left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=2}^{n}\frac{r\left(r-1\right)}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &\quad +\left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=1}^{n}\frac{r}{n^2}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \end{aligned}$$

$$\begin{aligned} &= \left(\frac{n+\beta_2}{n}\right)^{n}\frac{n-1}{n}\sum_{r=0}^{n-2}\binom{n-2}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r+2}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2} \\ &+\left(\frac{n+\beta_2}{n}\right)^{n}\frac{1}{n}\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r+1}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r-2} \\ &= \frac{(n-1)}{n}\left(\frac{n+\beta_2}{n}\right)^{2}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{2}+\left(\frac{n+\beta_2}{n}\right)\frac{1}{n}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right). \end{aligned} \tag{9}$$

By the definitions (3) and (5) we have

$$\begin{aligned} T_{n,\alpha,\beta}\left(t;x\right) &= \left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=0}^{n}\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &= \alpha_3 x\left(\frac{n+\beta_2}{n}\right)^{n}\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &\quad +\left(\frac{n+\beta_2}{n}\right)^{n}\frac{n\beta_3}{n+\beta_1}b_n\sum_{r=0}^{n}\frac{r}{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &\quad +\left(\frac{n+\beta_2}{n}\right)^{n}\frac{\beta_3\alpha_1}{n+\beta_1}b_n\sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^{r}\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &= \alpha_3 x T^{*}_{n,\alpha,\beta}\left(1;x\right)+\frac{n\beta_3}{n+\beta_1}b_n T^{*}_{n,\alpha,\beta}\left(t;x\right)+T^{*}_{n,\alpha,\beta}\left(1;x\right)\frac{\beta_3\alpha_1}{n+\beta_1}b_n. \end{aligned}$$

Taking into account (6) and (8) we get

$$T_{n,\alpha,\beta}\left(t;x\right)=\alpha_3 x+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3 x+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n.$$

Therefore

$$\begin{aligned} &\lim_{n\to\infty}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\frac{\left|T_{n,\alpha,\beta}\left(t;x\right)-x\right|}{1+x^2} \\ &\le \lim_{n\to\infty}\left|\alpha_3+\left(\frac{n+\beta_2}{n+\beta_1}\right)\beta_3-1\right|+\left(\frac{\alpha_1-\alpha_2}{n+\beta_1}\right)\beta_3 b_n \\ &= \alpha_3+\beta_3-1=0. \end{aligned} \tag{10}$$

Similarly

$$\begin{aligned} T_{n,\alpha,\beta}\left(t^2;x\right) &= \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left(\alpha_3 x+\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &= (\alpha_3 x)^2\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left(2x\alpha_3\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r} \\ &+\left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n}\left(\beta_3\frac{r+\alpha_1}{n+\beta_1}b_n\right)^2\binom{n}{r}\left(\frac{x}{b_n}-\frac{\alpha_2}{n+\beta_2}\right)^r\left(\frac{n+\alpha_2}{n+\beta_2}-\frac{x}{b_n}\right)^{n-r}. \end{aligned}$$

Again using (5) we can write

$$\begin{aligned} T_{n,\alpha,\beta}\left(t^2;x\right) &= (\alpha_3 x)^2\, T^*_{n,\alpha,\beta}(1;x)+\left(2x\alpha_3\beta_3\frac{n}{n+\beta_1}b_n\right)T^*_{n,\alpha,\beta}(t;x) \\ &+\left(2x\alpha_3\beta_3\frac{\alpha_1}{n+\beta_1}b_n\right)T^*_{n,\alpha,\beta}(1;x)+\left(\frac{n\beta_3}{n+\beta_1}b_n\right)^2 T^*_{n,\alpha,\beta}\left(t^2;x\right) \\ &+\left(\frac{2\beta_3^2\alpha_1 n}{(n+\beta_1)^2}b_n^2\right)T^*_{n,\alpha,\beta}(t;x)+\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2 T^*_{n,\alpha,\beta}(1;x) \end{aligned}$$

and from (6), (8) and (9) we get

$$\begin{aligned} T_{n,\alpha,\beta}\left(t^2;x\right) = &\left((\alpha_3 x)^2+2x\alpha_3\beta_3\frac{\alpha_1}{n+\beta_1}b_n+\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2\right)+ \\ &+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2}{n+\beta_2}b_n\right) \\ &\times\left(2x\alpha_3+\frac{2\beta_3\alpha_1}{n+\beta_1}b_n+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(x-\frac{\alpha_2 b_n}{n+\beta_2}\right)+\frac{\beta_3}{n+\beta_1}b_n\right) \end{aligned} \tag{11}$$

Obviously

$$\begin{aligned} &\lim_{n\to\infty}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\frac{\left|T_{n,\alpha,\beta}\left(t^2;x\right)-x^2\right|}{1+x^2} \\ &=\lim_{n\to\infty}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\left\{\frac{x^2}{1+x^2}\left[\alpha_3^2+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(2\alpha_3+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\right)-1\right]\right. \\ &+\frac{x}{1+x^2}\left[\frac{2\alpha_3\beta_3\alpha_1}{n+\beta_1}b_n+\beta_3\left(\frac{n+\beta_2}{n+\beta_1}\right)\left(\frac{2\beta_3\alpha_1}{n+\beta_1}b_n-\frac{\beta_3(n-1)}{n}\left(\frac{\alpha_2 b_n}{n+\beta_1}\right)+\frac{\beta_3}{n+\beta_1}b_n\right)\right. \\ &\left.-\beta_3\left(\frac{\alpha_2 b_n}{n+\beta_1}\right)\left(2\alpha_3+\frac{\beta_3(n-1)}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)\right)\right]+\frac{1}{1+x^2}\left[\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2\right. \\ &\left.\left.-\beta_3\left(\frac{\alpha_2 b_n}{n+\beta_1}\right)\left(\frac{2\beta_3\alpha_1}{n+\beta_1}b_n-\frac{\beta_3}{n+\beta_1}b_n\left(\frac{n-1}{n}\alpha_2-1\right)\right)\right]\right\} \\ &=\alpha_3^2+2\alpha_3\beta_3+\beta_3^2-1=0. \end{aligned} \tag{12}$$

If we use the operators

$$T_n(f;x)=\begin{cases} f(x) & \text{if } 0\le x\le \frac{\alpha_2}{n+\beta_1}b_n \\ T_{n,\alpha,\beta}(f;x) & \text{if } \frac{\alpha_2}{n+\beta_2}b_n\le x\le \frac{n+\alpha_2}{n+\beta_2}b_n \\ f(x) & \text{if } \frac{n+\alpha_2}{n+\beta_2}b_n\le x<\infty \end{cases} \tag{13}$$

then we can write

$$\lim_{n\to\infty}\left\|T_n(f)-f\right\|_2=\lim_{n\to\infty}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\frac{\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|}{1+x^2}. \tag{14}$$

From (7), (10) and (12) it follows that

$$\lim_{n\to\infty}\|T_n(t^\nu;)-x^\nu\|_2=0.$$

As an immediate application of Theorem 1.1, we have

$$\lim_{n\to\infty}\left\|T_n(f)-f\right\|_2=0$$

for $f\in C_2^*[0,\infty)$. From (14), we have desired result. □

Theorem 2.2. *We have*

$$\lim_{n\to\infty}\frac{1}{\sqrt{b_n}}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\frac{\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|}{1+x^2}=0$$

for any function $f\in C_2^*[0,\infty)$.

Proof. Since $f\in C_2^*[0,\infty)$ we can write $\lim_{x\to\infty}\frac{|f(x)|}{1+x^2}=K_f$. It is sufficient to study with the functions satisfying the condition $\lim_{x\to\infty}\frac{|f(x)|}{1+x^2}=0\left(\text{for example } \varphi(x)=f(x)-K_f\left(1+x^2\right)\right)$. That is, there exist a sufficiently large number $x_0>0$ such that $\frac{|f(x)|}{1+x^2}<\varepsilon$ for $x>x_0$. Considering the operators defined in (13), we get

$$\begin{aligned}\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2} &\le \frac{1}{\sqrt{b_n}}\sup_{0\le x\le x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}\\ &\le \frac{1}{\sqrt{b_n}}\left\|T_n(f)-f\right\|_{C[0,\,x_0]}+\frac{1}{\sqrt{b_n}}\left\|f\right\|_2\sup_{x>x_0}\frac{T_n\left(1+t^2;x\right)}{1+x^2}\\ &\quad+\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{\left|f(x)\right|}{1+x^2}.\end{aligned}$$

The first term of above summation tends to zero as $n\to\infty$ by the Korovkin's theorem. Since $\frac{|f(x)|}{1+x^2}<\varepsilon$ for $x>x_0$, the last term of above summation tends to zero as $n\to\infty$. Also we can write from (11)

$$\begin{aligned}\frac{1}{\sqrt{b_n}}\sup_{x>x_0}\frac{T_n\left(1+t^2;x\right)}{1+x^2} &\le \frac{\alpha_3^2+2\alpha_3\beta_3}{\sqrt{b_n}}+2\frac{\alpha_3\beta_3}{\sqrt{b_n}}\frac{\alpha_1}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\left(\frac{\beta_3\alpha_1}{n+\beta_1}b_n\right)^2\\ &\quad+\frac{1}{\sqrt{b_n}}\frac{2\beta_3^2\alpha_1}{n+\beta_1}b_n+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2(n-1)}{n}+\frac{1}{\sqrt{b_n}}\frac{\beta_3^2}{n+\beta_1}b_n\end{aligned}$$

which tend to zero as $n\to\infty$. Since

$$\lim_{n\to\infty}\frac{1}{\sqrt{b_n}}\sup_{0\le x<\infty}\frac{\left|T_n(f;x)-f(x)\right|}{1+x^2}=\lim_{n\to\infty}\frac{1}{\sqrt{b_n}}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\le x\le\frac{n+\alpha_2}{n+\beta_2}b_n}\frac{\left|T_{n,\alpha,\beta}(f;x)-f(x)\right|}{1+x^2}, \tag{15}$$

the proof is completed. □

3. Convergence of Derivative of $T_{n,\alpha,\beta}(f;x)$

In this section we choose $\alpha_3 = 0$. Convergence of derivative of Bernstein operators was given in [3].

Let h be a positive real number. The forward differences $\Delta_h^r$of f at the abscissas $x_0, x_1, \ldots, x_n$ are defined recursively as

$$\begin{aligned}\Delta_h^0 f\left(x_j\right) &= f\left(x_j\right)\\ \Delta_h^{r+1} f\left(x_j\right) &= \Delta_h^r f\left(x_{j+1}\right) - \Delta_h^r f\left(x_j\right)\end{aligned}$$

for $r \geq 0$. It follows that we can express the forward differences in terms a summation as

$$\Delta_h^k f\left(x_j\right) = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f\left(x_{j+s}\right),$$

where $x_j = x_0 + jh$.

The derivative of the $T_{n,\alpha,\beta}(f;x)$ may be expressed in terms of kth forward differences of f as following:

Lemma 3.1. *For any integer $k \geq 0$, we have*

$$\begin{aligned}\mathcal{T}_{n+k,\alpha,\beta}^{(k)}(f;x) &= \frac{(n+k)!}{n!}\left(\frac{1}{b_{n+k}}\right)^k \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k} \sum_{r=0}^{n} \Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1} b_{n+k}\right)\binom{n}{r}\\ &\quad \times \left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n-r},\end{aligned}$$

where Δ_h^k is kth differences operator of f with step $h = b_{n+k}/n+k+\beta_1$.

Proof. According to (3) we write

$$\begin{aligned}\mathcal{T}_{n+k,\alpha,\beta}(f;x) &= \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k} \sum_{r=0}^{n+k} f\left(\frac{r+\alpha_1}{n+k+\beta_1} b_{n+k}\right)\binom{n+k}{r}\\ &\quad \times \left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n+k-r}\end{aligned}$$

and differentiate k times, we have

$$\mathcal{T}_{n+k,\alpha,\beta}^{(k)}(f;x) = \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k} \sum_{r=0}^{n+k} f\left(\frac{r+\alpha_1}{n+k+\beta_1} b_{n+k}\right)\binom{n+k}{r} P(x),$$

where

$$P(x) = \frac{d^k}{dx^k}\left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n+k-r}.$$

If we use the Leibnitz rule for $P(x)$ with

$$\frac{d^s}{dx^s}\left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^r = \begin{cases} \frac{r!}{(r-s)!}\left(\frac{1}{b_{n+k}}\right)^s \left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^{r-s} & , r-s \geq 0 \\ 0 & , r-s < 0 \end{cases}$$

and

$$\frac{d^{k-s}}{dx^{k-s}}\left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n+k-r} = \begin{cases} \frac{(n+k-r)!}{(n+s-r)!}\left(\frac{-1}{b_{n+k}}\right)^{k-s} \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n+s-r} & , r-s \leq n \\ 0 & , r-s > n \end{cases}.$$

then we have

$$P(x) = \left(\frac{1}{b_{n+k}}\right)^k \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} \frac{r!}{(r-s)!} \frac{(n+k-r)!}{(n+s-r)!} \times \left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^{r-s} \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n+s-r}.$$

Since

$$\binom{n+k}{r} \frac{r!}{(r-s)!} \frac{(n+k-r)!}{(n+s-r)!} = \frac{(n+k)!}{n!} \binom{n}{r-s}$$

we can write the kth derivative of $\mathcal{T}_{n+k,\alpha,\beta}(f;x)$

$$\frac{(n+k)!}{n!} \left(\frac{1}{b_{n+k}}\right)^k \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k} \sum_{r=0}^{n} \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f\left(\frac{r+s+\alpha_1}{n+k+\beta_1} b_{n+k}\right) \binom{n}{r} \times \left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^{r} \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n-r}.$$

Since

$$\sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f\left(\frac{r+s+\alpha_1}{n+k+\beta_1} b_{n+k}\right) = \Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1} b_{n+k}\right)$$

where the operator Δ_h is applied with step $h = b_{n+k}/n+k+\beta_1$, we have desired result. □

Theorem 3.2. *Let the function f be a $(k-1)$-times continuous differentiable on $[0,\infty)$ and its kth derivative belongs to $Lip_M\alpha$, $0 < \alpha \le 1$ for some integer $k \ge 1$. Then we have*

$$\lim_{n\to\infty} \sup_{\frac{\alpha_2}{n+k+\beta_2} b_{n+k} \le x \le \frac{n+k+\alpha_2}{n+k+\beta_2} b_{n+k}} \frac{\left|\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x) - f^{(k)}(x)\right|}{1+x^\alpha} = 0.$$

Proof. We know that

$$\Delta_h^k f\left(\frac{r+\alpha_1}{n+k+\beta_1} b_{n+k}\right) = f^{(k)}(\xi_r) \frac{(b_{n+k})^k}{(n+k+\beta_1)^k}$$

where$(r+\alpha_1) b_{n+k}/(n+k+\beta_1) < \xi_r < (r+\alpha_1+k) b_{n+k}/(n+k+\beta_1)$. If we take

$$\xi_r = \frac{r+\alpha_1+\theta_r k}{n+k+\beta_1} b_{n+k}, \quad 0 < \theta_r < 1$$

we can write by Lemma 3.1

$$\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x) = \frac{(n+k)!}{n!(n+k+\beta_1)^k} \left(\frac{n+k+\beta_2}{n+k}\right)^{n+k} \sum_{r=0}^{n} f^{(k)}\left(\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1} b_{n+k}\right) \binom{n}{r} \times \left(\frac{x}{b_{n+k}} - \frac{\alpha_2}{n+k+\beta_2}\right)^{r} \left(\frac{n+k+\alpha_2}{n+k+\beta_2} - \frac{x}{b_{n+k}}\right)^{n-r}.$$

We can easily verify that

$$\lim_{n\to\infty} \frac{(n+k)!}{n!(n+k+\beta_1)^k} = 1.$$

Thus we have

$$\begin{aligned}
&\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\\
= \; & \frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^{n+k}\sum_{r=0}^{n}\left(f^{(k)}\left(\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}\right)-f^{(k)}(x)\right)\binom{n}{r}\right.\\
&\left.\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\right\}\\
&+f^{(k)}(x)\left(\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right).
\end{aligned}$$

Since $f^{(k)}\in Lip_M\alpha$ we have

$$\begin{aligned}
&\left|\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|\\
\leq \; & M\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^{n}\sum_{r=0}^{n}\left|\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}-x\right|^{\alpha}\binom{n}{r}\right.\\
&\left.\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\right\}\\
&+\left|f^{(k)}(x)\right|\left|\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|.
\end{aligned}$$

Applying Hölder's inequality and use the inequality

$$\left|f^{(k)}(x)\right|\leq\left|f^{(k)}(0)\right|+Mx^{\alpha}\leq M_f\left(1+x^{\alpha}\right)$$

we have

$$\begin{aligned}
&\left|\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|\\
\leq \; & M\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\left\{\left(\frac{n+k+\beta_2}{n+k}\right)^{n}\sum_{r=0}^{n}\left(\frac{r+\alpha_1+\theta_r k}{n+k+\beta_1}b_{n+k}-x\right)^{2}\binom{n}{r}\right.\\
&\left.\times\left(\frac{x}{b_{n+k}}-\frac{\alpha_2}{n+k+\beta_2}\right)^r\left(\frac{n+k+\alpha_2}{n+k+\beta_2}-\frac{x}{b_{n+k}}\right)^{n-r}\right\}^{\alpha/2}\\
&+M_f\left(1+x^{\alpha}\right)\left|\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
\left|\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right| \;\leq\; & M\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\left[\tilde{\mathcal{T}}_{n,\alpha,\beta}\left((t-x)^2;x\right)\right]^{\frac{\alpha}{2}}\\
&+M_f\left(1+x^{\alpha}\right)\left|\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|,
\end{aligned}$$

where

$$\tilde{\mathcal{T}}_{n,\alpha,\beta}(f;x)=\left(\tfrac{n+k+\beta_2}{n+k}\right)^n\sum_{r=0}^{n}f\left(\tfrac{r+\alpha_1+k}{n+k+\beta_1}b_{n+k}\right)\binom{n}{r}\left(\tfrac{x}{b_{n+k}}-\tfrac{\alpha_2}{n+k+\beta_2}\right)^r\left(\tfrac{n+k+\alpha_2}{n+k+\beta_2}-\tfrac{x}{b_{n+k}}\right)^{n-r}.$$

By calculating $\tilde{\mathcal{T}}_{n,\alpha,\beta}\left((t-x)^2;x\right)$, we get

$$\begin{aligned}\frac{\left|\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|}{1+x^\alpha} &\leq M\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k\frac{1}{1+x^\alpha}\left[x^2\gamma_n+x\sigma_n+\tau_n\right]^{\frac{\alpha}{2}}\\ &+M_f\left|\frac{(n+k)!}{n!\,(n+k+\beta_1)^k}\left(\frac{n+k+\beta_2}{n+k}\right)^k-1\right|,\end{aligned}$$

where

$$\gamma_n:=\left[\left(\left(\frac{n}{n+k}\right)\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)-1\right)^2-n\left(\frac{1}{n+k}\right)^2\left(\frac{n+k+\beta_2}{n+k+\beta_1}\right)\right],$$

$$\begin{aligned}\sigma_n\ \ :\ &=\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\left[\left(\frac{2n}{n+k}\right)\alpha_2-2(\alpha_1+k)-2\left(\frac{n}{n+k}\right)^2\frac{(n+k+\beta_2)}{n+k+\beta_1}\alpha_2\right.\\ &\left.+2(\alpha_1+k+1)\left(\frac{n}{n+k}\right)\frac{(n+k+\beta_2)}{n+k+\beta_1}\right],\end{aligned}$$

$$\tau_n:=\left(\frac{b_{n+k}}{n+k+\beta_1}\right)^2\left\{\left[\left(\frac{n}{n+k}\right)\alpha_2-(\alpha_1+k)\left(\frac{b_{n+k}}{n+k+\beta_1}\right)\right]^2+2\alpha_2\left(\frac{n}{n+k}\right)\right\}.$$

Taking supremum overall $x\in\left[\frac{\alpha_2}{n+k+\beta_2}b_{n+k},\frac{n+k+\alpha_2}{n+k+\beta_2}b_{n+k}\right]$ and passing to limit with $n\to\infty$ respectively, we obtain

$$\lim_{n\to\infty}\sup_{\frac{\alpha_2}{n+\beta_2}b_n\leq x\leq\frac{n+\alpha_2}{n+\beta_2}b_n}\frac{\left|\mathcal{T}^{(k)}_{n+k,\alpha,\beta}(f;x)-f^{(k)}(x)\right|}{1+x^\alpha}=0,$$

which is desired. □

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References

[1] I. Chlodowsky, Sur le developpement des fonctions definies dans un intervalle infini en series de polynomes de M. S. Bernstein, Compositio Math. 4 (1937) 380–393.
[2] A. D. Gadjiev and A. M. Ghorbanalizadeh, Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables, Appl. Math. Compt.216 (2010) 890–901.
[3] M. S. Floater, On the convergence of derivatives of Bernstein approximation, J. Approx. Theory 134 (2005) 130–135.
[4] A.D. Gadjiev, I. Efendiev, E. Ibikli, Generalized Bernstein-Chlodowsky polynomials, Rocky Mount. J. Math. V.28 No 4 (1998) 1267–1277 .
[5] A.D. Gadzhiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P.P. Korovkin, English translated, Sov. Math. Dokl. Vol 15 No 5 (1974).
[6] A.D. Gadzhiev, P.P Korovkin type theorems, Mathem. Zametki Vol 20 No: 5 (1976) Engl. Transl., Math. Notes 20 995–998.
[7] E. A. Gadjieva, E. Ibikli, On generalization of Bernstein-Chlodowsky polynomials, Hacet. Bull. Nat. Sci. Eng. 24 (1995) 31–40.
[8] E. A. Gadjieva and E. Ibikli, Weigted approximation by Bernstein-Chlodowsky polynomials, Indian J. Pure Appl. Math. 30 (1997) 83–87.
[9] D. D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rew. Roum. Math. Pure. Appl. 13 (1968) 1173–1194.