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Interpolation in the spaces N^p $(1 < p < \infty)$

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Abstract. For $p > 1$ the Privalov space N^p consists of all holomorphic functions f on the open unit disk $\mathbb{D}$ for which

$$\sup_{0\le r<1}\int_0^{2\pi}(\log^+|f(re^{i\theta})|^p\,\frac{d\theta}{2\pi}<\infty.$$

We study the interpolation problems for the spaces N^p. Our results and methods are similar to those obtained by N. Yanagihara in [24] for the Smirnov class N^+.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T})$ $(0 < q \le \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$. The *Privalov space* N^p $(1 < p < \infty)$ consists of all holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{0\le r<1}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p\,\frac{d\theta}{2\pi}<+\infty. \tag{1.1}$$

Here, as usual, we set $x^+ = \max(x, 0)$. These spaces were introduced by I. I. Privalov in the first edition of his book [18, p. 93], where N^p is denoted as A_q (with $q = p > 1$). Recall that the condition (1.1) with $p = 1$ defines the *Nevanlinna class* N of holomorphic functions on $\mathbb{D}$. Furthermore, the *Smirnov class* N^+ consists of those functions $f \in N$ holomorphic on $\mathbb{D}$ such that

$$\lim_{r\to 1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,\frac{d\theta}{2\pi}=\int_0^{2\pi}\log^+|f^*(e^{i\theta})|\,\frac{d\theta}{2\pi}<+\infty,$$

where f^* is the boundary function of f on T , i.e.,

$$f^*(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$$

is the radial limit of f which exists for almost every $e^{i\theta}$.

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Recall that the *Hardy space* H^q $(0 < q \leq \infty)$ consists of all functions f holomorphic on $\mathbb{D}$, which satisfy

$$\left(\|f\|_q\right)^{\max(1,q)} := \sup_{0\leq r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^q \frac{d\theta}{2\pi} < \infty$$

if $0 < q < \infty$, and which are bounded when $q = \infty$:

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

It is known (see [15]) that

$$N^q \subset N^p \ (q > p), \quad \bigcup_{q>0} H^q \subset \bigcap_{p>1} N^p, \quad \text{and} \quad \bigcup_{p>1} N^p \subset N^+,$$

where the above inclusion relations are proper.

Let $\varphi : \mathbb{R} \to [0, \infty)$ be a convex, nondecreasing function satisfying

(i) $\lim_{t\to\infty} \varphi(t)/t = \infty$ and

(ii) $\widetilde{\Delta}_2$ − condition : $\varphi(t + 2) \leq M\varphi(t) + K, t \geq t_0$ for some constants $M, K \geq 0$ and $t_0 \in \mathbb{R}$.

Such a function φ is called *strongly convex*, and one can associate with it the corresponding *Hardy-Orlicz space* $\mathcal{H}_\varphi$ defined as

$$\mathcal{H}_\varphi = \{f \in N^+ : \int_0^{2\pi} \varphi(\log|f^*(e^{i\theta})|) \frac{d\theta}{2\pi} < \infty\}.$$

First, it is well known that the union of all Hardy-Orlicz spaces corresponds to the Smirnov class (see e.g., [19, Chapter 3, Part II]). Furthermore, notice that for $\varphi(t) = e^{qt}$ with $0 < q < \infty$, $\mathcal{H}_\varphi$ becomes the usual Hardy space H^q on $\mathbb{D}$, and for $\varphi(t) = (t^+)^p$ with $1 < p < \infty$, $\mathcal{H}_\varphi$ becomes the Privalov space N^p defined above.

The study on the spaces N^p was continued by Stoll's work [21] in 1977. Namely, Stoll [21] was introduced the metric topology in the space $N^p (1 < p < \infty)$ (with the notation $(\log^+ H)^\alpha$ in [21]) and was proved the following result.

Theorem A. ([21, Theorem 4.2]). *The space N^p with the topology given by the metric d_p defined by*

$$d_p(f, g) = \Big(\int_0^{2\pi} \Big(\log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|) \Big)^p \frac{d\theta}{2\pi} \Big)^{1/p}, \quad f, g \in N^p, \tag{1.2}$$

becomes an F-algebra, that is an F-space in which multiplication is continuous.

Further, the linear topological structure and functional properties of the Privalov spaces on the unit disk $\mathbb{D}$ and theirs Fréchet envelopes were investigated in [4], [5], [12], [13], [14] and [21].

Remark 1. Note that (1.2) with $p = 1$ defines the Yanagihara's metric $\rho_1 = \rho$ on N^+ that makes N^+ into an F-algebra (see [22] and [23]).

It is well known [3, p. 26] that a function f holomorphic on $\mathbb{D}$ belongs to the Smirnov class N^+ if and only if it can be factorized as

$$f(z) = B(z)S(z)F(z), \quad z \in \mathbb{D}, \tag{1.4}$$

where B is the Blaschke product with respect to zeros $\{z_n\}_n \subset \mathbb{D}$ of f, S is the singular inner functions, and F is an outer function, i.e.,

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \tag{1.5}$$

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) \right),$$

with positive singular measure $d\mu$, and

$$F(z) = \omega \exp\left(\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|F^*(e^{it})| \frac{dt}{2\pi} \right),$$

where $\log|F^*| \in L^1(\mathbb{T})$, and ω is a constant of unit modulus.

Theorem B. ([18, pp. 98-100], also see [5]). *A function $f \in N^+$ factorized by* (1.4) *belongs to N^p if and only if* $\log^+|F^*| \in L^p(\mathbb{T})$.

Motivated by the investigations of N. Yanagihara given in [24] for the interpolation problems for the Smirnov class N^+, and related investigations for the Hardy spaces $H^q(0 < q < \infty)$ (see e.g., [1], [3], [11], [16] and [20]), here we consider the corresponding problems for the Privalov spaces $N^p(1 < p < \infty)$.

It is interesting to notice that many recent works on interpolation in different Hardy-Orlicz spaces on the unit disk $\mathbb{D}$ investigated a so-called *free interpolation* in these spaces. A Free interpolation problem in Hardy-Orlicz spaces was investigated by A. Hartman in [6], [7] and [8], and in particular, related problems in Nevanlinna and Smirnov classes were studied by A. Hartman et alteri in [10]. Similar interpolation problems for Fréchet envelope of certain Hardy-Orlicz spaces were studied by A. Hartman and X. Massaneda in [9]. For more information on the Fréchet and Banach envelope of Hardy-Orlicz spaces see [4], [5], [13] and [17].

This paper is organized as follows. In Section 2 we define the interpolation problem and present the main interpolation theorem for the Hardy spaces $H^q(0 < q \le \infty)$. This result (Theorem C) is basic for our study of related interpolation problems for Privalov spaces given in the next section. In this section, we present and prove three interpolation theorems (Theorems 1-3) for mentioned spaces. Our results and methods are similar to those on N^+ obtained by Yanagihara in [24].

2. The Interpolation problems

Let $l^q(T)$ $(0 < q < \infty)$ denote, as usual the set of all complex sequences $\{a_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |a_n|^q < \infty,$$

and let l^∞ be the set of all bounded sequences. Here, as always in the sequel, for a sequence $Z = \{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$, we suppose that $z_n \neq 0$, $z_n \neq z_m$ if $n \neq m$ and that *the Blaschke condition* is satisfied:

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty. \tag{2.1}$$

For such a sequence $\{z_n\}_{n=1}^{\infty}$ and $n \in \mathbb{N}$ denote by $B_n(z)$ the infinite product defined as

$$B_n(z) = \prod_{m \neq n} \frac{|z_m|}{z_m} \frac{z_m - z}{1 - \bar{z}_m z}. \tag{2.2}$$

A sequence $\{z_n\}_{n=1}^{\infty}$ is said to be *uniformly separated* if there is a number $\delta > 0$ such that

$$|B_n(z_n)| = \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| \ge \delta > 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.3}$$

For a fixed $0 < q < \infty$ and a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$, we denote by l_z^q the set of all complex sequences $\{c_n\}_{n=1}^{\infty}$ for which

$$\sum_{n=1}^{\infty}(1-|z_n|^2)|c_n|^q < \infty. \tag{2.4}$$

For given a sequence $\{z_n\}_{n=1}^{\infty}$, let T_q be the linear operator on the Hardy space $H^q(0 < q \leq \infty)$ defined by

$$T_q(f) = \left\{(1-|z_n|^2)^{1/q} f(z_n)\right\}_{n=1}^{\infty} \quad \text{for} \quad 0 < q < \infty,$$

and

$$T_\infty(f) = \{f(z_n)\}_{n=1}^{\infty} \quad \text{for} \quad q = \infty.$$

The following statement is in fact the main interpolation theorem for the class H^q.

Theorem C ([3, p. 149, Theorem 9.1]). *Let $0 < q \leq \infty$ be any fixed. Then $T_q(H^q) = l^q$ if and only if $\{z_n\}_{n=1}^{\infty}$ is uniformly separated.*

Recall that the above result is proved by Carleson [1] for $q = \infty$, by Shapiro and Shields [20] for $1 \leq q < \infty$, and by Kabaila [11] for $0 < q < 1$.

3. The main results

By analogy with the sequences $\tilde{l}_z^q$ defined by (2.4), and the sequence l_z^+ given by (2.7) in [24], for a given complex sequences $Z = \{z_n\}_{n=1}^{\infty}$ satisfying the condition (2.1) and for $p > 1$, we define the set l_z^p as the set of all complex sequences $\{c_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty}(1-|z_n|^2)\,(\log^+|c_n|)^p < \infty. \tag{3.1}$$

We say that Z is a *universal interpolation sequence* for the pair (N^p, l_z^p) if for every sequence $\{c_n\}_{n=1}^{\infty}$ in l_z^p there exists a function $f \in N^p$ such that $f(z_n) = c_n$ for all $n \in \mathbb{N}$. We write it simply as u.i.s. for (N^p, l_z^p). The following two theorems are the N^p-analogue of Theorems 1 and 2 in [24] obtained for the Smirnov class.

The following result of N. Mochizuki will be useful for our purposes.

Lemma 1. ([15, the inequality (2)]). *For any function $f \in N^p$ there holds*

$$\log(1+|f(z)|) \leq 2^{1/p} d_p(f,0)(1-|z|)^{-1/p}, \quad z \in \mathbb{D}.$$

For the proof of the second part of the Theorem 1, we will need the following lemma.

Lemma 2. *The class l_z^p is an F-space with respect to the metric σ_p given by*

$$\sigma_p(u,v) = \left(\sum_{n=1}^{\infty}(1-|z_n|^2)\log^p(1+|c_n(u)-c_n(v)|)\right)^{1/p},$$

for $u = \{c_n(u)\}$ and $v = \{c_n(v)\}$.

Proof. We first observe that the triangle inequality follows from Minkowski's inequality. The proof of Lemma 2 is quite similar to the corresponding result for the class N^+ given in [23, Theorem 1] (cf. [24, Lemma 2]), and therefore may be omitted. □

Theorem 1. *In order that a sequence* $Z = \{z_n\}_{n=1}^{\infty}$ *in* $\mathbb{D}$ *be a u.i.s. for* (N^p, l_z^p) *it is sufficient that* $\{z_n\}_{n=1}^{\infty}$ *is uniformly separated, and is necessary that*

$$(1 - |z_n|^2) \log^p \left(\frac{1}{|B_n(z_n)|} \right) \to 0 \quad as \quad n \to \infty. \tag{3.2}$$

Proof. Sufficieny. Suppose $\{z_n\}_{n=1}^{\infty}$ is uniformly separated. For given sequence $\{c_n\}_{n=1}^{\infty}$ in l_z^p, put

$$b_n = \log |c_n| \quad \text{if} \quad |c_n| \geq 1; \quad b_n = 0 \quad \text{if} \quad |c_n| < 1. \tag{3.3}$$

Then the sequence $\{(1 - |z_n|^2)^{1/p} b_n\}$ is in l^p, or equivalently, a sequence $\{b_n\}_{n=1}^{\infty}$ is in $\tilde{l}_z^p$. Hence, by Theorem A, there exists a function $g \in H^p$ such that $g(z_n) = b_n$ for all $n \in \mathbb{N}$. Define the function f_1 as

$$f_1(z) = \exp\left(g(z)\right), \quad z \in \mathbb{D}.$$

Then $f_1 \in N^p$ with

$$f_1(z_n) = |c_n| \quad \text{if} \quad |c_n| \geq 1; \quad f_1(z_n) = 1 \quad \text{if} \quad |c_n| < 1. \tag{3.4}$$

Put

$$c_n' = \frac{c_n}{|c_n|} \quad \text{if} \quad |c_n| \geq 1; \quad c_n' = c_n \quad \text{if} \quad |c_n| < 1. \tag{3.5}$$

Then $\{c_n'\} \in l^{\infty}$, and by Theorem C, there is a bounded holomorphic function f_2 with

$$f_2(z_n) = c_n' \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.6}$$

Thus, if we define the function f by $f(z) = f_1(z) f_2(z)$, $z \in \mathbb{D}$, then $f \in N^p$ and from (3.3)-(3.6) we see that $f(z_n) = c_n$ for all $n \in \mathbb{N}$, as desired.

Necessity. We follow the proof of the same part of Theorem 1 in [24]. Let K be the set of functions $f \in N^p$ such that $f(z_n) = 0$ for all $n \in \mathbb{N}$. From the inequality of Lemma 1 it is easy to deduce that K is a closed subspace of N^p. Consider the quotient space

$$\tilde{N}^p = N^p / K, \quad \bar{f} = f + K \in \tilde{N}^p \quad \text{for} \quad f \in N^p.$$

Define the metric $\bar{d}_p$ on $\tilde{N}^p$ in the usual manner as

$$\bar{d}_p(\bar{f}, \bar{0}) = \inf_{f \in \bar{f}} d_p(f, 0), \quad \bar{d}_p(\bar{f}, \bar{g}) = \bar{d}_p(\overline{f - g}, \bar{0}).$$

Then $\tilde{N}^p$ is an F-space u with respect to the metric $\bar{d}_p$.

For each sequence $u = \{c_n\}_{n=1}^{\infty}$ in l_z^p there corresponds a unique $\bar{f} \in \tilde{N}^p$ such that

$$f(z_n) = c_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and for each} \quad f \in \bar{f}. \tag{3.7}$$

Therefore, the mapping $\overline{T}(u) = \bar{f}$, $u \in l_z^p$ is well defined from l_z^p onto $\tilde{N}^p$. Obviously, $\overline{T}$ is linear. In the same manner as in the proof of Theorem 1 in [24, p. 433], we can prove that the operator $\overline{T}$ is closed. By the closed graph theorem [2, p. 57], we see that $\overline{T}$ is continuous.

Let $u_n = \{c_k(u_n)\}_{k=1}^{\infty}$ be a sequence such that

$$c_k(u_n) = 0 \quad \text{if} \quad k \neq n; \quad c_n(u_n) = 1.$$

Since $\sigma_p(u_n, 0) \to 0$ as $n \to \infty$, it follows that $\bar{d}_p(\bar{h}_n, \bar{0}) \to 0$, where $\bar{h}_n = \overline{T}(u_n)$. Hence, there exist $h_n \in \bar{h}_n$ such that $d_p(h_n, 0) \to 0$ as $n \to \infty$. Since $h_n \in N^p$ and by (3.7) $h_n(z_k) = 0$ for all $k \neq n$, from the canonical factorization theorem for the class N^p (Theorem B), h_n can be factorized as

$$h_n(z) = B_n(z)F_n(z), \quad z \in \mathbb{D},$$

where B_n is the Blaschke product defined by (2.3) and $F_n \in N^p$. Thus, $|F_n^*(e^{i\theta})| = |h_n^*(e^{i\theta})|$ almost every on T, whence it follows that $d_p(F_n, 0) = d_p(h_n, 0)$. Since by (3.7) we have $h_n(z_n) = 1$ for all $n \in \mathbb{N}$, using the estimate from Lemma 1, we have

$$\begin{aligned} (1-|z_n|)^{1/p} \log\left(\frac{1}{|B_n(z_n)|}\right) &\leq (1-|z_n|)^{1/p} \log\left(1 + \frac{|h_n(z_n)|}{|B_n(z_n)|}\right) \\ &= (1-|z_n|)^{1/p} \log(1 + |F_n(z_n)|) \\ &\leq 2^{1/p} d_p(F_n, 0) \\ &= 2^{1/p} d_p(h_n, 0). \end{aligned}$$

Since $d_p(h_n, 0) \to 0$, the above inequality completes the proof of the theorem. □

Remark 2. Obviously, the fact that $\{z_n\}_{n=1}^\infty$ is uniformly separated implies the condition (3.2), but the example $\{z_n\} = \{1 - n^{-2}\}$ shows that the converse is false.

Remark 3. Notice also that the "sufficiency part" of Theorem 1 related to a larger family of Hardy-Orlicz spaces is quite recently proved by M. H. M. Marzuq in [12, Theorem 3.1].

We denote by $\bar{l}_z^p$ $(1 < p < \infty)$ the set of all strictly positive sequences $\{c_n\}_{n=1}^\infty$ such that

$$\sum_{n=1}^{\infty} (1 - |z_n|^2) |\log c_n|^p < \infty, \tag{3.8}$$

and denote by $\bar{N}^p$ the set of all zero-free holomorphic on $\mathbb{D}$ functions f $f(0) > 0$ and $\phi(z) = \log f(z) \in H^p$, where we take as $\phi(0)$ is a real number. Clearly, $\bar{l}_z^p \subset l_z^p$ and $\bar{N}^p \subset N^p$.

The following assertion is the $\bar{N}^p$-analogue of Theorem 2 in [24].

Theorem 2. *A sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D}$ is a u.i.s. for $(\bar{N}^p, \bar{l}_z^p)$, in the sense that for any sequence $\{c_n\}_{n=1}^\infty$ in $\bar{l}_z^p$ there exists $f \in \bar{N}^p$ with $\log f(z_n) = \log c_n$ for all $n \in \mathbb{N}$, if and only if $\{z_n\}_{n=1}^\infty$ is uniformly separated.*

Proof. Sufficiency. Take an arbitrary sequence $\{c_n\}_{n=1}^\infty$ in $\bar{l}_z^p$. If we put $b_n = \log c_n$ $(\arg(c_n) = 0)$, then by the theorem of Shapiro and Shields [20], there is a function $g \in H^p$ with $g(0) = 0$ and $g(z_n) = b_n$ for all $n \in \mathbb{N}$. If we put $f(z) = \exp(g(z))$, $z \in \mathbb{D}$, then we have $f \in \bar{N}^p$ and $f(z_n) = c_n$ for all $n \in \mathbb{N}$.

Necessity. The space $\bar{l}_z^p$ can be considered as a real Banach space with respect to the addition and scalar multiplication defined as follows:

(i) $\{\{c_n\} + \{b_n\}\}_{n=1}^\infty$ is defined to be the sequence $\{c_n b_n\}_{n=1}^\infty$.

(ii) For a real number λ, $\lambda\{c_n\}$ is a sequence defined as $\{(c_n)^\lambda\}_{n=1}^\infty$.

(iii) The norm of $\|\cdot\|$ of $\{c_n\}_{n=1}^\infty$ is defined as

$$\|\{c_n\}\| = \left(\sum_{n=1}^{\infty} (1 - |z_n|^2) |\log c_n|^p\right)^{1/p}. \tag{3.9}$$

Analogously, $\bar{N}^p$ can also be viewed as a real Banach space with respect to the addition and scalar multiplication defined as follows:

(i)′ $f + g$ is defined to be the function whose the value at $z \in \mathbb{D}$ equals $f(z)g(z)$, i.e., $(f + g)(z) = f(z)g(z)$.

(ii)′ For a real number λ, λf is defined to be the function whose the value at $z \in \mathbb{D}$ equals $(f(z))^\lambda$, i.e.,

$(\lambda f)(z) = (f(z))^\lambda$, $(\lambda f)(0) > 0$.

(iii)′ The norm of f is defined as

$$\|f\| = \sup_{0\le r<1} \left(\int_0^{2\pi} |\log f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} = \left(\int_0^{2\pi} |\log f^*(e^{i\theta})| \frac{d\theta}{2\pi} \right)^{1/p}, \tag{3.10}$$

where we take $\arg(f(0)) = 0$.

For a given sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D}$, let P be the set of all functions $f \in \bar{N}^p$ such that $\log f(z_n) = 0$ for all $n \in \mathbb{N}$. P is obviously a closed subspace of $\bar{N}^p$. Put

$$N_*^p = \bar{N}^p / P \quad \text{and} \quad \bar{f} = f + P \quad \text{for} \quad f \in \bar{N}^p.$$

Then N_*^p is a real Banach space with the norm $\|\bar{f}\| = \inf_{f\in\bar{f}} \|f\|$. For each $u = \{c_n(u)\}_{n=1}^\infty = \{c_n\}_{n=1}^\infty$ in $\bar{l}_z^p$ there corresponds a unique $\bar{f} \in N_*^p$ such that

$$\log f(z_n) = \log c_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and for all} \quad f \in \bar{f}.$$

Write this correspondence as $\bar{S}$, i.e., $\bar{f} = \bar{S}(u)$. Obviously, $\bar{S}$ is a linear operator. In the same manner as in the proof of (ii), Theorem 1 in [24], we can show that $\bar{S}$ is a closed operator. Hence, by the closed graph theorem [2, p. 57], $\bar{S}$ is continuous. Thus, we have

$$\|f\| \le M'\|u\| \tag{3.11}$$

with a positive constant M', for an $f \in \bar{f} = \bar{S}(u)$.

Since $\log f \in H^p$, by the subharmonicity of the function $v(z) = |\log f(z)|^p$ on $\mathbb{D}$ we get

$$v(re^{i\theta}) \le \int_0^{2\pi} P(r, \theta - t) v^*(e^{i\theta}) \frac{d\theta}{2\pi} \le \frac{2}{1-r} \int_0^{2\pi} |v^*(e^{i\theta})| \frac{d\theta}{2\pi}, \quad 0 \le r < 1,$$

whence because of (3.10) it follows immediately that

$$(1 - |z|^2)^{1/p} |\log f(z)| \le M''\|f\| \tag{3.12}$$

with a constant $M'' = 4^{1/p}$. Let $u_n = \{c_k(u_n)\}_{k=1}^\infty$ be a positive sequence such that

$$c_k(u_n) = 1 \quad \text{if} \quad k \ne n; \quad c_n(u_n) = e.$$

Then by (iii) we have $\|u_n\| = (1 - |z_n|^2)^{1/p}$.

Let f_n be a function in $\bar{S}(u_n)$ satisfying the condition (3.11). Put $\arg(B_n(0)) = \alpha_n$ and

$$F_n(z) = \exp\left(\frac{\log f_n(z)}{e^{-i\alpha_n} B_n(z)} \right), \quad z \in \mathbb{D}.$$

Then $F_n \in \bar{N}^p$ and $|\log F_n^*(e^{i\theta})| = |\log f_n^*(e^{i\theta})|$ almost every on T. Therefore, from (3.11) and (3.12) we obtain

$$\begin{aligned}
(1 - |z_n|^2)^{1/p} |\log F_n(z_n)| &\le M''\|F_n\| \\
&= M''\|f_n\| \\
&\le M'M''(1 - |z_n|^2)^{1/p}.
\end{aligned}$$

Since

$$|\log F_n(z_n)| = |\log f_n(z_n)| / |B_n(z_n)| = 1/|B_n(z_n)|,$$

from the above inequality we infer that

$$|B_n(z_n)| \ge 1/M'M''.$$

Hence, $\{z_n\}$ is uniformly separated. This completes the proof of the theorem. □

The following result is the N^p-analogue of Theorem 3 in [24] concerning the Nevanlinna class N.

Theorem 3. *Suppose that a sequence* $\{z_n\}_{n=1}^{\infty}$ *in* $\mathbb{D}$ *is uniformly separated, and let* $f \in N^p$ *be a function for which* $\log|f^*| \in L^p(\mathbb{T})$. *Then*

$$\sum_{n=1}^{\infty}(1-|z_n|^2)\left(\log^+|f(z_n)|\right)^p < +\infty.$$

On the other hand, we can find a uniformly separated sequence $\{z_n\}_{n=1}^{\infty}$ *in* $\mathbb{D}$ *and a complex sequence* $\{c_n\}_{n=1}^{\infty}$ *which satisfy*

$$\sum_{n=1}^{\infty}(1-|z_n|^2)\left(\log^+|c_n|\right)^{p-\delta} < +\infty \quad \textit{for any} \quad 0<\delta\le 1, \tag{3.13}$$

while there is no function $f \in N^p$ *with* $f(z_n)=c_n$ *for all* $n\in\mathbb{N}$.

Proof. Let $f \in N^p$ with $\log|f^*| \in L^p(\mathbb{T})$. Then by Theorem B, f can be factorized as

$$f(z)=B(z)S(z)F(z), \quad z\in\mathbb{D},$$

where $B(z)$ is the Blaschke product with respect to zero points of f, S is a singular inner function, and F is an outer function. If we write $g(z)=S(z)F(z)$, it is easily see that a function $\log|g(z)|$ can be represented by a *Poisson-Stieltjes integral*. Hence, by ([3, p. 35, Corollary]) it follows that $\log g(z)$ belongs to H^q for any $0<q<1$. From this and the fact that $|g^*(e^{i\theta})| = |f^*(e^{i\theta})|$ almost every we conclude that $\log g(z)$ belongs to H^p. Hence, by ([3, p. 149, Theorem 9.1]) we obtain

$$\sum_{n=1}^{\infty}(1-|z_n|^2)|\log g(z_n)|^p < +\infty,$$

whence it follows that

$$\sum_{n=1}^{\infty}(1-|z_n|^2)\left(\log^+|f(z_n)|\right)^p \le \sum_{n=1}^{\infty}(1-|z_n|^2)|\log g(z_n)|^p < +\infty.$$

This proves the first part of the theorem.

For the proof of the second part let b be a real number such that $0<b<1$, and put $z_n = 1-b^n$, $c_n=\exp(n/(b^{n/p}))$, $n\in\mathbb{N}$. Then $\{z_n\}_{n=1}^{\infty}$ satisfies the condition (2.1) as well as (2.3), i.e., the sequence $\{z_n\}_{n=1}^{\infty}$ is uniformly separated. A sequence $\{c_n\}_{n=1}^{\infty}$ satisfies the condition (3.13) for any δ, $0<\delta\le 1$, and

$$(1-|z_n|)\left(\log^+|c_n|\right)^p \to \infty \quad \text{as} \quad n\to\infty. \tag{3.14}$$

From the inequality $\log^+|x| \le \log(1+|x|)$ and Lemma 1 it follows immediately that

$$(1-|z|)\left(\log^+|f(z)|\right)^p = O(1). \tag{3.15}$$

Finally, from (3.14) and (3.15) we conclude that there is no function $f\in N^p$ with $f(z_n)=c_n$ for all $n\in\mathbb{N}$. This completes the proof of the theorem. □

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