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Stability and boundedness in multi delay vector Liénard equation

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Abstract. In this paper, we consider a vector Liénard equation with the multiple deviating arguments. Based on the Lyapunov-Krasovskii functional approach, the asymptotic stability of the zero solution and the boundedness of all solutions are discussed. We give an example to illustrate the theoretical analysis made in this work and to show the effectiveness of the method utilized here.

1. Introduction

In mathematics, more specifically in the study of dynamical systems and differential equations, Liénard equation is a second order differential equation, named after the French physicist Alfred-Marie Liénard. During the development of radio and vacuum tube technology, Liénard and modified Liénard equations were intensely studied as they can be used to model oscillating circuits. Today, in applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, etc. are associated with Liénard equation or modified Liénard equation. By this time, the qualitative properties (stability, boundedness, convergence, existence of periodic of solutions, etc.) of scalar Liénard or modified Liénard equation with and without a deviating argument have been intensively discussed and are still being investigated in the literature (Ahmad and Rama Mohana Rao [1], Anh et al. [2], Barnett [3], Burton ([5], [6]), Burton and Zhang [7], Caldeira-Saraiva [8], Cantarelli [9], Èl'sgol'ts and Norkin [11], Gao and Zhao [12], Hale [13], Hara and Yoneyama ([14], [15]), Heidel ([16], [17]), Huang and Yu [18], Jitsuro and Yusuke [19], Kato ([20], [21]), Kolmanovskii and Myshkis [22], Krasovskii [23], LaSalle and Lefschetz [24], Li [25], Liu and Huang ([26], [27]), C. J. Liu and Xu [28], Z.R. Liu [29], Long and Zhang [30], Luk [31], Lyapunov [32], Malyseva [33], Muresan [34], Nápoles Valdés [35], Sugie [36], Sugie and Amano [37], Sugie et al. [38], Tunç ([39], [40], [41], [42], [43], [44], [45], [46], [47], [48]), Yang [49], Ye et al. [50], Yu and Xiao [51], Yoshizawa [52], B. Zhang ([53], [54]), X.S. Zhang and Yan [55], X. Zhou and Jiang [56], J. Zhou and Z. R. Liu [57], J. Zhou and Xiang [58], Wei and Huang [59], Wiandt [60]).

It should be mentioned that Tunç [48] considered the nonlinear vector Liénard equation with the constant delay $\tau > 0$:

$$X''(t) + F(X(t), X'(t))X'(t) + H(X(t-\tau)) = 0.$$

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The author established certain sufficient conditions under which the zero solution of the above equation is unstable.

However, to the best of our knowledge from the literature, the stability and boundedness of solutions of vector Liénard equation and modified vector Liénard equation with multi delay have not been discussed in the literature, yet.

In this paper, we consider the vector Liénard equation with the multiple constant deviating arguments, $\tau_i > 0$:

$$X''(t) + F(X(t), X'(t))X'(t) + G(X(t)) + \sum_{i=1}^{n} H_i(X(t-\tau_i)) = P(t), \tag{1}$$

in which $t \in \Re^+$, $\Re^+ = [0, \infty)$, $t - \tau_i > 0$, and $X \in \Re^n$; F is a continuous symmetric $n \times n-$ matrix, $G : \Re^n \to \Re^n$, $H_i : \Re^n \to \Re^n$ and $P : \Re^n \to \Re^n$ are continuous, G and H_i are also differentiable with $G(0) = H_i(0) = 0$. It is assumed the existence and the uniqueness of the solutions of Eq. (1) ([10, pp. 13, 14]).

We write Eq. (1) in the differential system form:

$$\begin{aligned} X' &= Y, \\ Y' &= -F(X,Y)Y - G(X) - \sum_{i=1}^{n} H_i(X) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds + P(t), \end{aligned} \tag{2}$$

which was obtained by setting $X' = Y$, where $X(t)$ and $Y(t)$ are, respectively, abbreviated as X and Y, and throughout the paper.

The Jacobian matrices of $G(X)$ and $H_i(X)$ are given by

$$J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right), J_{H_1}(X) = \left(\frac{\partial h_{1i}}{\partial x_j}\right), ..., J_{H_n}(X) = \left(\frac{\partial h_{ni}}{\partial x_j}\right), \ (i, \ j = 1, \ 2, ..., n),$$

respectively, where $(x_1, ..., x_n)$, $(g_1, ..., g_n)$ and $(h_{1i}), ..., (h_{ni})$ are the components of X, G and H_i, respectively. It is also assumed that the Jacobian matrices $J_G(X)$ and $J_{H_i}(X)$ exist and are continuous.

The symbol $\langle X, Y \rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X \rangle = \|X\|^2$, and $\lambda_i(\Omega)$ are the eigenvalues of the real symmetric $n \times n-$ matrix Ω. The matrix Ω is said to be negative-definite, when $\langle \Omega X, X \rangle \leq 0$ for all nonzero X in $\Re^n$.

The motivation of this paper has been inspired by the results established in the above mentioned papers. This paper is also a first attempt to obtain certain sufficient conditions on the stability and boundedness of solutions of a vector Liénard equation with multiple deviating arguments, and it has a contribution to the subject in the literature and may be useful for researchers working on the qualitative behaviors of solutions.

We need the following preliminary result.

Lemma 1.1. *([4]). Let A be a real symmetric $n \times n-$ matrix and*

$$\bar{a} \geq \lambda_i(A) \geq a > 0, (i = 1, 2, ..., n),$$

where $\bar{a}$ and a are constants.
Then

$$\bar{a} \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$\bar{a}^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle .$$

For a given number $r \geq 0$, let C^n denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathfrak{R}^n$ and for $\phi \in C^n$, $\|\phi\| = \sup_{-r\leq\phi\leq 0} \|\phi(\theta)\|$. C_H^n will denote the set of ϕ in C^n for which $\|\phi\| < H$. For any continuous function $x(u)$ defined on $-r \leq u \leq B$, $B > 0$, any fixed t, $0 \leq t \leq B$, the symbol x_t will denote the function $x(t + \theta)$, $-r \leq \theta \leq 0$.

If $f(\phi)$ is a functional defined for every ϕ in C_H^n and $x'(t)$ is the right side derivative of $x(t)$, we consider the autonomous functional differential equation:

$$x'(t) = f(x_t), t \geq 0. \tag{3}$$

We say $x(\phi)$ is a solution of Eq. (3) with the initial condition ϕ in C_H^n at $t = 0$ if there is a constant $B > 0$ such that $x(\phi)$ is a function from $[-r, B)$ into $\mathfrak{R}^n$ such that $x_t(\phi)$ is in C_H^n for $0 \leq t < B$, $x_0(\phi) = \phi$ and $x(\phi)(t)$ satisfies Eq. (3) for $0 \leq t < B$.

Definition 1.2. *([6]) A continuous function $W : \mathfrak{R}^n \to \mathfrak{R}^+$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i , where i is an integer.)*

Definition 1.3. *([13]). Let V be a continuous scalar functional in C_H^n. The derivative of V along the solutions of Eq. (3) will be defined by*

$$\dot{V}(\phi) = \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi))}{h}.$$

Lemma 1.4. *([13]). Suppose $f(0) = 0$. Let V be a continuous functional defined on C_H^n with $V(0) = 0$ and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all ϕ in C_H^n, $u(\|\phi(0)\|) \leq V(\phi)$, $V'(\phi) \leq 0$, then the solution $x = 0$ of Eq. (3) is stable.*

Let $R \subset C_H^n$ be a set of all functions $\phi \in C_H^n$ where $V'(\phi) = 0$. If $\{0\}$ is the largest invariant set in R, then the solution $x = 0$ of Eq. (3) is asymptotically stable.

Let us also consider the non-autonomous functional differential equation:

$$x' = g(t, x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0, \tag{4}$$

where $g : \mathfrak{R}^+ \times C_H \to \mathfrak{R}^n$ is a continuous mapping, $g(t, 0) = 0$, and we suppose that g takes closed bounded sets into bounded sets of $\mathfrak{R}^n$. Here $(C, \ \|.\|)$ is the Banach space of continuous functions $\phi : [-r, \ 0] \to \mathfrak{R}^n$ with supremum norm, $r > 0$; C_H is the open $H-$ ball in C; $C_H := \{\phi \in (C[-r, 0], \ \mathfrak{R}^n) : \|\phi\| < H\}$. Let S be the set of $\varphi \in C$ such that $\|\varphi\| \geq H$. We shall denote by $S^\bullet$ the set of all functions $\varphi \in C$ such that $|\varphi(0)| \geq H$, where H is large enough.

Definition 1.5. *([6]). Let D be an open set in $\mathfrak{R}^n$ with $0 \in D$. A function $V : [0, \infty) \times D \to [0, \infty)$ is called positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$, and is called a decrescent function if there is a wedge W_2 with $V(t, x) \leq W_2(|x|)$.*

Theorem 1.6. *([6]). If there is a Lyapunov functional for (4) and wedges satisfying*

(i) *$W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(\|\varphi\|)$, (where $W_1(r)$ and $W_2(r)$ are wedges),*

(ii) *$\dot{V}(t, \varphi) \leq 0$,*

then the zero solution of Eq. (4) is uniformly stable.

Theorem 1.7. *([52]). Suppose that there exists a continuous Lyapunov functional $V(t, \varphi)$ defined for all $t \in \mathfrak{R}^+$ and $\varphi \in S^\bullet$, which satisfies the following conditions;*

(i) *$a(|\varphi(0)|) \leq V(t, \varphi) \leq b_1(|\varphi(0)|) + b_2(\|\varphi\|)$,*

where $a(r)$, $b_1(r)$, $b_2(r) \in CI$, (CI denotes the families of continuous increasing functions), and are positive for $r > H$ and $a(r) - b_2(r) \to \infty$ as $r \to \infty$,

(ii) *$\dot{V}(t, \varphi) \leq 0$.*

Then, the solutions of Eq. (4) are uniform-bounded.

2. Main results

Let $P(.) \equiv 0$ in Eq. (1).

Our first result is the following theorem.

Theorem 2.1. *We assume that there exist positive constants a_1, a_2, α_i and β_i such that the following conditions hold in Eq. (1):*

(i) *The matrix F is symmetric, $\lambda_i(F(.))) \geq a_1$ for all X, $Y \in \Re^n$, $G(0) = 0$, $G(X) \neq 0$, $(X \neq 0)$, $J_G(X)$ is symmetric, and $\lambda_i(J_G(X)) \geq a_2$ for all $X \in \Re^n$,*

(ii) *$H_i(0) = 0$, $H_i(X) \neq 0$, $(X \neq 0)$, $J_{H_i}(X)$ are symmetric and $\alpha_i \leq \lambda_i(J_{H_i}(X)) \leq \beta_i$ for all $X \in \Re^n$.*

If

$$\tau < \frac{a_1}{\sqrt{n}\sum\limits_{i=1}^{n}\beta_i},$$

then the zero solution of Eq. (1) is asymptotically stable.

Proof. We define a Lyapunov-Krasovskii functional $V(.) = V(X_t, Y_t)$ by

$$\begin{aligned} V(.) \quad &= \quad \sum_{i=1}^{n}\int_0^1 \langle H_i(\sigma X), X\rangle d\sigma + \int_0^1 \langle G(\sigma X), X\rangle d\sigma + \frac{1}{2}\langle Y, Y\rangle \\ &\quad + \sum_{i=1}^{n}\mu_i \int_{-\tau_i}^{0}\int_{t+s}^{t} \|Y(\theta)\|^2 \, d\theta ds, \end{aligned}$$

where s is a real variable such that the integrals $\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t}\|Y(\theta)\|^2 \, d\theta ds$ are non-negative, and μ_i are certain positive constants to be determined later in the proof.

It is clear that $V(0,0) = 0$. Using the estimates $H_i(0) = G(0) = 0$, $\frac{\partial}{\partial\sigma}H_i(\sigma X) = J_{H_i}(\sigma X)X$, $\frac{\partial}{\partial\sigma}G(\sigma X) = J_G(\sigma X)X$, $\lambda_i(J_{H_i}(X)) \geq \alpha_i$ and $\lambda_i(J_G(X)) \geq a_2$, $(i = 1, 2, ..., n)$, we obtain

$$H_i(X) = \int_0^1 J_{H_i}(\sigma X) X d\sigma$$

and

$$G(X) = \int_0^1 J_G(\sigma X) X d\sigma$$

so that

$$\begin{aligned} \int_0^1 \langle H_1(\sigma X), X\rangle d\sigma \quad &= \quad \int_0^1\int_0^1 \langle \sigma_1 J_{H_1}(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 \\ &\geq \quad \int_0^1\int_0^1 \langle \sigma_1\alpha_1 X, X\rangle d\sigma_2 d\sigma_1 \geq \frac{\alpha_1}{2}\|X\|^2, \end{aligned}$$

$$\begin{aligned}\int_0^1 \langle H_2(\sigma X), X\rangle d\sigma &= \int_0^1\int_0^1 \langle \sigma_1 J_{H_2}(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 \\ &\geq \int_0^1\int_0^1 \langle \sigma_1 \alpha_2 X, X\rangle d\sigma_2 d\sigma_1 \geq \frac{\alpha_2}{2}\|X\|^2,\end{aligned}$$

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$$\begin{aligned}\int_0^1 \langle H_n(\sigma X), X\rangle d\sigma &= \int_0^1\int_0^1 \langle \sigma_1 J_{H_n}(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 \\ &\geq \int_0^1\int_0^1 \langle \sigma_1 \alpha_n X, X\rangle d\sigma_2 d\sigma_1 \geq \frac{\alpha_n}{2}\|X\|^2,\end{aligned}$$

$$\begin{aligned}\int_0^1 \langle G(\sigma X), X\rangle d\sigma &= \int_0^1\int_0^1 \langle \sigma_1 J_{G}(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 \\ &\geq \int_0^1\int_0^1 \langle \sigma_1 a_2 X, X\rangle d\sigma_2 d\sigma_1 \geq \frac{a_2}{2}\|X\|^2.\end{aligned}$$

On gathering all these estimates into $V(.) = V(X_t, Y_t)$, we deduce that

$$\begin{aligned}V(.) &\geq \frac{1}{2}(\sum_{i=1}^{n}\alpha_i)\|X\|^2 + a_2\|X\|^2 + \frac{1}{2}\|Y\|^2 + \sum_{i=1}^{n}\mu_i \int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds \\ &\geq D_1(\|X\|^2 + \|Y\|^2) + \sum_{i=1}^{n}\lambda_i \int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds,\end{aligned}$$

where $D_1 = \min\{\frac{1}{2}(\sum\limits_{i=1}^{n}\alpha_i) + a_2,\ \frac{1}{2}\}$.

Hence, we can find a continuous function $u(s)$ such that

$$u(\|\phi(0)\|) \leq V(\phi), u(\|\phi(0)\|) \geq 0.$$

Using a basic calculation, the time derivative of the functional $V(.)$ along the solutions of (2) yields

$$\begin{aligned}\dot{V}(.) &= -\langle F(X,Y)Y, Y\rangle + <\sum_{i=1}^{n}\int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds, Y> \\ &+ < \sum_{i=1}^{n}(\mu_i\tau_i)Y, Y> - \sum_{i=1}^{n}\mu_i\int_{t-\tau_i}^{t}\|Y(\theta)\|^2 d\theta.\end{aligned}$$

Using the assumptions $\lambda_i(F(X,Y)) \geq a_1$, $\lambda_i(J_{H_i}(X)) \leq \beta_i$ and the estimate $2|a|\ |b| \leq a^2 + b^2$ (with a and b are real numbers) combined with the classical Cauchy-Schwartz inequality, it follows that

$$-\ \langle F(X,Y)Y,Y\rangle \leq -a_1 \|Y\|^2 \,,$$

$$\begin{aligned}
&< \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds, Y > \leq \|Y\| \left\| \int_{t-\tau_i}^{t} J_{H_i}(X(s))Y(s)ds \right\| \\
&\leq \sqrt{n}\beta_i \|Y\| \left\| \int_{t-\tau_i}^{t} Y(s) \right\| ds \\
&\leq \sqrt{n}\beta_i \|Y\| \int_{t-\tau_i}^{t} \|Y(s)\| \, ds \\
&\leq \frac{1}{2}\sqrt{n}\beta_i \int_{t-\tau_i}^{t} (\|Y(t)\|^2 + \|Y(s)\|^2) ds \\
&\leq \frac{1}{2}\sqrt{n}\beta_i\tau_i \|Y\|^2 \ + \frac{1}{2}\sqrt{n}\beta_i \int_{t-\tau_i}^{t} \|Y(s)\|^2 \, ds
\end{aligned}$$

so that

$$\begin{aligned}
\dot{V}(.) \quad &\leq \quad -a_1 \|Y\|^2 + (\sum_{i=1}^{n} \mu_i\tau_i) \|Y\|^2 + \frac{1}{2}(\sqrt{n} \sum_{i=1}^{n} \beta_i\tau_i) \|Y\|^2 \\
&\quad - \sum_{i=1}^{n} (\mu_i - \frac{1}{2}\sqrt{n}\beta_i) \int_{t-\tau_i}^{t} \|Y(s)\|^2 \, ds.
\end{aligned}$$

Let

$$\mu_i = \frac{1}{2}\sqrt{n}\beta_i.$$

Then

$$\dot{V}(.) \leq \ \{-a_1 + \ \sqrt{n} \sum_{i=1}^{n} (\beta_i\tau_i)\} \|Y\|^2 \,.$$

Let $\tau = \max \tau_i$. Hence, we get

$$\dot{V}(.) \leq \{-a_1 + \ \sqrt{n} \sum_{i=1}^{n} \beta_i\tau\} \|Y\|^2 \,.$$

If $\tau < \frac{a_1}{\sqrt{n}\sum_{i=1}^{n}\beta_i}$, then we have for some positive constant k that

$$\dot{V}(.) \leq -\ k \|Y\|^2 \leq 0.$$

We also observe from the previous estimate of $\dot{V}(.)$ that $\dot{V}(.) = 0 \Rightarrow Y = 0$ and $X = \xi$ (a constant vector) for all $t \geq 0$. That is, we get

$$X = \xi, Y = 0, (t \geq 0).$$

Substituting foregoing estimates in system (2), we have $H_i(\xi) = 0$, which necessarily implies that $\xi = 0$ since $H_i(0) = 0$ and $H_i(\xi) \neq 0$, when $\xi \neq 0$. Thus, we obtain

$$X = Y = 0 \text{ for all } t \geq 0.$$

The last estimate shows that the largest invariant setin Z is $Q = \{0\}$, where $Z = \{\phi \in C_H : \dot{V}(\phi) = 0\}$. Further, it can be seen that the only solution of Eq. (1) for which $\dot{V}(.) \equiv 0$ is the solution $X \equiv 0$. Thus, subject to the above discussion, if $\tau < \frac{a_1}{\sqrt{n}\sum_{i=1}^{n}\beta_i}$, then we can conclude that the zero solution of Eq. (1) is asymptotically stable.

The proof of Theorem 2.1 is completed. □

Corollary 2.2. *Let us assume that the assumptions of Theorem 2.1 hold. Then, the zero solution of Eq. (1) is uniformly stable.*

Let $P(.) \neq 0$ in Eq. (1).

Our second result is the following theorem

Theorem 2.3. *Let all the assumptions of Theorem 2.1 and the assumption*

$$\|P(t)\| \leq Q(t)$$

hold, where $Q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of integrable Lebesgue functions.

If

$$\tau < \frac{a_1}{\sqrt{n}\sum_{i=1}^{n}\beta_i},$$

then there exists a positive constant K such that the solution X of Eq. (1) defined by the initial function

$$X(t) = \psi(t), X'(t) = \psi'(t), t_0 - \tau \leq t \leq t_0,$$

satisfies the estimates

$$\|X(t)\| \leq K, \|X'(t)\| \leq K$$

for all $t \geq t_0$, where $\psi \in C^1([t_0 - \tau, \ t_0], \Re)$.

Proof. We reconsider Lyapunov-Krasovskii functional, which is defined above. Then, using the assumptions of Theorem 2.3, it can be easily seen that

$$V(.) \geq D_1(\|X\|^2 + \|Y\|^2) \geq D_1 \|Y\|^2 .$$

Since $P(.) \neq 0$, the time derivative of $V(.)$ can be revised as the following:

$$\begin{aligned} \dot{V}(.) &\leq -\alpha \|Y\|^2 + \langle Y, P(t)\rangle \\ &\leq \|Y\| \ \|P(t)\| \\ &\leq \|Y\| \ Q(t). \end{aligned}$$

Using the estimate

$$\|Y\| \leq 1 + \|Y\|^2 ,$$

it follows that

$$\begin{aligned} V(.) &\leq Q(t) + Q(t)\,\|Y\|^2 \\ &\leq Q(t) + D_2 Q(t) V(.), \end{aligned}$$

where $D_2 = D_1^{-1}$.

Integrating the last estimate from 0 to t, $(t \geq 0)$, we obtain

$$V(X_t, Y_t) - V(X_0, Y_0) \leq \int_0^t Q(s)ds + D_2 \int_0^t V(X_s, Y_s)Q(s)ds.$$

Let $D_3 = V(X_0, Y_0) + \int_0^t Q(s)ds$. Using Gronwall-Bellman inequality, we get

$$V(X_t, Y_t) \leq D_3 \exp(D_2 \int_0^t Q(s)ds).$$

In view of the above discussion, it follows that

$$D_1(\|X\|^2 + \|Y\|^2) \leq V(X_t, Y_t) \leq D_3 \exp(D_2 \int_0^t Q(s)ds).$$

Let $\exp(D_2 \int_0^\infty Q(s)ds) = K_1(> 0)$, since $Q \in L^1(0, \infty)$.

Hence

$$\|X\|^2 + \|Y\|^2 \leq K_2,$$

where $K_2 = D_1^{-1} D_3 \exp(D_2 K)$.

The proof of Theorem 2.3 is completed. □

Corollary 2.4. *Let us assume that the assumptions of Theorem 2.3 hold. Then all solutions of Eq. (1) are uniform-bounded.*

Example 2.5. *As a special case of Eq. (1) for $n = 2$, we choose*

$$F(X, X') = \begin{bmatrix} 4 + \frac{x_1^2 + x_1'^2}{1+\exp(x_1^2 + x_1'^2)} & 0 \\ 0 & 4 + \frac{x_1^2 + x_1'^2}{1+\exp(x_1^2 + x_1'^2)} \end{bmatrix},$$

$$G(X) = \begin{bmatrix} x_1 + arctgx_1 \\ x_2 + arctgx_2 \end{bmatrix},$$

$$H_1(X(t - \tau_1)) = \begin{bmatrix} x_1(t - \tau_1) + arctgx_1(t - \tau_1) \\ x_2(t - \tau_1) + arctgx_2(t - \tau_1) \end{bmatrix},$$

$$H_2(X(t - \tau_2)) = \begin{bmatrix} x_1(t - \tau_2) + arctgx_1(t - \tau_2) \\ x_2(t - \tau_1) + arctgx_2(t - \tau_2) \end{bmatrix}$$

and

$$P(t) = \begin{bmatrix} \frac{\sin t}{1+t^2} \\ \\ \frac{\cos t}{1+t^2} \end{bmatrix}.$$

It follows that

$$\lambda_1(F(.)) = 4 + \frac{x_1^2 + x_1'^2}{1 + \exp(x_1^2 + x_1'^2)},$$

$$\lambda_2(F(.)) = 4 + \frac{x_1^2 + x_1'^2}{1 + \exp(x_1^2 + x_1'^2)},$$

$$\lambda_i(F(.)) \geq 4 = a_1 > 0,$$

$$J_G(X) = \begin{bmatrix} 1 + (1 + x_1^2)^{-1} & 0 \\ 0 & 1 + (1 + x_2^2)^{-1} \end{bmatrix},$$

$$\lambda_i(J_G(.)) \geq 1 = a_2 > 0,$$

$$J_{H_1}(X) = \begin{bmatrix} 1 + (1 + x_1^2(t - \tau_1))^{-1} & 0 \\ \\ 0 & 1 + (1 + x_2^2(t - \tau_1))^{-1} \end{bmatrix},$$

$$J_{H_2}(X) = \begin{bmatrix} 1 + (1 + x_1^2(t - \tau_2))^{-1} & 0 \\ \\ 0 & 1 + (1 + x_2^2(t - \tau_2))^{-1} \end{bmatrix},$$

$$1 \leq \lambda_i(J_{H_i}(.)) \leq 2,$$

$$\|P(t)\| = \left\| \begin{matrix} \frac{\sin t}{1+t^2} \\ \\ \frac{\cos t}{1+t^2} \end{matrix} \right\| \leq \frac{2}{1+t^2} = Q(t)$$

and

$$\int_0^\infty Q(s)ds = 2\int_0^\infty \frac{1}{1+s^2}ds = \pi,$$

that is, $Q \in L^1(0, \infty)$.
Thus, it is necessary as remark that all the conditions of Theorem 2.1 and Theorem 2.3 hold, whenever $\tau < \frac{1}{\sqrt{2}}$.

3. Conclusion

A vector Liénard equation with multiple constant deviating arguments is considered. The stability and boundedness of solutions of this equation is discussed. In proving our results, we employ the Lyapunov-Krasovskii functional approach by defining a new Lyapunov- Krasovskii functional. An example is also constructed to illustrate our theoretical findings.

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