Filomat 27:3 (2013), 499–513
DOI 10.2298/FIL1303499D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Exact number of positive solutions for quasilinear boundary value problems with $p-$convex nonlinearity

Mohammed Derhab[a], Hafidha Sebbagh[b]

[a]Department of Mathematics, Faculty of Sciences, University Abou-Bekr Belkaid Tlemcen, B.P.119, Tlemcen, 13000, Algeria
[b]Department of Mathematics, Faculty of Sciences, University Abou-Bekr Belkaid Tlemcen, B.P.119, Tlemcen, 13000, Algeria,

Abstract. By using the quadrature method, we study the exact number of positive solutions of the following quasilinear boundary value problem :

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' &= \lambda f(u) \text{ in } (0,1)\,, \\ u &> 0 \text{ in } (0,1)\,, \\ u(0) = u(1) &= 0, \end{cases}$$

where $\varphi_p(y) = |y|^{p-2}\,y$, $y \in \mathbb{R}$, $p > 1$, $\lambda > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class C^2 and p-convex function.

1. Introduction

The purpose of this work is to study the exact number of positive solutions of the following quasilinear boundary value problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' &= \lambda f(u) \text{ in } (0,1)\,, \\ u &> 0 \text{ in } (0,1)\,, \\ u(0) = u(1) &= 0, \end{cases} \tag{1}$$

where $\varphi_p(y) = |y|^{p-2}\,y$, $y \in \mathbb{R}$, $p > 1$, $\lambda > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class C^2 and p-convex function.

Boundary value problems with convex nonlinearity or p-convex nonlinearity have been studied by several authors. Let us recall some of them.

In [15], the author consider the following problem

$$\begin{cases} -u'' &= \lambda f(u) \text{ in } (0,1)\,, \\ u &> 0 \text{ in } (0,1)\,, \\ u(0) = u(1) &= 0. \end{cases} \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 34B15-34C10.
Keywords. $p-$Laplacian; positive solutions; quadrature method; $p-$convex nonlinearity.
Received: 30 November 2012; Revised: 13 February, 2013; Accepted: 03 March 2013
Communicated by Eberhard Malkowsky
Research partially supported by a MESRS-DRS grant B02020100075 and PNR project N^0 43/28/2011
Email addresses: derhab@yahoo.fr (Mohammed Derhab), hfsb87@hotmail.com (Hafidha Sebbagh)

By using the quadrature method, he proved when f is convex, with $f(0) > 0$, then (2) has for each $\lambda > 0$, either zero, one, or two positive solutions on the interval $(0, 1)$. For such f, the authors determined a number $\mu_1 \geq 0$ and proved the existence of a number $\lambda^* \geq \mu_1$ such that

(i) If $\mu_1 = 0$, then $\lambda^* > 0$ and (2) has two solutions for $0 < \lambda < \lambda^*$, one for $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$;

(ii) If $0 < \mu_1 < \lambda^*$, then (2) has one solution for $0 < \lambda \leq \mu_1$, and $\lambda = \lambda^*$, two for $\mu_1 < \lambda < \lambda^*$, and none for $\lambda > \lambda^*$;

(iii) If $0 < \mu_1 = \lambda^* < +\infty$, then (2) has one solution for $0 < \lambda < \lambda^*$ and none for $\lambda > \lambda^*$;

(iv) If $\mu_1 = +\infty$, then $\lambda^* = +\infty$ and (2) has solutions for all $\lambda > 0$.

The author in [15] gave also a necessary and sufficient condition for $\lambda^* > \mu_1$.

In [11], the authors studied the following boundary value problem

$$\begin{cases} -u'' &= f(u) \text{ in } (-R, R), \\ u &> 0 \text{ in } (-R, R), \\ u(-R) = u(R) &= 0, \end{cases} \tag{3}$$

where f is a strictly convex C^2 function on $[0, +\infty)$ and $R > 0$.

By using the time-map technique (see e.g. [17]), the authors proved that if $\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty$, then we have the following results

(i) If $f(u) > 0$ $(u \in [0, +\infty))$, then there exists $R_{\sup} > 0$ such that (3) has two solutions for $R < R_{\sup}$, one solution for $R = R_{\sup}$, and none for $R > R_{\sup}$;

(ii) If $f(0) = 0$ and $f'(0) > 0$, then there exists $R_{\sup} > 0$ such that (3) has one solution for $R < R_{\sup}$ and none for $R \geq R_{\sup}$;

(iii) If $f(0) < 0$, then there exists $R_{\sup} > 0$ such that (3) has one solution for $R \leq R_{\sup}$ and none for $R \geq R_{\sup}$.

The authors studied also the case when $\lim_{u \to +\infty} \frac{f(u)}{u} = L$, with $L > 0$.

In [12], the authors investigated the exact number of positive solutions for the following quasilinear boundary value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' &= f(u) \text{ in } (-R, R), \\ u &> 0 \text{ in } (-R, R), \\ u(-R) = u(R) &= 0, \end{cases} \tag{4}$$

where $\varphi_p(y) = |y|^{p-2} y$, $y \in \mathbb{R}$, $p > 2$ and f is a C^1 function on $[0, +\infty)$ whose roots are isolated and $p-$convex.

By using the time-map technique, the authors proved that if f is strictly $p-$convex and $\lim_{u \to +\infty} \frac{f(u)}{\varphi_p(u)} = +\infty$, then we have the following results

(i) If $f(0) = 0$ and $\lim_{u \to 0} \frac{f(u)}{\varphi_p(u)} = 0$, then for all $R > 0$ problem (4) has a unique solution,

(ii) If $f(0) = 0$ and $\lim_{u \to 0} \frac{f(u)}{\varphi_p(u)} = m \in (0, +\infty)$, then there exists $R_0 > 0$ such that (4) has a unique solution for $R < R_0$ and none for $R \geq R_0$;

(iii) If $f(0) = 0$ and $f'(0) < 0$ or $f(0) < 0$, then there exists $R_1 > 0$ such that (4) has a unique solution for $R < R_1$ and none for $R \geq R_1$.

The authors in [12] gave also a classification of positive solutions when the nonlinearity f is strictly $p-$convex, $f(0) > 0$ and has one root or two roots and $\lim_{u\to+\infty} \frac{f(u)}{\varphi_p(u)} = +\infty$.

One of the methods used to study problems of type (1) is the time- map approach. This method is used to study uniqueness and multiplicity of solutions for boundary value problems of type (1).

We note that if the time map is monotone, problems of type (1) has at most one solution and if the time map is convex, or concave, then the problem (1) has at most two solutions (see [1], [11], [14], [17] and [19]).

In this work we prove that if the nonlinearity f is super p-linear at 0 and at $+\infty$ and p-convex, then the time map is concave. Our results improve and generalize those obtained in the literature.

The paper is organized as follows: In section 2, we state our main result. In section 3, we state the method used to prove our main result. Some preliminary lemmas are the aim of section 4. Section 5 is devoted to the proof of our main result. Finally in section 6, we give an example

2. Main result

We consider the following boundary value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' &= \lambda f(u) \text{ in } (0,1), \\ u &> 0 \text{ in } (0,1), \\ u(0) = u(1) &= 0, \end{cases} \tag{5}$$

where $\varphi_p(y) = |y|^{p-2} y$, $y \in \mathbb{R}$, $p > 1$, $\lambda > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following hypothesis

(**H1**) $\lim_{u\to 0^+} \frac{f(u)}{\varphi_p(u)} = \lim_{u\to+\infty} \frac{f(u)}{\varphi_p(u)} = +\infty$,

(**H2**) $f'(u) > 0$, for all $u > 0$,

(**H3**) $f''(u) > 0$, for all $u > 0$,

(**H4**) $\lim_{u\to+\infty} ((p-1)f'(u) - uf'(u)) < 0$,

(**H5**) $(p-2)f'(u) - uf''(u) < 0$, for all $u > 0$ and $p > 2$.

(**H6**) $\lim_{u\to+\infty} (pF(u) - uf(u)) < 0$, where $F(u) = \int_0^u f(s)\,ds$, for all $u > 0$.

Remark 2.1. *We note that if* $1 < p \leq 2$, *then by (**H2**) and (**H3**), we have* $(p-2)f'(u) - uf''(u) < 0$, *for all* $u > 0$.

To state our result, define

$$S^+ = \left\{u \in C^1([0,1]); u > 0 \text{ in } (0,1),\ u(0) = u(1) = 0 \text{ and } u'(0) > 0\right\}.$$

Let A^+ be the subset of S^+ composed by the functions u satisfying:

- u is symmetrical about $\frac{1}{2}$.
- The derivative of u vanishes once and only once in $(0,1)$.

The main result of this work is

Theorem 2.2. *Assume that* $p = 2$ *or* $p \geq 4$, $\lambda > 0$ *and* f *satisfies the hypothesis (**H1**)- (**H6**), then there exists* $\lambda_* > 0$ *such that*

i) If $\lambda > \lambda_*$, *then the problem (5)admits no positive solution,*

ii) If $\lambda = \lambda_*$, *then the problem (5) admits a unique positive solution and it belongs to* A^+,

iii) If $\lambda < \lambda_*$, *then the problem (5) admits exactly two positive solutions and they belong to* A^+.

3. Time-mapping approach

In this section we introduce the well-know time mapping approach (see for instance [2]-[10], [15]-[16] and the monograph [14] for further developments of this topic).

Consider the following boundary value problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = g\left(u\right) \text{ in } (0,1), \\ u\left(0\right) = u\left(1\right) = 0, \end{cases} \tag{6}$$

where $g \in C\left(\mathbb{R}_+, \mathbb{R}\right)$.

Define $G\left(s\right) := \int_0^s g\left(t\right) dt$.

For any $E \geq 0$ and $p > 1$, let

$$X_+\left(p, E\right) = \left\{s > 0;\ E^p - \frac{p}{p-1} G\left(\zeta\right) > 0,\ \forall\, \zeta,\ 0 < \zeta < s\right\},$$

$$S_+\left(p, E\right) = \begin{cases} 0 \text{ if } X_+\left(p, E\right) = \varnothing, \\ \operatorname{Sup} X_+\left(p, E\right) \text{ otherwise,} \end{cases}$$

$$D = \left\{E \geq 0;\ 0 < S_+\left(p, E\right) < +\infty \text{ and } g\left(S_+\left(p, E\right)\right) > 0\right\},$$

and we define the following time-map

$$T_+\left(p, E\right) = \int_0^{S_+(p,E)} \left[E^p - \frac{p}{p-1} G\left(u\right)\right]^{-\frac{1}{p}} du.$$

We now state the following well know theorem without proof (See for instance [10]).

Theorem 3.1. *Assume that $g \in C\left(\mathbb{R}_+, \mathbb{R}\right)$, $E \geq 0$ and $p > 1$. Then the problem (6) admits a solution $u \in A^+$ satisfying $u'\left(0\right) = E$ if and only if $E \in D \cap (0, +\infty)$ and $T_+\left(p, E\right) = \frac{1}{2}$. In this case the solution is unique and its sup-norm is equal to $S_+\left(p, E\right)$.*

4. Preliminary lemmas

Lemma 4.1. *Consider the equation in $s \in \mathbb{R}_+$*

$$E^p - \frac{p}{p-1} \lambda F(s) = 0, \tag{7}$$

where $E > 0$, $p > 1$ and $\lambda > 0$.

Then for any $E > 0$, equation (7) admits a unique positive zero $r_+\left(p, \lambda, E\right)$. Moreover

(i) The function $E \mapsto r_+\left(p, \lambda, E\right)$ is C^1 in $(0, +\infty)$ and

$$\frac{\partial r_+}{\partial E}\left(p, \lambda, E\right) = \frac{\left(p-1\right) E^{p-1}}{\lambda f(r_+\left(p, \lambda, E\right))} > 0,\ \forall p > 1,\ \forall \lambda > 0 \text{ and } \forall E > 0.$$

(ii) $\lim_{E \to 0^+} r_+ (p, \lambda, E) = 0.$

(iii) $\lim_{E \to +\infty} r_+ (p, \lambda, E) = +\infty.$

Proof. The proof of this lemma is similar to that of Lemma 4.1 in [4] or Lemma 8 in [2]. So, it is omitted. □

Now, we are ready, for any $p > 1$, $\lambda > 0$ and $E > 0$, to compute $X_+ (p, \lambda, E)$ as defined in section 3. In fact

$$X_+ (p, \lambda, E) =]0, r_+ (p, \lambda, E)[.$$

Then

$$S_+ (p, \lambda, E) = r_+ (p, \lambda, E) .$$

On other hand, we have

$$\begin{aligned} D &= \{E > 0,\ 0 < S_+ (p, \lambda, E) < +\infty \text{ and } f (S_+ (p, \lambda, E)) > 0\} \\ &=]0, +\infty[. \end{aligned}$$

By lemma 4.1, we have

$$\lim_{E \to 0^+} S_+ (p, \lambda, E) = 0, \tag{8}$$

$$\lim_{E \to +\infty} S_+ (p, \lambda, E) = +\infty, \tag{9}$$

and

$$\frac{\partial S_+}{\partial E} (p, \lambda, E) = \frac{(p-1) E^{p-1}}{\lambda f (S_+ (p, \lambda, E))} > 0,\ \forall p > 1,\ \forall \lambda > 0 \text{ and } \forall E > 0.$$

At present, we define, for any $p > 1$, $\lambda > 0$ and $E > 0$, the time-map T_+ by

$$T_+ (p, \lambda, E) = \int_0^{S_+(p,\lambda,E)} \left[E^p - \frac{p}{p-1} \lambda F(s) \right]^{-\frac{1}{p}} du. \tag{10}$$

Now if we put the change of variables $u = S_+ (p, \lambda, E)\, t$ in (10), we obtain that

$$T_+ (p, \lambda, E) = \left(\frac{p}{p-1} \lambda \right)^{\frac{-1}{p}} \int_0^1 [F(s_+(p, \lambda, E)) - F(s_+(p, \lambda, E)t)]^{-\frac{1}{p}} dt.$$

We observe that

$$T_+ (p, \lambda, E) = G (p, \lambda, S_+ (p, \lambda, E)) ,\ \text{for all } p > 1,\ \lambda > 0 \text{ and } E > 0,$$

where

$$G (p, \lambda, \rho) = \left(\frac{p}{p-1} \lambda \right)^{\frac{-1}{p}} \int_0^1 [F(\rho) - F(\rho t)]^{-\frac{1}{p}} dt, \tag{11}$$

for all $p > 1$, $\lambda > 0$ and $\rho > 0$.

Since the function $E \mapsto S_+(p,\lambda,E)$ is an increasing C^1-diffeomorphism from $(0,+\infty)$ onto itself it follows that if we put, for all $p > 1$ and $\lambda > 0$,

$$J_1(p,\lambda) := \left\{E > 0 : T_+(p,\lambda,E) = \frac{1}{2}\right\},$$

and

$$J_2(p,\lambda) := \left\{\rho > 0 : G(p,\lambda,\rho) = \frac{1}{2}\right\},$$

then

$$Card(J_1(p,\lambda)) = Card(J_2(p,\lambda)), \text{ for all } p > 1 \text{ and } \lambda > 0.$$

Hence, from now, we will focus our attention in counting the number of solutions of the equation $G(p,\lambda,\rho) = \frac{1}{2}$ in the variable $\rho > 0$, instead of the equation $T_+(p,\lambda,E) = \frac{1}{2}$ in the variable $E > 0$.

Proposition 4.2. *If u is a positive solution of problem (5), then $u \in A^+$.*

Proof. The proof is similar to that of Proposition 4.2 in [7]. So, it is omitted. □

Lemma 4.3. *For any $p > 1$ and $\lambda > 0$, we have*

(i) $\lim_{\rho\to 0^+} G(p,\lambda,\rho) = 0.$

(ii) $\lim_{\rho\to +\infty} G(p,\lambda,\rho) = 0.$

Proof. Let $p > 1$ and $\lambda > 0$ be fixed.

(i) By assumption (**H1**), we have for all $A_1 > 0$, there exists $A_2 > 0$ such that

$$f(u) > A_1\varphi_p(u), \text{ for all } 0 < u < A_2.$$

Let $t \in (0,1)$ and $\rho > 0$, we have

$$\begin{aligned}
\frac{F(\rho)-F(\rho t)}{\rho^p} &= \frac{\int_{\rho t}^{\rho} f(u)du}{\rho^p} \\
&= \int_t^1 \frac{f(\rho\tau)}{\varphi_p(\rho)}d\tau \\
&= \int_t^1 \frac{f(\rho\tau)}{\varphi_p(\rho\tau)}\tau^{p-1}d\tau \\
&> A_1\int_t^1 \tau^{p-1}d\tau, \text{ for all } 0 < \rho < A_2 \\
&= \frac{A_1}{p}(1-t^p).
\end{aligned}$$

Therefore for all $0 < \rho < A_2$, we have

$$\frac{F(\rho)-F(\rho t)}{\rho^p} > \frac{A_1}{p}(1-t^p). \tag{12}$$

By (11) and (12), one has

$$\begin{aligned} G(p,\lambda,\rho) &\le \left(\frac{p}{p-1}\lambda\right)^{\frac{-1}{p}} \int_0^1 \left[\frac{A_1}{p}(1-t^p)\right]^{-\frac{1}{p}} dt, \text{ for all } 0<\rho<A_2 \\ &= A_1^{-\frac{1}{p}} \lambda^{-\frac{1}{p}} \frac{(p-1)^{\frac{1}{p}}}{p} \beta\left(1-\frac{1}{p}, \frac{1}{p}\right), \text{ for all } 0<\rho<A_2, \end{aligned}$$

where $\beta(.,.)$ is the Euler Beta function defined by

$$\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \text{ for all } x>0 \text{ and } y>0.$$

Then for all $A_1 > 0$, there exists $A_2 > 0$ such that

$$G(p,\lambda,\rho) \le A_1^{-\frac{1}{p}} \lambda^{-\frac{1}{p}} \frac{(p-1)^{\frac{1}{p}}}{p} \beta\left(1-\frac{1}{p}, \frac{1}{p}\right), \text{ for all } 0<\rho<A_2.$$

Which means that

$$\lim_{\rho\to 0^+} G(p,\lambda,\rho) = 0.$$

(ii) The proof is similar to that of (i). So, it is omitted.
□

Lemma 4.4. *For any $p > 1$ and $\lambda > 0$, we have*

(i) There exists $\rho_1 > 0$ such that the function $\rho \mapsto G(p,\lambda,\rho)$ is strictly increasing on $(0,\rho_1)$.

(ii) There exists $\rho_2 > \rho_1$ such that he function $\rho \mapsto G(p,\lambda,\rho)$ is strictly decreasing on $(\rho_2,+\infty)$.

Proof. Let $p > 1$ and $\lambda > 0$ be fixed.
Differentiating (11) with respect to ρ, we obtain

$$\frac{\partial G}{\partial \rho}(p,\lambda,\rho) = \frac{1}{p\rho}\left(\frac{\lambda p}{p-1}\right)^{-\frac{1}{p}} \int_0^\rho \frac{H(\rho)-H(u)}{[F(\rho)-F(u)]^{\frac{p+1}{p}}} du, \tag{13}$$

where

$$H(u) = pF(u) - uf(u).$$

We have

$$H'(u) = (p-1) f(u) - uf'(u),$$

and

$$H''(u) = (p-2) f'(u) - uf''(u).$$

Since f satisfies (**H3**), (**H4**), (**H5)** and (**H6**) and by Remark 2.1, it follows that there exists two real numbers ρ_1 and ρ_2 with $0 < \rho_1 < \rho_2$ such that

$$H'(u) > 0 \text{ on } (0,\rho_1), H'(\rho_1) = 0 \text{ and } H'(u) < 0 \text{ on } (\rho_1,+\infty),$$

and

$$H(u) > 0 \text{ on } (0,\rho_2), H(\rho_2) = 0 \text{ and } H(u) < 0 \text{ on } (\rho_2,+\infty).$$

So, it follows that

$$\frac{\partial G}{\partial \rho}(p,\lambda,\rho) > 0 \text{ for all } p > 1,\ \lambda > 0 \text{ and } \rho \in (\rho_1,+\infty),$$

and

$$\frac{\partial G}{\partial \rho}(p,\lambda,\rho) < 0 \text{ for all } p > 1,\ \lambda > 0 \text{ and } \rho \in (\rho_2,+\infty).$$

Which means that for all $p > 1$ and $\lambda > 0$, the function $\rho \mapsto G(p,\lambda,\rho)$ is strictly increasing on $(0,\rho_1)$ and strictly decreasing on $(\rho_2,+\infty)$. □

Now we are going to prove that the time map $\rho \mapsto G(p,\lambda,\rho)$ admits a unique critical point on (ρ_1,ρ_2). For this, we consider the following Cauchy problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda f(u) \\ u\left(\frac{1}{2}\right) = \rho,\ u'\left(\frac{1}{2}\right) = 0, \end{cases} \tag{14}$$

where $\rho > 0$.

We note that since each positive solution of the problem (5) belongs to A^+, then we can define the time-map $\rho \mapsto G(p,\lambda,\rho)$ by the following implicit equation

$$u(G(\rho),\lambda,\rho) = 0 \text{ and } u(x,\lambda,\rho) > 0, \text{ for all } x \in \left[\frac{1}{2}, G(\rho)\right),$$

where $x \mapsto u(x,\lambda,\rho)$ is a positive solution of the Cauchy problem (14) and $G(\rho) := G(p,\lambda,\rho)$.

Now if we put by definition $v(x) := \frac{\partial u}{\partial \rho}(x,\lambda,\rho)$, $\omega(x) := \frac{\partial^2 u}{\partial \rho^2}(x,\lambda,\rho)$ and since $\left(\varphi_p(u')\right)' = (p-1)|u'|^{p-2}u''$ for all $p \geq 2$, then if $p = 2$ or $p \geq 3$, we have

$$\begin{cases} -(p-1)(|u'|^{p-2}v')' &= \lambda f'(u)v, \\ v\left(\frac{1}{2}\right) = 1,\ v'(\frac{1}{2}) &= 0, \end{cases}$$

and if $p = 2$ or $p \geq 4$, we have

$$\begin{cases} (p-1)(|u'|^{p-2}\omega')' + (p-1)(p-2)(|u'|^{p-4}u'v'^2)' + \lambda f''(u)v^2 + \lambda f'(u)\omega = 0, \\ \omega\left(\frac{1}{2}\right) = 0,\ \omega'(\frac{1}{2}) = 0 \end{cases}$$

We have the following results

Proposition 4.5. *Assume that $p = 2$ or $p \geq 3$ and f satisfies the hypothesis **(H1)-(H6)**, then v has at most one root on $[0, G(\rho)]$.*

Proof. The proof is similar to that of Proposition 11 in [12]. So, it is omitted. □

Proposition 4.6. *Assume that $p = 2$ or $p \geq 4$, f satisfies the hypothesis **(H1)-(H6)** and suppose that $v(G(\rho)) = 0$, then $\omega(G(\rho)) < 0$.*

Proof. We consider the following problem

$$\begin{cases} (p-1)(|u'|^{p-2}\omega')' \\ +(p-1)(p-2)(|u'|^{p-4}u'v'^{2})' + \lambda f''(u)v^2 + \lambda f'(u)\omega = 0, \\ \omega\left(\frac{1}{2}\right) = 0,\ \omega'(\frac{1}{2}) = 0. \end{cases} \tag{15}$$

Multiplying the equation in (15) by u' and integrating the resulting equation over $(\frac{1}{2}, x)$, we obtain

$$\begin{aligned} J \quad : \quad &= (p-1)\int_{\frac{1}{2}}^{x}(|u'|^{p-2}\omega')'u'(t)dt + (p-1)(p-2)\int_{\frac{1}{2}}^{x}(|u'|^{p-4}u'v'^{2})'u'(t)dt \\ &\quad +\lambda\int_{\frac{1}{2}}^{x} f''(u)v^2(t)\,u'(t)dt + \lambda\int_{\frac{1}{2}}^{x} f'(u)u'(t)\omega(t)\,dt \\ &= (p-1)I_1 + (p-1)(p-2)I_2 + I_3 + I_4, \end{aligned}$$

where

$$I_1 = \int_{\frac{1}{2}}^{x}(|u'|^{p-2}\omega')'u'(t)dt,$$
$$I_2 = \int_{\frac{1}{2}}^{x}(|u'|^{p-4}u'v'^{2})'u'(t)dt,$$
$$I_3 = \int_{\frac{1}{2}}^{x} f''(u)v^2(t)\,u'(t)dt,$$

and

$$I_4 = \int_{\frac{1}{2}}^{x} f'(u)u'(t)\omega(t)\,dt.$$

Easy computations shows that

$$\begin{aligned} I_1 &= |u'(x)|^{p-2}u'(x)\omega'(x) - \tfrac{1}{p-1}\int_{\frac{1}{2}}^{x}(|u'(t)|^{p-2}u'(t))'\omega'(t)dt \\ &= |u'(x)|^{p-2}u'(x)\omega'(x) + \frac{\lambda}{p-1}\int_{\frac{1}{2}}^{x} f(u(t))\,\omega'(t)dt, \\ I_2 &= |u'(x)|^{p-2}v'^2(x) - \int_{\frac{1}{2}}^{x}|u'(t)|^{p-4}u'(t)u''(t)v'^2(t)dt \\ &= |u'(x)|^{p-2}v'^2(x) - \frac{1}{p-2}\int_{\frac{1}{2}}^{x}(|u'(t)|^{p-2})'v'^2(t)dt, \\ I_3 &= \lambda v^2(x)f'(u(x)) - \lambda f'(\rho) - 2\lambda\int_{\frac{1}{2}}^{x} f'(u(t))v(t)v'(t)dt \\ &= \lambda v^2(x)f'(u(x)) - \lambda f'(\rho) + 2(p-1)\int_{\frac{1}{2}}^{x}(|u'(t)|^{p-2}v'(t))'v'(t)dt \\ &= \lambda v^2(x)f'(u(x)) - \lambda f'(\rho) + 2(p-1)\int_{\frac{1}{2}}^{x}\left[(|u'(t)|^{p-2})'v'^2(t) + |u'(t)|^{p-2}v''(t)\,v'(t)\right]dt, \end{aligned}$$

and

$$I_4 = \lambda f(u(x))\omega(x) - \lambda \int_{\frac{1}{2}}^{x} f(u(t))\omega'(t)dt.$$

Then, we have

$$\begin{aligned} J = & (p-1)|u'(x)|^{p-2} u'(x)\omega'(x) + (p-1)(p-2)|u'(x)|^{p-2} v'^2(x) + \lambda v^2(x) f'(u(x)) - \lambda f'(\rho) \\ & + \lambda f(u(x))\omega(x) + (p-1)|u'(x)|^{p-2} v'^2(x). \end{aligned}$$

Since $J = 0$ and using the previous equality, we obtain

$$\tilde{E}(x) = \overline{E}(x), \tag{16}$$

where $\tilde{E}$ and $\overline{E}$ are defined on $\left[\frac{1}{2}, G(\rho)\right]$ by

$$\tilde{E}(x) = (p-1)(p-2)|u'(x)|^{p-2} v'(x) + \lambda f'(u(x))v^2 + (p-1)|u'(x)|^{p-2} v'^2(x), \tag{17}$$

and

$$\overline{E}(x) = (p-1)|u'(x)|^{p-2} u'(x)\omega'(x) - \lambda f'(\rho) + \lambda f(u(x))\omega(x), \tag{18}$$

Let $x \in \left[\frac{1}{2}, G(\rho)\right]$ be fixed, we have

$$\begin{aligned} \tilde{E}'(x) = & (p-1)(p-2)^2 |u'(x)|^{p-4} u'(x)u''(x)v'(x) + (p-1)(p-2)|u'(x)|^{p-2} v''(x) \\ & + \lambda f''(u(x))u'(x)v^2(x) + 2\lambda f'(u(x))v(x)v'(x) + (p-1)(p-2)|u'(x)|^{p-4} u'(x)u''(x)v'^2(x) \\ & + 2(p-1)|u'(x)|^{p-2} v'(x)v''(x) \\ = & (p-1)(p-2)^2 |u'(x)|^{p-4} u'(x)u''(x)v'(x) + (p-1)(p-2)|u'(x)|^{p-2} v''(x) \\ & + \lambda f''(u(x))u'(x)v^2(x) - (p-1)(p-2)|u'(x)|^{p-4} u'(x)u''(x)v'^2(x) \\ = & -(p-2)\lambda f'(u(x))v(x) + \lambda f''(u(x))u'(x)v^2(x) - (p-1)(p-2)|u'(x)|^{p-4} u'(x)u''(x)v'^2(x). \end{aligned}$$

Since $p \geq 2$, $u'(x) \leq 0$ for all x in $[\frac{1}{2}, G(\rho)]$ and using the hypothesis (**H2**) and (**H3**) and by the proposition 4.5, we obtain that

$$\tilde{E}'(x) \leq 0, \text{ for all } x \in [\frac{1}{2}, G(\rho)],$$

which implies that

$$\tilde{E}'(x) \leq \tilde{E}'(\frac{1}{2}), \text{ for all } x \in [\frac{1}{2}, G(\rho)].$$

Which means that for all $x \in [\frac{1}{2}, G(\rho)]$, we have

$$(p-1)(p-2)|u'(x)|^{p-2} v'(x) + \lambda f'(u(x))v^2 + (p-1)|u'(x)|^{p-2} v'^2(x) \leq \lambda f'(\rho).$$

By using this inequality and (16), (17) and (18), we obtain

$$(p-1)|u'(x)|^{p-2} u'(x)w'(x) + \lambda f(u(x))w(x) \geq 0, \text{ for all } x \in [\frac{1}{2}, G(\rho)].$$

Dividing this last inequality by $|u'(x))|^{p-2} u'(x)$ and since $u' < 0$ on $\left(\frac{1}{2}, G(\rho)\right]$, we have

$$(p-1)w'(x) + \frac{\lambda f(u(x))}{|u'(x))|^{p-2} u'(x)} w(x) \leq 0, \text{ for all } x \in \left(\frac{1}{2}, G(\rho)\right]. \tag{19}$$

Now by using the following energy equality

$$\frac{p-1}{p}\left[|u'(x)|^p - \left|u'(\frac{1}{2})\right|^p\right] = F(\rho) - F(u(x)), \text{ for all } x \in [\frac{1}{2}, G(\rho)],$$

we obtain that

$$u'(x) = -\left(\frac{p}{p-1}\right)^{\frac{1}{p}} [F(\rho) - F(u(x))]^{\frac{1}{p}}, \text{ for all } x \in [\frac{1}{2}, G(\rho)]. \tag{20}$$

By (19) and (20), one has

$$(p-1)w'(x) - \frac{\lambda f(u(x))}{\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} [F(\rho) - F(u(x))]^{\frac{p-1}{p}}} w(x) \leq 0, \text{ for all } x \in \left(\frac{1}{2}, G(\rho)\right].$$

That is

$$\frac{d}{dx}\left(e^{-A(x)}.w(x)\right) \leq 0, \text{ for all } x \in \left(\frac{1}{2}, G(\rho)\right], \tag{21}$$

where

$$A(x) = \frac{1}{p-1}\int_{\varepsilon}^{x} \frac{f(u)}{\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} [F(\rho) - F(u)]^{\frac{p-1}{p}}} du, \text{ for all } (\epsilon, x) \subset \left(\frac{1}{2}, G(\rho)\right].$$

Now, we are going to prove that $w(G(\rho)) < 0$. By (15), one has

$$(p-1)(|u'|^{p-2}\omega')'(x) + (p-1)(p-2)(|u'|^{p-4}u'v'^2)'(x) + \lambda f''(u(x))v^2(x) + \lambda f'(u(x))\omega(x) = 0, \text{ for all } x \in \left(\frac{1}{2}, G(\rho)\right).$$

That is

$$\begin{aligned}(p-1)(|u'|^{p-2}\omega')'(x) &= -(p-1)(p-2)\times \\ &\quad \times\left[(p-3)|u'(x)|^{p-4}u''(x)v'^2(x) + 2|u'(x)|^{p-4}u'(x)v'(x)v''((x))\right] \\ &\quad -\lambda f''(u(x))v^2(x) - \lambda f'(u(x))\omega(x), \text{ for all } x \in \left(\tfrac{1}{2}, G(\rho)\right).\end{aligned}$$

Since $p = 2$ or $p \geq 4$, $u'\left(\frac{1}{2}\right) = \omega\left(\frac{1}{2}\right) = 0$ and $v\left(\frac{1}{2}\right) = 1$, we obtain that

$$\begin{aligned}\lim_{x\to\frac{1}{2}^+}(p-1)(|u'|^{p-2}\omega')'(x) &= -\lambda f''(\rho)v^2\left(\frac{1}{2}\right) - \lambda f'(\rho)\omega\left(\frac{1}{2}\right) \\ &= -\lambda f''(\rho) < 0,\end{aligned}$$

which implies that there exists $\alpha > 0$ such that

$$(p-1)(|u'|^{p-2}\omega')'(x) < 0, \text{ for all } x \in \left(\frac{1}{2}, \frac{1}{2} + \alpha\right).$$

Then, we have

$$|u'(x)|^{p-2}\omega'(x) < \left|u'\left(\frac{1}{2}\right)\right|^{p-2}\omega'\left(\frac{1}{2}\right) = 0, \text{ for all } x \in (\frac{1}{2}, \frac{1}{2} + \alpha),$$

which implies that

$$w'(x) < 0, \text{ for all } x \in (\frac{1}{2}, \frac{1}{2} + \alpha).$$

Since $\omega\left(\frac{1}{2}\right) = 0$, we obtain

$$w(x) < 0, \text{ for all } x \in (\frac{1}{2}, \frac{1}{2} + \alpha).$$

Now, if we choose $\varepsilon \in (\frac{1}{2}, \frac{1}{2} + \alpha)$, we have

$$w(\varepsilon) < 0. \tag{22}$$

By (21) and (22), it follows that

$$e^{-A(x)}.w(x) \leq w(\varepsilon) < 0, \text{ for all } x \in \left(\frac{1}{2}, G(\rho)\right].$$

If we put $x = G(\rho)$, we obtain that

$$w(G(\rho)) < 0.$$

□

Remark 4.7. *The proof of this proposition is a generalization to that of Lemma 2.4 in [8].*

Theorem 4.8. *If there exists $\rho_* \in (0, +\infty)$ such that $G'(\rho_*) = 0$, then $G''(\rho_*) < 0$.*

Proof. Since the time map G is defined by the following implicit equation

$$u(G(\rho), \lambda, \rho) = 0 \text{ and } u(x, \lambda, \rho) > 0, \text{ for all } x \in \left[\frac{1}{2}, G(\rho)\right),$$

then differentiating the equation $u(G(\rho), \lambda, \rho) = 0$ with respect to ρ, we obtain that

$$\frac{\partial u(G(\rho), \lambda, \rho)}{\partial x} G'(\rho) + \frac{\partial u(G(\rho), \lambda, \rho)}{\partial \rho} = 0.$$

Differentiating the resulting equation, one has

$$\left(\frac{\partial^2 u(G(\rho), \lambda, \rho)}{\partial x^2} G'(\rho) + \frac{\partial^2 u(G(\rho), \lambda, \rho)}{\partial \rho \partial x}\right) G'(\rho) + \frac{\partial u(G(\rho), \lambda, \rho)}{\partial x} G''(\rho) + \frac{\partial^2 u(G(\rho), \lambda, \rho)}{\partial x \partial \rho} G'(\rho) + \frac{\partial^2 u(G(\rho), \lambda, \rho)}{\partial \rho^2} = 0.$$

Now if there exists $\rho_* \in (0, +\infty)$ such that $G'(\rho_*) = 0$, we have

$$\frac{\partial u(G(\rho_*), \lambda, \rho_*)}{\partial x} G''(\rho_*) + \frac{\partial^2 u(G(\rho_*), \lambda, \rho_*)}{\partial \rho_*^2} = 0.$$

Since $\dfrac{\partial u(G(\rho_*), \lambda, \rho_*)}{\partial x} < 0$ and by proposition 4.6, we have $w(G(\rho_*)) = \dfrac{\partial^2 u(G(\rho_*), \lambda, \rho_*)}{\partial \rho_*^2} < 0$, then it follows that

$$G''(\rho_*) < 0.$$

□

5. Proof of theorem 2.2

Assume that $p = 2$ or $p \geq 4$ and $\lambda > 0$.

From the preceding lemmas one has the following picture about the function $\rho \mapsto G(p, \lambda, \rho)$ which is defined on $(0, +\infty)$

- $\lim\limits_{\rho\to 0^+} G(p,\lambda,\rho) = \lim\limits_{\rho\to +\infty} G(p,\lambda,\rho) = 0,$

  and

- $G(p, \lambda, .)$ admits a unique critical point ρ_* at which admits its maximum value $G(p, \lambda, \rho_*)$.

So, by theorem 3.1, one has

i) If $G(p, \lambda, \rho_*) < \frac{1}{2}$, problem (5) admits no positive solution,

ii) If $G(p, \lambda, \rho_*) = \frac{1}{2}$, problem (5) admits a unique positive solution and it's belongs to A^+,

iii) If $G(p, \lambda, \rho_*) > \frac{1}{2}$, problem (5) admits exactly two positive solutions and they belong to A^+.

If we put $\lambda_* = (\frac{p-1}{p})\left[2\int\limits_0^1 [F(\rho_*) - F(\rho_* t)]^{-\dfrac{1}{p}} dt\right]^p$, then by theorem 3.1, it follows that

(i) If $\lambda < \lambda_*$, problem (5) admits no positive solution,

(ii) If $\lambda = \lambda_*$, problem (5) admits a unique positive solution and it belongs to A^+,

(iii) If $\lambda > \lambda_*$, problem (5) admits exactly two positive solutions and they belong to A^+.

Remark: We conjecture that the result obtained in this work remain valid if $p \in]1, 2[\cup]2, 4[$.

6. Example

In this section, we apply the previous result to the following problem

$$\begin{cases} -\left(\varphi_p(u')\right)' &= u^\alpha + c \text{ in } (0,1), \\ u &> 0 \text{ in } (0,1), \\ u(0) = u(1) &= 0, \end{cases} \tag{23}$$

where $\varphi_p(y) = |y|^{p-2} y$, $y \in \mathbb{R}$, $p = 2$ or $p \geq 4$, $\alpha > p - 1$ and $c > 0$.
It is easy to check that

$$\lim_{u\to 0^+} \frac{f(u)}{\varphi_p(u)} = \lim_{u\to+\infty} \frac{f(u)}{\varphi_p(u)} = +\infty,$$

$$f'(u) = \alpha u^{\alpha-1} > 0, \text{ for all } u > 0,$$

$$f''(u) = \alpha(\alpha - 1)u^{\alpha-2} > 0, \text{ for all } u > 0,$$

$$(p-1)f(u) - uf'(u) = (p-1-\alpha)u^{\alpha} + (p-1)c, \text{ for all } u > 0.$$

Then

$$\lim_{u\to+\infty}(p-1)f(u) - uf'(u) = -\infty,$$

$$(p-2)f'(u) - uf''(u) = \alpha(p-\alpha-1)u^{\alpha+1}, \text{ for all } u > 0.$$

Then

$$(p-2)f'(u) - uf''(u) < 0, \text{ for all } u > 0,$$

and

$$pF(u) - uf(u) = \left(\frac{p-\alpha-1}{\alpha+1}\right)u^{\alpha+1} + (p-1)cu, \text{ for all } u > 0.$$

Then

$$\lim_{u\to+\infty} pF(u) - uf(u) = -\infty.$$

Then by theorem 2.2, it follows that there exists $\lambda_* > 0$ such that

i) If $\lambda > \lambda_*$, then the problem (23) admits no positive solution,

ii) If $\lambda = \lambda_*$, then the problem (23) admits a unique positive solution and it's belongs to A^+,

iii) If $\lambda < \lambda_*$, then the problem (23) admits exactly two positive solutions and they belong to A^+.

Acknowledgments
The authors thanks the anonymous referee for several comments and suggestions which contributed to improve this paper.

References

[1] T.Adamowicz & P.Korman, Remarks on time map quasilinear equations, Journal of Mathematical Analysis and Applications, Vol 376, Issue 2, (2011) 686-695.

[2] I. Addou & A. Benmezai, Boundary value problems for the one-dimensional $p-$Laplacian with even superlinearity, Electronic Journal of Differential Equations 9 (1999) 1-29.

[3] I. Addou, S. M. Bouguima, A. Benmezai & M. Derhab, Exactness results for generalized Ambrosetti-Brezis-Cerami and related one-dimensional elliptic equations, Electronic Journal of Differential Equations 66 (2000) 1-34.

[4] I. Addou, S. M. Bouguima, M. Derhab & Y. S. Raffed, On the number of solutions of a quasilinear elliptic class of B. V. P. with jumping nonlinearities, Dynamic Syst. Appl. 7 (4) (1998) 575-599.

[5] H. Amann, T. Laetsch, positive solutions of convex Nonlinear Eigenvalue problems, Indiana Univ. Math, J. 25 (1976), pp. 259-270.

[6] A. Ambrosetti, On the exact number of positive solution of convex nonlinear problem, Boll. U.M.I. (5) 15-A 610-615.

[7] M. Derhab & M. Megnafi, Exact number of positive solutions for a class of quasilinear boundary value problems, Communications on Applied Nonlinear Analysis, Volume 15 (4) (2008) 15-37.

[8] S.Gadam & Joseph A. Iaia, Exact multiplicity of positive solutions in semipositone problems with concave-convex type nonlinearities, Electronic Journal of Qualitative Theory of Differential Equations, 4 (2001) 1-9.

[9] D. D. Joseph, Non-linear heat generation and stability of the temperature distribution in conducting solids, Int. J. Heat Mass Transfer, 8 (1965) 281-288.

[10] M. Guedda & L. Veron, Bifurcation phenomena associate to the p-Laplacian operator, Trans. Amer. Math. Soc. 310 (1988) 419-431.

[11] J.Karàtson & P.L Simon, Bifurcation for semilinear elliptic equations with convex nonlinearity, Electronic Journal of Differential Equations 43 (1999) 1-16.
[12] J.Karàtson & P.L Simon, Exact multiplicity for degenerate two-point boundary value problems with $p-$convex nonlinearity, Nonlinear Anal. T. M. A. 52 (6) (2003) 1569-1590.
[13] H. B. Keller & D. S. Cohen, Some positone problems suggested by nonlinear heat generation, J. Math. Mech., 16 (1967) 1361-1376.
[14] P.Korman, Global solution branches and exact multiplicity of solutions for two point boundary value problems, Handbook of Differential Equations, Ordinary Differential Equations, Vol. 3, Edited by A. Canada, P. Drabek and A. Fonda, Elsevier Science, North Holland (2006) 547-606.
[15] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math J. 20 (1971) 1-13.
[16] T. Laetsch, The theory of forced, convex, autonomous two point boundary value problems, Rocky Mountain J. Math. 15 (1985) 133-154.
[17] R.Schaaf, Global Solution Branches of Two Point Boundary Value Problems, Lecture Notes in Mathematics, N^0 1458, Springer-Verlag 1990.
[18] J. A. Smoller & A. G. Wassemann, Global bifurcation of steady state solutions, Journal of Differential Equations 39 (1981) 269-290.
[19] J. A. Smoller & A. G. Wassemann, On the monotonicity of the time-map, Journal of Differential Equations 77 (1989) 287-303.