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RH-regular Transformations Which Sums a Given Double Sequence

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Abstract. In 1946 P. Erdos and P. C Rosenbloom presented the following theorem that arose out of discussions they had with R. P. Agnew. Let $\{x_n\}$ be a bounded divergent sequence. Suppose that $\{x_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$. Then $\{y_n\}$ is of the form $\{cx_n + a_n\}$ where $\{a_n\}$ is convergent. The goals of the paper includes the presentation of a multidimensional analog of Erdos and Rosenbloom results in [1].

1. Definitions, Notations and Preliminary Results

Definition 1.1 (Pringsheim, 1900). *A double sequence $x = [x_{k,l}]$ has* **Pringsheim limit** *L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $\left|x_{k,l} - L\right| < \epsilon$ whenever $k, l > N$. Such an x is describe more briefly as "P-convergent".*

Definition 1.2 (Patterson, 2000). *The double sequence y is a* **double subsequence** *of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $x_j = x_{n_j,k_j}$, then y is formed by*

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

In [6] Robison presented the following notion of regular four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 1.3. *The four-dimensional matrix A is said to be* **RH-regular** *if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Theorem 1.4. *(Hamilton [2], Robison [6]) The four dimensional matrix A is RH-regular if and only if*

RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
RH_2: P-$\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: P-$\lim_{m,n} \sum_{k=1}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each l;*

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RH_4: P-$\lim_{m,n}\sum_{l=1}^{\infty}\left|a_{m,n,k,l}\right| = 0$ for each k;
RH_5: $\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|$ is P-convergent;
RH_6: there exist finite positive integers Δ and Γ such that
$\sum_{k,l>\Gamma}\left|a_{m,n,k,l}\right| < \Delta$.

2. Main Results

Theorem 2.1. *Let $\{x_{k,l}\}$ be a bounded P-divergent double sequence. Suppose that $\{y_{k,l}\}$ is P-summable by every RH-regular summability matrix which sums $\{x_{k,l}\}$. Then $\{y_{k,l}\}$ is of the form $\{cx_{k,l} + a_{k,l}\}$ where $\{x_{k,l}\}$ is P-convergent.*

Proof. Let $\{x_{m_k,n_l} : k, l = 1, 2, 3, \ldots\}$ be a P-convergent subsequence of $\{x_{m,n}\}$. Then $\{x_{m,n}\}$ is summable by

$$a_{m,n,k,l} = \begin{cases} 1, & \text{if} \quad m = m_k \ n = n_l \\ 0, & \text{if} \quad \text{otherwise} \end{cases}.$$

Therefore $\{y_{m_k,n_l}\}$ is also P-convergent. Let $\{(m_k^{'}, n_l^{'}) : k, l = 1, 2, 3, \ldots\}$ and $\{(m_k^{''}, n_l^{''}) : k, l = 1, 2, 3, \ldots\}$ be index sequences with $m_k^{'} \neq m_k^{''}$ and $n_k^{'} \neq n_k^{''}$ for all (k, l) and let

$$\mathrm{P} - \lim_{k,l} x_{m_k^{'},n_l^{'}} = A$$

and

$$\mathrm{P} - \lim_{k,l} x_{m_k^{''},n_l^{''}} = B$$

with $A \neq B$. The double sequences $\{y_{m_k^{'},n_l^{'}}\}$ and $\{y_{m_k^{''},n_l^{''}}\}$ are also P-convergent double sequences say to α and β, respectively. Let $\{x_{m_k,n_l}\}$ be any double subsequence of $\{x_{m,n}\}$ with

$$\mathrm{P} - \lim_{k,l} x_{m_k,n_l} = C.$$

Let λ and ρ be such that

$$\lambda + \rho = 1 \text{ and } \lambda A + \rho B = C;$$

and define A as follows

$$a_{m,n,k,l} = \begin{cases} \lambda, & \text{if} \quad m = m_k^{'} \ n = n_l^{'} \text{ with } k \text{ and } l \text{ are both even} \\ \rho, & \text{if} \quad m = m_k^{''} \ n = n_l^{''} \text{ with } k \text{ and } l \text{ are both even} \\ 1, & \text{if} \quad m = m_k \ n = n_l \text{ with } k \text{ and } l \text{ are both odd} \\ 0, & \text{if} \quad \text{otherwise of all } m, n, k, \text{ and } l \end{cases}.$$

Then $\mathrm{P} - \lim_{k,l}(Ax)_{k,l} = C$. Therefore A also sums $\{y_{m,n}\}$ that is

$$\begin{aligned} \mathrm{P} - \lim_{k,l} y_{m_k,n_l} &= \mathrm{P} - \lim_{k,l}\left(\lambda y_{m_k^{'},n_{n_l}^{'}} + \rho y_{m_k^{''},n_{n_l}^{''}}\right) \\ &= \lambda\Delta + \rho\Delta^{'} \end{aligned} \tag{1}$$

where Δ and $\Delta^{'}$ depend only on C. Thus $\mathrm{P} - \lim_{k,l} y_{m_k,n_l}$ depend only on C. Infact C is a linear function. We now must determine R and T such that $\Delta = RA + T$ and $\Delta^{'} = RB + T$. Let m_k and n_k be positive index sequences and let $m_k^{'''}$ and $n_k^{'''}$ be subsequences of m_k and n_k, respectively, such that

$$\mathrm{P} - \lim_{k,l} x_{m_k^{'''},n_l^{'''}} = C.$$

Thus by the determination of λ and ρ above we obtain the following:

$$\begin{aligned} \mathrm{P}-\lim_{k,l}\left(y_{m_k''',n_l'''}+Rx_{m_k''',n_l'''}\right) &= \lambda\Delta+\rho\Delta'-RC \\ &= \lambda(RA+T)+\rho(RB+T)-RC \\ &= T(\lambda+\rho). \end{aligned} \tag{2}$$

Therefore every double sequence of $\{y_{m,n} - Rx_{m,n}\}$ contain a double sequence that is P-converges to T. Therefore

$$\mathrm{P}-\lim_{m,n}(y_{m,n}-Rx_{m,n}) = T.$$

□

The following is clearly a corollary of the above theorem.

Corollary 2.2. *If $\{x_{m,n}\}$ and $\{y_{m,n}\}$ are P-divergent double sequences and $\{y_{m,n}\}$ is summable by every RH-regular summability matrix which sums $\{x_{m,n}\}$ then $\{x_{m,n}\}$ is summable by every RH-regular summability matrix method which sums $\{y_{m,n}\}$.*

By a theorem of Patterson in [4] there are no single four dimensional summability method which has the double sequences of the form $\{cx_{m,n}+a_{m,n}\}$ as its P-convergence field. Note, however, that the theorem above grant us that this set of double sequences is the common part of the P-convergence field of summability methods that sum $\{x_{m,n}\}$.

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