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On a new class of operators and Weyl type theorems

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Abstract. In the present article, we introduce a new class of operators which will be called the class of k-quasi $*$-paranormal operators that includes $*$-paranormal operators. A part from other results, we show that following results hold for a k-quasi $*$-paranormal operator T:

(i) T has the SVEP.

(ii) Every non-zero isolated point in the spectrum of T is a simple pole of the resolvent of T.

(iii) All Weyl type theorems hold for T.

(iv) Comments and some open problems are also presented.

1. Introduction

In what follows H will be an infinite dimensional separable complex Hilbert space. By an operator on H, we mean a bounded linear transformation from H to H. Let $B(H)$ be the Banach algebra of operators on H. We call an operator T to be hyponormal if $T^*T \geq TT^*$; quasi-hyponormal if $T^{*2}T^2 \geq (T^*T)^2$; paranormal if $\|T^2x\|\|x\| \geq \|Tx\|^2$ for all $x \in H$; k-paranormal if $\|T^kx\|\|x\| \geq \|Tx\|^k$ for all $x \in H$. According to [1], an operator T is called $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ and T is called k^*-paranormal if $\|T^*x\|^k \leq \|T^kx\|$ for all unit vector x in H where k is a natural number with $k \geq 2$. The class of $*$-paranormal operators and more generally the class of k^*-paranormal operators was originally introduced in [21] and [22] with different names as k-hyponormal or operators of class $(H;k)$. For more results for such operators, one can refer [5], [9], [12], and [27]. The following inclusion are well known an proper [22].

$$\{\text{Hyponormal operator} \subseteq *\text{-paranormal operator} \subseteq \text{normaloid operator}\}.$$

The classes of paranormal operators and $*$-paranormal operators are independent subclasses of normaloid operators [22, Theorem 3]. In the present article we introduce a new class of k-quasi $*$-paranormal operators defined as follows:

Definition 1.1. *An operator T is called k-quasi $*$- paranormal if it satisfies the following inequality:*

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|$$

for all unit vector $x \in H$ where k is a natural number.

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Since for A, B and C in $B(H)$,

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \text{ for all } \lambda > 0 \Leftrightarrow \|Bx\|^2 \leq \|Ax\|\|Cx\| \text{ for all } x \in H,$$

we find T to be k-quasi $*$-paranormal if and only if

$$T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0, \text{ for all } \lambda > 0(*).$$

The following implications are obvious

$$\text{Hyponormal} \Rightarrow *\text{-paranormal} \Rightarrow k\text{-quasi} * \text{-paranormal}.$$

If $T \in B(H)$, we shall write $N(T)$ and ran(T) for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively.

Definition 1.2. *An operator $T \in B(H)$ is said to have Bishop's property (β) if $(T - z)f_n(z) \to 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D, then $f_n(z) \to 0$ uniformly on every compact subset of D.*

Definition 1.3. *$T \in B(H)$ is said to have the single valued extension property, abbreviated, T has SVEP if $f(z)$ is an analytic vector valued function on some open set $D \subset \mathbf{C}$ such that $(T - z)f(z) = 0$ for all $z \in D$, then $f(z) = 0$ for all $z \in D$.*

2. Basic Properties

In what follows, the symbol $Q^*(k)$ will be used for the class of k-quasi $*$-paranormal operators.

Proposition 2.1. *For $T \in Q^*(k)$, $N(T - zI) \subseteq N(T^* - \bar{z})$ for each non-zero complex number z.*

Proof. Suppose $Tx = zx$. Since

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|$$

for all unit vector $x \in H$.

Thus $\|T^*x\| \leq |z|$. Hence

$$\|T^*x - \bar{z}x\|^2 = \|T^*x\|^2 - 2\Re\langle T^*x, \bar{z}x\rangle + |z|^2\|x\|^2 = \|T^*x\|^2 - 2\Re\langle x, \bar{z}Tx\rangle$$
$$+|z|^2\|x\|^2 = \|T^*x\|^2 - |z|^2\|x\|^2 \leq |z|^2 - |z|^2 = 0.$$

Hence $T^*x = \bar{z}x$. □

Remark 2.2. *The above proposition is not valid for $z = 0$. To see this, let T be nilpotent of index $k+1$. Then $T \in Q^*(k)$ and $N(T)$ is not a subset of $N(T^*)$ (otherwise T will be a zero operator).*

Proposition 2.3. *If $T \in Q^*(k)$ and M is an invariant subspace of T. Then $T_{|M}$ (the restriction of T to M) is k-quasi-$*$-paranormal.*

Proof. Let P be the orthogonal projection on M. Then

$$T^kP = (PTP)^k = PT^kP.$$

This leads to

$$P(T^{*k}TT^*T^k)P \geq PT^{*k}TPT^*T^kP = PT^{*k}PTPT^*T^kP$$
$$= (PT^*P)^k(PTP)(PT^*P)(PTP)^k.$$

Therefore the inequality

$$T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0 \text{ for all } \lambda > 0$$

implies

$$PT^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^kP \geq 0 \text{ for all } \lambda > 0,$$

and hence

$$T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1T_1^* + \lambda^2)T_1^k \geq 0 \text{ for all } \lambda > 0.$$

This proves the result.

□

Proposition 2.4. *Every quasi-hyponormal operator is quasi $*$-paranormal.*

Proof. Let T be quasi-hyponormal. Since every quasi-hyponormal is paranormal, we have

$$\|T^*Tx\|^2 \leq \|T^2x\|^2$$

$$= \|T(\frac{Tx}{\|Tx\|})\|^2\|Tx\|^2$$

$$\leq \|T^2(\frac{Tx}{\|Tx\|})\|\|Tx\|^2$$

$$= \|T^3x\|\|Tx\|$$

for all $x \in H$. This proves the result. □

Proposition 2.5. *Every quasi $*$-paranormal is 3-paranormal (and hence normaloid).*

Proof. Suppose T is quasi $*$-paranormal. Then

$$\|Tx\|^4 = \langle T^*Tx, x\rangle^2$$

$$\leq \|T^*Tx\|^2\|x\|^2 \leq \|T^3x\|\|Tx\|\|x\|^2.$$

Hence

$$\|Tx\|^3 \leq \|T^3x\|\|x\|$$

or T is 3-paranormal. □

In order to obtain some spectral properties of class $Q^*(k)$, we shall need the following result.

Lemma 2.6. *Let $T \in B(H)$ be k-quasi $*$-paranormal such that* ran T^k *is not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad H = \overline{\text{ran } T^k} \oplus N(T^{*k}).$$

Then T_1 is $*$-paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{\text{ran } T^k} \oplus N(T^{*k})$$

and let P be the orthogonal projection of T onto $\overline{\text{ran } T^k}$. Since $P^2 = P$ and $P \leq I$, we have $P(TT^*)P \geq (PTP)(PT^*P)$. Since T is k-quasi $*$-paranormal,

$$P(T^{*2}T^2P - 2\lambda TT^* + \lambda^2)P \geq 0 \text{ for all } \lambda > 0,$$

$$P(T^{*2}T^2)P - 2\lambda P(TT^*)P + \lambda^2 \geq 0 \text{ for all } \lambda > 0.$$

Therefore $T_1^{*2}T_1^2 - 2\lambda T_1 T_1^* + \lambda^2 \geq 0$ for all $\lambda > 0$. This shows that T_1 is $*$-paranormal on $\overline{\operatorname{ran} T^k}$. Further, we have

$$\langle T_3^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{*k}(I-P)x\rangle = 0,$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H$. Thus $T^{*k} = 0$. We have $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup G$, where G is the union of certain holes in $\sigma(T)$ which is a subsets of $\sigma(T_1) \cap \sigma(T_3)$ [14, Corollary 7]. Since $\sigma(T_1) \cap \sigma(T_3)$ has no interior points, we have

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

□

Throughout our exposition, we will exploit the representation of $T \in Q^*(k)$ given in the preceding lemma.

As a consequence of Lemma 2.1, we obtain.

Corollary 2.7. *Let $T \in B(H)$ be a k-quasi $*$-paranormal operator. If T_1 is invertible, then T is similar to a direct sum of a $*$-paranormal and a nilpotent operator.*

Proof. Since T_1 is invertible, we have $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. Then there exists an operator S such that $T_1 S - S T_3 = T_2$ [23]. Hence

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

□

Lemma 2.8. *If $T \in B(H)$ is k-quasi $*$-paranormal, then* $\operatorname{asc}(T - \lambda) \leq 1$ for all complex numbers λ.

Proof. Proposition 2.1 implies $(T-\lambda)^{-1}(0) \perp (T-\lambda)H$. Hence, if $x \in (T-\lambda)^{-2}(0)$ and $x \notin (T-\lambda)^{-1}(0)$, then $x = 0$. This implies $\operatorname{asc}(T-\lambda) \leq 1$. □

Corollary 2.9. *If T is a k-quasi-$*$-paranormal operator, then T has SVEP.*

Proof. Lemma 2.8 implies that a k-quasi-$*$-paranormal operator has SVEP [1, Theorem 3.8]. □

Recently it is proved in [12] that every isolated point in the spectrum of a $*$-paranormal operator T is a simple pole of the resolvent of T. More generally, for k-quasi $*$-paranormal operators, we have.

Proposition 2.10. *Let $T \in B(H)$ be k-quasi $*$-paranormal. If μ is a non-zero isolated point in $\sigma(T)$, then it is a simple pole of the resolvent of T. Hence T is polaroid.*

Proof. In case $\operatorname{ran}(T^k)$ is dense then T is $*$-paranormal and so the result follows [12]. So we assume that $\operatorname{ran} T^k$ is not dense. Then by Lemma 2.6, the operator T can be decomposed as:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } H = \overline{\operatorname{ran}(T^k)} \oplus N(T^{*k}),$$

where A is $*$-paranormal and $C^k = 0$. Now if μ is a non-zero isolated point of $\sigma(T)$, then $\mu \in \operatorname{iso}\sigma(A)$ because $\sigma(T) = \sigma(A) \cup \{0\}$. Therefore μ is a simple pole of the resolvent of A [12, Theorem 2.9] and the $*$-paranormal operator A can be written as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } \overline{\mathrm{ranT^k}} = N(A-\mu) \oplus \operatorname{ran}(A-\mu),$$

where $\sigma(A_1) = \{\mu\}$. Therefore

$$T - \mu I = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & A_2 - \mu & B_2 \\ 0 & 0 & C - \mu \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & F \end{pmatrix} \text{ on } H = N(A-\mu) \oplus \operatorname{ran}(A-\mu) \oplus N(T^{*k}),$$

where

$$F = \begin{pmatrix} A_2 - \mu & B_2 \\ 0 & C - \mu \end{pmatrix}.$$

We claim that F is an invertible operator on $\mathrm{ran}(A - \mu) \oplus N(T^{*k})$. First we verify that $A_2 - \mu I$ is invertible. If not, then μ will be an isolated point in $\sigma(A_2)$. Since A_2 is $*$-paranormal and $*$-paranormal is isoloid, hence μ is an eigenvalue of A_2 and so $A_2 x = \mu x$ for some non-zero vector x in $\mathrm{ran}(A - \mu I)$. On the other hand, $Ax = A_2 x$ implying x is in $N(A - \mu I)$. Hence x must be a zero vector. This contradiction shows that $A_2 - \mu I$ is invertible. Since $C - \mu I$ is also invertible, it follows that F is invertible [13, Problem 71]. Since $T - \mu I$ is invertible, $T - \mu I$ has finite ascent and descent. It is easy to show that $p(T - \mu I) = q(T - \mu I) = 1$. Hence μ is a simple pole of the resolvent of T. □

Corollary 2.11. *A k-quasi $*$-paranormal operator is isoloid.*

More generally, for k-quasi $*$-paranormal operators, we have

Theorem 2.12. *Let A be a k-quasi-$*$-paranormal operator and λ be a non-zero isolated point of $\sigma(A)$. Then, the Riesz idempotent E for λ is self-adjoint and*

$$EH = N(A - \lambda) = N(A - \lambda)^*.$$

Proof. If A is k-quasi-$*$-paranormal, then λ is an eigenvalue of A and $EH = N(A - \lambda)$ by Corollary 2.11. Since $N(A - \lambda) \subset N(A - \lambda)^*$ by Proposition 2.1, it suffices to show that $N(A - \lambda)^* \subset N(A - \lambda)$. Since $N(A - \lambda)$ is a reducing subspace of A by Proposition 2.1 and the restriction of a k-quasi-$*$-paranormal operator to its reducing subspaces is also a k-quasi-$*$-paranormal operator by Proposition 2.2, hence A can be written as follows:

$$A = \lambda \oplus A_1 \text{ on } H = N(A - \lambda) \oplus (N(A - \lambda))^\perp,$$

where A_1 is k-quasi-$*$-paranormal with $N(A_1 - \lambda) = \{0\}$. Since

$$\lambda \in \sigma(A) = \{\lambda\} \cup \sigma(A_1)$$

is isolated, the only two cases occur, one is $\lambda \notin \sigma(A_1)$ and the other is that λ is an isolated point of $\sigma(A_1)$ and this contradicts the fact that $N(A_1 - \lambda) = \{0\}$. Since A_1 is invertible as an operator on $(N(A - \lambda))^\perp$, $N(A - \lambda) = N(A - \lambda)^*$.

Next, we show that E is self-adjoint. Since

$$EH = N(A - \lambda) = N(A - \lambda)^*,$$

we have

$$((z - A)^*)^{-1} E = \overline{(z - \lambda)^{-1}} E.$$

Therefore

$$E^* E = -\frac{1}{2\pi i} \int_{\partial D} ((z - A)^*)^{-1} E d\bar{z} = -\frac{1}{2\pi i} \int_{\partial D} \overline{(z - A)^{-1}} E d\bar{z} =$$

$$\overline{(\frac{1}{2\pi i} \int_{\partial D} (z - A)^{-1} dz)} E = E.$$

This completes the proof. □

3. Weyl type theorems

An operator T is called Fredholm if ran(T) is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim \mathcal{H}/\text{ran(T)} < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup \text{acc } \sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso $K = K \setminus \text{acc } K$, then we let

$$\pi_{00}(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T).$$

More generally, M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators ([6–8]). In a recent paper [19] the author showed that generalized Weyl's theorem holds for (p,k)-quasi-hyponormal operators. Recently, X. Cao, M. Guo and B. Meng [11] proved Weyl type theorems for p-hyponormal operators. M. Berkani investigated B-Fredholm theory as follows (see [1, 6–8]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $\text{ran}(\text{T}^{\text{n}})$ is closed and the induced operator

$$T_{[n]} : \text{ran}(\text{T}^{\text{n}}) \ni \text{x} \to \text{Tx} \in \text{ran}(\text{T}^{\text{n}})$$

is Fredholm, i.e., $\text{ran}(\text{T}_{[\text{n}]}) = \text{ran}(\text{T}^{\text{n}+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim \text{ran}(\text{T}^{\text{n}})/\text{ran}(\text{T}_{[\text{n}]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$. The following results is due to M. Berkani and M. Sarih [8].

Proposition 3.1. *Let $T \in B(H)$.*

(1) *If* $\text{ran}(\text{T}^{\text{n}})$ *is closed and* $T_{[n]}$ *is Fredholm, then* $\text{rm}(\text{T}^{\text{m}})$ *is closed and* $T_{[m]}$ *is Fredholm for every* $m \geq n$. *Moreover,* $\text{ind } T_{[m]} = \text{ind } T_{[n]} (= \text{ind } T)$.

(2) *An operator T is B-Fredholm (B-Weyl) if and only if there exist T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $T = T|\mathcal{M} \oplus T|\mathcal{N}$ where $T|\mathcal{M}$ is Fredholm (Weyl) and $T|\mathcal{N}$ is nilpotent.*

The B-Weyl spectrum $\sigma_{BW}(T)$ are defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [7]. Recently in [6] M. Berkani and A. Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T.

We define $T \in SF_+^-$ if ran(T) is closed, $\dim N(T) < \infty$ and ind $T \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim N(T - \lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \{\lambda \mid T - \lambda \notin SF_+^-\} \subset \sigma_W(T)$. We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

V. Rakočević [24, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

We define $T \in SBF_+^-$ if there exists a positive integer n such that $\mathrm{ran}(T^n)$ is closed, $T_{[n]} : \mathrm{ran}(T^n) \ni x \to Tx \in \mathrm{ran}(T^n)$ is upper semi-Fredholm (i.e., $\mathrm{ran}(T_{[n]}) = \mathrm{ran}(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap \mathrm{ran}(T^n) < \infty$) and $0 \geq \mathrm{ind}\, T_{[n]}(= \mathrm{ind}\, T)$ ([8]). We define $\sigma_{SBF_+^-}(T) = \{\lambda \mid T - \lambda \notin SBF_+^-\} \subset \sigma_{SF_+^-}(T)$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim N(T - \lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T).$$

M. Berkani and J.J. Koliha [7] proved that if generalized a-Weyl's theorem holds for T, then a-Weyl's theorem holds for T.

If a Banach space operator T has SVEP (everywhere), the single-valued extension property, then T and T^* satisfy Browder's (equivalently, generalized Browder's) theorem and a-Browder's (equivalently, generalized a- Browder's) theorem. A sufficient condition for an operator T satisfying Browder's (generalized Browder's) theorem to satisfy Weyl's (resp., generalized Weyl's) theorem is that T is polaroid. Observe that if $T \in B(H)$ has SVEP, then $\sigma(T) = \sigma_a(T^*)$. Hence, if T has SVEP and is polaroid, then T^* satisfies generalized a-Weyl's (so also, a-Weyl's) theorem [2].

Theorem 3.2. *Let $T \in B(H)$.*
i) If T^* is a k-quasi-$*$-paranormal operator, then also T satisfies generalized a-Weyl's theorem.
ii) If T is a k-quasi-$*$-paranormal operator, then generalized a-Weyl's theorem holds for T^*.

Proof. (i) it is well known that T is polaroid if and only if T^* is polaroid [2, Theorem 2.11]. Now since a k-quasi-$*$-paranormal operator is polaroid by Proposition 2.10 and has SVEP by Corollary 2.9, [2, Theorem 3.10] gives us the result of the theorem. For (ii) we can also apply [2, Theorem 3.10]. □

Since the polaroid condition entails $E(T) = \pi(T)$ and the SVEP for T entails that generalized Browders theorem holds for T [3, Theorem 3.2], i.e. $\sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T)$ denotes the Drazin spectrum of T. Therefore,

$$E(T) = \pi(T) = \sigma(T) \setminus \sigma_D(T) = \sigma(T) \setminus \sigma_{BW}(T).$$

Thus we have the following corollary.

Corollary 3.3. *If T is k-quasi-$*$-paranormal, then also T satisfies generalized Weyl's theorem.*

Remark 3.4. 1. *Recall [2] that if T is polaroid, then T satisfies generalized Weyl's theorem (resp. generalized a-Weyl's) theorem if and only if T satisfies Weyl's theorem (resp. a-Weyl's theorem). Hence if T is a k-quasi-$*$-paranormal operator, the above equivalences hold.*

2. *Let $f(z)$ be an analytic function on $\sigma(T)$. If T is polaroid, then $f(T)$ is polaroid too [2].*

i) If T^* is k-quasi-$*$-paranormal, then $f(T)$ satisfies generalized a-Weyl's theorem. Indeed, since T^* is polaroid, the result holds by [2, Theorem 3.12]

ii) If T is k-quasi-$*$-paranormal, then $f(T^*)$ satisfies generalized a-Weyl's theorem. Indeed, since T is polaroid, the result holds by [2, Theorem 3.12].

4. Comments and Some open problems

In [12, Proposition 2.4], the authors showed that a $*$-paranormal operator has Bishop's property β. In the proof of this proposition, the authors have used [26, Theorem 3.5] which is not correct. Because the proof of this theorem is false. Indeed, let $f_n(z) = z^n$ for $z \in B(2;1) = \{w \in \mathbb{C} : |w-2| < 1\}$. Then $\|f_n\|_{\overline{B(z_0;r)}} = (r + |z_0|)^n$ and $sup_n \|f_n\|_{\overline{B(z_0;r)}} = 1$, for all $z_0 \in B(2;1)$ and $0 < r < 1 - |2 - z_0|$. Let $R_1 = \frac{1-|z_0-2|}{2}$ and $g_n = \frac{f_n}{1+\|f_n\|_{\overline{B(z_0;R_1)}}}$. Then $\|g_n\|_{\overline{B(z_0;\frac{R_1}{2})}} \to 0$, but this says nothing about $\|f_n\|_{\overline{B(z_0;\frac{R_1}{2})}}$. Thus [12, Proposition 2.4] and [12, Corollary 2.8] are not correct and still open problems.

Note that if a $*$-paranormal operators has property (β), then a k-quasi-$*$-paranormal operator also has Bishop's property (β) by [20, Theorem 2.5].

The proof of Lemma 2.2 in [18] depends on the polaroid operator A (similarly, B) with a finite set F of isolated points having invariant subspaces M_1 and M_2 such that $A_1 = A|_{M_1}$ is algebraic, $\sigma(A_1) = F$ and $A_2 = A|_{M_2}$ invertible on F. This is not always true, for the raison that Riesz decomposition theorem does guarantee subspaces M_1 and M_2, and operators A_1 and A_2, as above but more is required for A_1 to be algebraic. Thus the proof is not correct and Lemma 2.2 in [18] still an open problem. Spectral properties of paranormal operators have been investigated by a number of authors. Notably, a published proof that paranormal operators have Bishop's property (β), [26], has been retracted this year. This error has in fact propagated through recent work in operator theory, and so a correct proof of this result would be useful and interesting. Also the proof of Lemma 2.4 in [18] is not true. Indeed the author use the fact that paranormal operator has property (β) [26, Corollary 3.6] but this is not true as mentioned above. Thus [26, Corollary 3.6] still an open problem and the rest of the results in [18] are not correct and are still open problems.

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