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Growth properties of modified α-potentials in the upper-half space

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Abstract. The aim of this paper is to discuss the behavior at infinity of modified α-potentials represented by the modified kernels in the upper-half space of the n-dimensional Euclidean space, which generalizes the growth properties of analytic functions, harmonic functions and superharmonic functions.

1. Introduction and main results

Let $\mathbf{R}^n$ ($n \geq 2$) denote the n-dimensional Euclidean space with points $x = (x_1, x_2, \cdots, x_{n-1}, x_n) = (x', x_n)$, where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open Ω of $\mathbf{R}^n$ are denoted by $\partial\Omega$ and $\overline{\Omega}$ respectively. The upper half-space is the set $H = \{x = (x', x_n) \in \mathbf{R}^n : \ x_n > 0\}$, whose boundary is ∂H. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x,\ y \in \mathbf{R}^n$ as $x = (x', x_n),\ y = (y', y_n)$, where $x' = (x_1, x_2, \cdots, x_{n-1}),\ y' = (y_1, y_2, \cdots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting $x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n,\ \ |x| = \sqrt{x \cdot x},\ \ |x'| = \sqrt{x' \cdot x'}$.

For $x \in \mathbf{R}^n$ and $r > 0$, let $B_n(x, r)$ denote the open ball with center at x and radius r in $\mathbf{R}^n$.

It is well known that (see, e.g. [4, Ch. 6]) the positive powers of the Laplace operator Δ can be defined by

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \mathcal{F}^{-1}(|\xi|^{\alpha} \hat{f}(\xi)), \tag{1}$$

where $\alpha > 0$, f is a Schwarz function and

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix\xi} dx.$$

It follows that we can extend the definition (1) to certain negative powers of $-\Delta$, $(-\Delta)^{-\frac{\alpha}{2}}$ for $0 < \alpha < n$ and define an operator I_α by

$$I_\alpha f = (-\Delta)^{-\frac{\alpha}{2}} f = \mathcal{F}^{-1}(|\xi|^{-\alpha} \hat{f}),$$

where $0 < \alpha < n$ and f is a function in the Schwartz class.

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If I_α is defined as the inverse Fourier transform of $|\xi|^{-\alpha}$ (in the sense of distributions), one can show that

$$I_\alpha(x) = \gamma_\alpha |x|^{\alpha-n},$$

where γ_α is a certain constant (see, e.g. [4, p. 414] for the exact value of γ_α).

The function I_α is known as the Riesz kernel. It follows immediately from the rules for manipulating Fourier transforms that any Schwartz function f can be written as a Riesz potential,

$$f(x) = I_\alpha g(x) = (I_\alpha * g)(x) = \gamma_\alpha \int_{\mathbf{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}} dy,$$

where $0 < \alpha < n$ and $g = (-\Delta)^{\frac{\alpha}{2}} f$.

This Riesz kernel I_α in $\mathbf{R}^n$ inspired us to introduce the modified Riesz kernel for H. To do this, we first set

$$E_\alpha(x) = \begin{cases} -\log|x| & \text{if } \alpha = n = 2, \\ |x|^{\alpha-n} & \text{if } 0 < \alpha < n. \end{cases}$$

Let $G_\alpha(x,y)$ be the green function of order α for H, that is

$$G_\alpha(x,y) = E_\alpha(x-y) - E_\alpha(x-y^*) \qquad x, y \in \overline{H},\ x \neq y,\ 0 < \alpha \leq n,$$

where * denotes reflection in the boundary plane ∂H just as $y^* = (y_1, y_2, \cdots, y_{n-1}, -y_n)$.

We define the Poisson kernel $P_\alpha(x, y')$ when $x \in H$ and $y' \in \partial H$ by

$$P_\alpha(x,y') = \left.\frac{\partial G_\alpha(x,y)}{\partial y_n}\right|_{y_n=0} = C_\alpha \frac{x_n}{|x-y'|^{n-\alpha+2}},$$

where $C_\alpha = 2(n-\alpha)$ if $0 < \alpha < n$ and $= 2$ if $\alpha = n = 2$.

We remark that in the case $\alpha = 2$, $G_2(x,y)$ and $P_2(x,y')$ are the classical Green function and Poisson kernel for H respectively (see, e.g. [5, p. 127]).

In case $\alpha = n = 2$, we consider the modified kernel function, which is defined by

$$E_{n,m}(x-y) = \begin{cases} E_n(x-y) & \text{if } |y| < 1, \\ E_n(x-y) + \Re\left(\log y - \sum_{k=1}^{m-1}(\frac{x^k}{ky^k})\right) & \text{if } |y| \geq 1, \end{cases}$$

In case $0 < \alpha < n$, we define

$$E_{\alpha,m}(x-y) = \begin{cases} E_\alpha(x-y) & \text{if } |y| < 1, \\ E_\alpha(x-y) - \sum_{k=0}^{m-1} \frac{|x|^k}{|y|^{n-\alpha+k}} C_k^{\frac{n-\alpha}{2}}\left(\frac{x \cdot y}{|x||y|}\right) & \text{if } |y| \geq 1, \end{cases}$$

where m is a non-negative integer, $C_k^\omega(t)$ ($\omega = \frac{n-\alpha}{2}$) is the ultraspherical (or Gegenbauer) polynomials([8]). The expression arises from the generating function for Gegenbauer polynomials

$$(1 - 2tr + r^2)^{-\omega} = \sum_{k=0}^{\infty} C_k^\omega(t) r^k, \tag{2}$$

where $|r| < 1$, $|t| \leq 1$ and $\omega > 0$. The coefficients $C_k^\omega(t)$ is called the ultraspherical (or Gegenbauer) polynomial of degree k associated with ω, the function $C_k^\omega(t)$ is a polynomial of degree k in t.

Then we define the modified Poisson kernel $P_{\alpha,m}(x,y')$ and Green function $G_{\alpha,m}(x,y)$ respectively by

$$P_{\alpha,m}(x,y') = \begin{cases} P_\alpha(x,y') & \text{if } |y'| < 1, \\ P_\alpha(x,y') - \sum_{k=0}^{m-1} \frac{C_\alpha x_n |x|^k}{|y'|^{n-\alpha+2+k}} C_k^{\frac{n-\alpha+2}{2}}\left(\frac{x \cdot y'}{|x||y'|}\right) & \text{if } |y'| \geq 1, \end{cases}$$

$$G_{\alpha,m}(x,y) = \begin{cases} E_{n,m+1}(x-y) - E_{n,m+1}(x-y^*) & \text{if } \alpha = n = 2, \\ E_{\alpha,m+1}(x-y) - E_{\alpha,m+1}(x-y^*) & \text{if } 0 < \alpha < n, \end{cases}$$

where $x, y \in \overline{H}$ and $x \neq y$.

Write

$$U_{\alpha,m}(x,\nu) = \int_{\partial H} P_{\alpha,m}(x,y')d\nu(y') \quad \text{and} \quad G_{\alpha,m}(x,\mu) = \int_H G_{\alpha,m}(x,y)d\mu(y),$$

where ν(resp. μ) is a non-negative measure on ∂H(resp. H). Here note that $U_{2,0}(x,\nu)$(resp. $G_{2,0}(x,\mu)$) is nothing but the general Poisson integral (resp. Green potential).

Let k be a non-negative Borel measurable function on $\mathbf{R}^n \times \mathbf{R}^n$, and set

$$k(y,\mu) = \int_E k(y,x)d\mu(x) \ \text{ and } \ k(\mu,x) = \int_E k(y,x)d\mu(y),$$

for a non-negative measure μ on a Borel set $E \subset \mathbf{R}^n$. As in [6], we define a capacity C_k by

$$C_k(E) = \sup \mu(\mathbf{R}^n), \quad E \subset H,$$

where the supremum is taken over all non-negative measures μ such that S_μ(the support of μ) is contained in E and $k(y,\mu) \leq 1$ for every $y \in H$.

For $\beta \leq 1$, we consider the function $k_{\alpha,\beta}$ defined by

$$k_{\alpha,\beta}(y,x) = \begin{cases} x_n^{-\beta} y_n^{-1} G_\alpha(x,y) & \text{for } x,y \in H, \\ \lim_{z\to y', z\in H} x_n^{-\beta} z_n^{-1} G_\alpha(x,z) = C_\alpha x_n^{1-\beta}|x-y'|^{\alpha-n-2} & \text{for } x \in H \text{ and } y' \in \partial H. \end{cases}$$

If $\beta = 1$, then $k_{\alpha,1}$ is extended to be continuous on $\overline{H} \times \overline{H}$ in the extended sense, where $\overline{H} = H \cup \partial H$.

Recently, Siegel and Talvila ([7]) proved the following.

Theorem 1.1. *Let f be a measurable function on $\mathbf{R}^{n-1}$ satisfying*

$$\int_{\mathbf{R}^{n-1}} \frac{|f(y')|}{(1+|y'|)^{n+m}} dy' < \infty.$$

Then the function $v(x) = \int_{\mathbf{R}^{n-1}} P_{2,m}(x,y')f(y')dy'$ satisfies

$$v \in C^2(H) \cap C^0(\overline{H}),$$

$$\Delta v = 0, \quad x \in H,$$

$$\lim_{x \to x'} v(x) = f(x') \text{ nontangentially a.e. } x' \in \partial H,$$

$$v(x) = o(x_n^{1-n}|x|^{m+n}) \quad \text{as } |x| \to \infty, \quad x \in H.$$

Our first aim is to establish the following theorem.

Theorem 1.2. *Let ν be a non-negative measure on ∂H satisfying*

$$\int_{\partial H} \frac{1}{(1+|y'|)^{n+m-\alpha+2}} d\nu(y') < \infty. \tag{3}$$

Then there exists a Borel set $E \subset H$ with properties:

$$(i) \lim_{|x|\to\infty, x \in H-E} x_n^{-\beta}|x|^{\beta-m-1} U_{\alpha,m}(x,\nu) = 0;$$

$$(ii) \sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta}}(E_i) < \infty,$$

where $E_i = \{x \in E : 2^i \leq |x| < 2^{i+1}\}$.

Theorem 1.3. *Let μ be a non-negative measure on H satisfying*

$$\int_H \frac{y_n}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) < \infty. \tag{4}$$

Then there exists a Borel set $E \subset H$ satisfying Theorem 1.2 (ii) such that

$$\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{\beta-m-1} G_{\alpha,m}(x,\mu) = 0.$$

We define the modified α-potentials on H by

$$R_\alpha(x) = U_{\alpha,m}(x,\nu) + G_{\alpha,m}(x,\mu), \tag{5}$$

where $0 < \alpha \leq n$ and ν (resp. μ) is a non-negative measure on ∂H (resp. H) satisfying (3) (resp. (4)). Clearly, $R_2(x)$ is a superharmonic function.

The following theorem follows readily from Theorems 1.2 and 1.3.

Theorem 1.4. *Let $R_\alpha(x)$ be defined by (5). Then there exists a Borel set $E \subset H$ satisfying Theorem 1.2 (ii) such that*

$$\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{\beta-m-1} R_\alpha(x) = 0.$$

Remark 1.5. *In the case $\alpha = 2$ and $0 \leq \beta \leq 1$, by using Lemma 2.5 below, we can easily show that Theorem 1.2 (ii) means that E is β-rarefied at infinity in the sense of [1]. In particular, This condition with $\alpha = 2$, $\beta = 1$ (resp. $\alpha = 2$, $\beta = 0$) means that E is minimally thin at infinity (resp. rarefied at infinity) in the sense of [3].*

Next we are concerned with the best possibility of Theorem 1.4 as to the size of the exceptional set.

Proposition 1.6. *Let $E \subset H$ be a Borel set satisfying Theorem 1.2 (ii) and $R_\alpha(x)$ be defined by (5). Then we can find a non-negative measure λ defined on $\overline{H}$ satisfying*

$$\int_{\overline{H}} \frac{1}{(1+|y|)^{n+m-\alpha+2}} d\lambda(y) < \infty,$$

such that

$$\limsup_{|x|\to\infty, x\in E} x_n^{-\beta}|x|^{\beta-m-1} R_\alpha(x) = \infty,$$

where $d\lambda(y') = d\nu(y')$ ($y' \in \partial H$) and $d\lambda(y) = y_n d\mu(y)$ ($y \in H$).

2. Some Lemmas

Throughout this paper, let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 2.1. *Let m be a non-negative integer and $M > 0$.*

(i) If $1 \leq |y'| \leq \frac{|x|}{2}$, then $|P_{\alpha,m}(x,y')| \leq M\frac{x_n|x|^{m-1}}{|y'|^{n+m-\alpha+1}}$.

(ii) If $|y'| \geq 2|x|$ and $|y'| \geq 1$, then $|P_{\alpha,m}(x,y')| \leq M\frac{x_n|x|^m}{|y'|^{n+m-\alpha+2}}$.

Lemma 2.2. *There exists a positive constant M such that $G_\alpha(x,y) \leq M\frac{x_n y_n}{|x-y|^{n-\alpha+2}}$, where $0 < \alpha \leq n$, $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in H.*

This can be proved by simple calculation.

Lemma 2.3. *Gegenbauer polynomials have the following properties:*
(i) $|C_k^{\omega}(t)| \leq C_k^{\omega}(1) = \frac{\Gamma(2\omega+k)}{\Gamma(2\omega)\Gamma(k+1)}, \ |t| \leq 1;$
(ii) $\frac{d}{dt}C_k^{\omega}(t) = 2\omega C_{k-1}^{\omega+1}(t), \ k \geq 1;$
(iii) $\sum_{k=0}^{\infty} C_k^{\omega}(1)r^k = (1-r)^{-2\omega};$
(iv) $|C_k^{\frac{n-\alpha}{2}}(t) - C_k^{\frac{n-\alpha}{2}}(t^*)| \leq (n-\alpha)C_{k-1}^{\frac{n-\alpha+2}{2}}(1)|t-t^*|, \ |t| \leq 1, \ |t^*| \leq 1.$

Proof. (i) and (ii) can be derived from [8]. (iii) follows by taking $t = 1$ in (2); (iv) follows by (1), (2) and the Mean Value Theorem for Derivatives. □

Lemma 2.4. *For x, $y \in \mathbf{R}^n$ $(\alpha = n = 2)$, we have the following properties:*
(i) $|\Im \sum_{k=0}^{m} \frac{x^k}{y^{k+1}}| \leq \sum_{k=0}^{m-1} \frac{2^k x_n |x|^k}{|y|^{k+2}};$
(ii) $|\Im \sum_{k=0}^{\infty} \frac{x^{k+m+1}}{y^k}| \leq 2^{m+1} x_n |x|^m;$
(iii) $|G_{n,m}(x,y) - G_n(x,y)| \leq M \sum_{k=1}^{m} \frac{k x_n y_n |x|^{k-1}}{|y|^{k+1}};$
(iv) $|G_{n,m}(x,y)| \leq M \sum_{k=m+1}^{\infty} \frac{k x_n y_n |x|^{k-1}}{|y|^{k+1}};$

The following lemma can be proved by using Fuglede (see [2, Théorèm 7.8],)

Lemma 2.5. *For any Borel set E in H, we have $C_{k_{\alpha,\beta}}(E) = \hat{C}_{k_{\alpha,\beta}}(E)$, where $\hat{C}_{k_{\alpha,\beta}}(E) = \inf \lambda(\overline{H})$, the infimum being taken over all non-negative measures λ on $\overline{H}$ such that $k_{\alpha,\beta}(\lambda, x) \geq 1$ for every $x \in E$.*

3. Proof of Theorem 1.2

For any $\varepsilon_1 > 0$, there exists $R_{\varepsilon_1} > 2$ such that

$$\int_{\{y' \in \partial H, |y'| \geq R_{\varepsilon_1}\}} \frac{1}{(1+|y'|)^{n+m-\alpha+2}} d\nu(y') < \epsilon_1.$$

For fixed $x \in H$ and $|x| \geq 2R_{\varepsilon_1}$, we write

$$\begin{aligned} U_{\alpha,m}(x,\nu) &= \int_{G_1} P_{\alpha,m}(x,y')d\nu(y') + \int_{G_2} P_{\alpha,m}(x,y')d\nu(y'))u(y)d\sigma(y) \\ &\quad + \int_{G_3} [P_{\alpha,m}(x,y') - P_\alpha(x,y')]d\nu(y') + \int_{G_4} P_\alpha(x,y')d\nu(y') + \int_{G_5} P_{\alpha,m}(x,y')d\nu(y') \\ &= U_1(x) + U_2(x) + U_3(x) + U_4(x) + U_5(x), \end{aligned}$$

where $G_1 = \{y' \in \partial H : |y'| < 1\}$, $\quad G_2 = \{y' \in \partial H : 1 \leq |y'| < \frac{|x|}{2}\}$,
$G_3 = G_4 = \{y' \in \partial H : \frac{|x|}{2} \leq |y'| < 2|x|\}$ and $G_5 = \{y' \in \partial H : |y'| \geq 2|x|\}$.
First note that

$$\begin{aligned} |U_1(x)| &\leq x_n(\frac{|x|}{2})^{\alpha-n-2} \int_{G_1} d\nu(y') \\ &\leq M x_n |x|^{\alpha-n-2}. \end{aligned} \tag{6}$$

By Lemma 2.1 (i), we have

$$\begin{aligned} |U_2(x)| &\leq M x_n |x|^{m-1} \int_{G_2} \frac{1}{|y'|^{n+m-\alpha+1}} d\nu(y') \\ &\leq M x_n |x|^m \int_{G_2} \frac{1}{|y'|^{n+m-\alpha+2}} d\nu(y'). \end{aligned} \tag{7}$$

Write

$$U_2(x) = U_{21}(x) + U_{22}(x),$$

where

$$U_{21}(x) = \int_{G_2 \bigcap B_{n-1}(0,R_{\epsilon_1})} P_{\alpha,m}(x,y')d\nu(y') \quad \text{and} \quad U_{22}(x) = \int_{G_2 - B_{n-1}(0,R_{\epsilon_1})} P_{\alpha,m}(x,y')d\nu(y').$$

If $|x| > 2R_{\epsilon_1}$, then

$$\begin{aligned} |U_{21}(x)| &\leq Mx_n|x|^{m-1}R_{\epsilon_1}\int_{B_{n-1}(0,R_{\epsilon_1})-B_{n-1}(0,1)} \frac{1}{|y'|^{n+m-\alpha+2}}d\nu(y') \\ &\leq MR_{\epsilon_1}x_n|x|^{m-1}. \end{aligned}$$

Moreover, by (7) we have

$$|U_{22}(x)| \leq M\epsilon_1 x_n|x|^m.$$

That is

$$|U_2(x)| \leq M\epsilon_1 x_n|x|^m. \tag{8}$$

We have by Lemma 2.3 (iii)

$$\begin{aligned} |U_3(x)| &\leq M\int_{G_3}\sum_{k=0}^{m-1}\frac{x_n|x|^k}{|y'|^{n-\alpha+2+k}}C_k^{\frac{n-\alpha+2}{2}}(1)d\nu(y') \\ &\leq Mx_n|x|^m\sum_{k=0}^{m-1}\frac{1}{2^k}C_k^{\frac{n-\alpha+2}{2}}(1)\int_{G_3}\frac{1}{|y'|^{n+m-\alpha+2}}d\nu(y') \\ &\leq M\varepsilon_1 x_n|x|^m. \end{aligned} \tag{9}$$

By Lemma 2.1 (ii), we obtain

$$\begin{aligned} |U_5(x)| &\leq Mx_n|x|^m\int_{G_5}\frac{1}{|y'|^{n+m-\alpha+2}}d\nu(y') \\ &\leq M\epsilon_1 x_n|x|^m. \end{aligned} \tag{10}$$

Note that $U_4(x) = x_n^\beta \int_{G_4} k_{\alpha,\beta}(y',x)d\nu(y')$. In view of (3), we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and $\sum_{i=1}^\infty a_i b_i < \infty$, where

$$b_i = \int_{\{y'\in\partial H: 2^{i-1}<|y'|<2^{i+2}\}}\frac{1}{|y'|^{n+m-\alpha+2}}d\nu(y').$$

Consider the sets

$$E_i = \{x \in H : 2^i \leq |x| < 2^{i+1}, x_n^{-\beta}U_4(x) \geq a_i^{-1}2^{-i(\beta-m-1)}\},$$

for $i = 1, 2, \cdots$. If μ is a non-negative measure on H such that $S_\mu \subset E_i$ and $k_{\alpha,\beta}(y',\mu) \leq 1$ for $y' \in \partial H$, then we have

$$\begin{aligned} \int_H d\mu &\leq a_i 2^{i(\beta-m-1)}\int x_n^{-\beta}U_4(x)d\mu(x) \\ &\leq Ma_i 2^{i(\beta-m-1)}\int_{\{y'\in\partial H:2^{i-1}<|y'|<2^{i+2}\}} k_{\alpha,\beta}(y',\mu)d\nu(y') \\ &\leq Ma_i 2^{i(\beta-m-1)}\int_{\{y'\in\partial H:2^{i-1}<|y'|<2^{i+2}\}} d\nu(y') \\ &\leq M4^{m+n-\alpha+2}2^{i(n-\alpha+\beta+1)}a_i b_i \end{aligned}$$

So that

$$C_{k_{\alpha,\beta}}(E_i) \le M4^{m+n-\alpha+2}2^{i(n-\alpha+\beta+1)}a_i b_i,$$

which yields

$$\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta}}(E_i) < \infty.$$

Setting $E = \bigcup_{i=1}^{\infty} E_i$, we see that Theorem 1.2 (ii) is satisfied and

$$\limsup_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{\beta-m-1}U_4(x) \le \limsup_{i\longrightarrow\infty} 2^{|\beta-m-1|}2^{i(\beta-m-1)}a_i^{-1}2^{-i(\beta-m-1)} = 0. \tag{11}$$

Combining (6), (8), (9), (10) and (11), we complete the proof of Theorem 1.2.

4. Proof of Theorem 1.3

For any $\varepsilon_2 > 0$, there exists $R_{\varepsilon_2} > 2$ such that

$$\int_{\{y\in H, |y|\ge R_{\varepsilon_2}\}} \frac{y_n}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) < \epsilon_2.$$

For fixed $x \in H$ and $|x| \ge 2R_{\varepsilon_2}$, we write

$$\begin{aligned} G_{\alpha,m}(x,\mu) &= \int_{H_1} G_\alpha(x,y)d\mu(y) + \int_{H_2} G_\alpha(x,y)d\mu(y) + \int_{H_3} [G_{\alpha,m}(x,y) - G_\alpha(x,y)]d\mu(y) \\ &\quad + \int_{H_4} G_{\alpha,m}(x,y)d\mu(y) + \int_{H_5} G_\alpha(x,y)d\mu(y) + \int_{H_6} [G_{\alpha,m}(x,y) - G_\alpha(x,y)]d\mu(y) \\ &\quad + \int_{H_7} G_{\alpha,m}(x,y)d\mu(y) \\ &= V_1(x) + V_2(x) + V_3(x) + V_4(x) + V_5(x) + V_6(x) + V_7(x), \end{aligned}$$

where $H_1 = \{y \in H : |y| \ge R_{\epsilon_2}, |x-y| \le \frac{|x|}{2}\}$, $H_2 = \{y \in H : |y| \ge R_{\epsilon_2}, \frac{|x|}{2} < |x-y| \le 3|x|\}$,
$H_3 = \{y \in H : |y| \ge R_{\epsilon_2}, |x-y| \le 3|x|\}$, $H_4 = \{y \in H : |y| \ge R_{\epsilon_2}, |x-y| > 3|x|\}$,
$H_5 = H_6 = \{y \in H : 1 \le |y| < R_{\epsilon_2}\}$ and $H_7 = \{y \in H : |y| < 1\}$.

We distinguish the following two cases.

Case 1. $0 < \alpha < n$.

Note that

$$V_1(x) = x_n^\beta \int_{H_1} k_{\alpha,\beta}(y,x)d\lambda(y),$$

where $d\lambda(y) = y_n d\mu(y)$.

By the lower semi-continuity of $k_{\alpha,\beta}(y,x)$, then we can prove the following fact in the same way as $U_4(x)$ in the proof of Theorem 1.2.

$$\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{\beta-m-1}V_1(x) = 0. \tag{12}$$

where $E = \bigcup_{i=1}^{\infty} E_i$, $E_i = \{x \in E : 2^i \le |x| < 2^{i+1}\}$ and $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta}}(E_i) < \infty$.

Moreover by Lemma 2.2

$$\begin{aligned} |V_2(x)| &\le Mx_n \int_{H_2} \frac{y_n}{|x-y|^{n-\alpha+2}} d\mu(y) \\ &\le Mx_n|x|^m \int_{H_2} \frac{y_n}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\ &\le M\epsilon_2 x_n |x|^m. \end{aligned} \tag{13}$$

Note that $C_0^{\omega}(t) \equiv 1$. By (iii) and (iv) in Lemma 2.3, we take $t = \frac{x \cdot y}{|x||y|}$, $t^* = \frac{x \cdot y^*}{|x||y^*|}$ in Lemma 2.3 (iv) and obtain

$$\begin{aligned}
|V_3(x)| &\leq \int_{H_3} \sum_{k=1}^{m} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{\frac{n-\alpha+2}{2}}(1) \frac{x_n y_n}{|x||y|} \frac{2|y|^{n+m-\alpha+2}}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\
&\leq M x_n |x|^m \sum_{k=1}^{m} \frac{1}{4^{k-1}} C_{k-1}^{\frac{n-\alpha+2}{2}}(1) \int_{H_3} \frac{y_n}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\
&\leq M \varepsilon_2 x_n |x|^m. \qquad (14)
\end{aligned}$$

Similarly, we have by (iii) and (iv) in Lemma 2.3

$$\begin{aligned}
|V_4(x)| &\leq \int_{H_4} \sum_{k=m+1}^{\infty} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{\frac{n-\alpha+2}{2}}(1) \frac{x_n y_n}{|x||y|} \frac{|y|^{n+m-\alpha+2}}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\
&\leq M x_n |x|^m \sum_{k=m+1}^{\infty} \frac{1}{2^{k-1}} C_{k-1}^{\frac{n-\alpha+2}{2}}(1) \int_{H_4} \frac{y_n}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\
&\leq M \varepsilon_2 x_n |x|^m. \qquad (15)
\end{aligned}$$

By Lemma 2.2, we have

$$|V_5(x)| \leq M \frac{R_{\epsilon_2}^{n-\alpha+2} x_n}{|x|^{n-\alpha+2}}. \qquad (16)$$

Similarly as $V_3(x)$, we obtain

$$\begin{aligned}
|V_6(x)| &\leq \int_{H_6} \sum_{k=1}^{m} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{\frac{n-\alpha+2}{2}}(1) \frac{x_n y_n}{|x||y|} \frac{2|y|^{n+m-\alpha+2}}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\
&\leq M x_n \sum_{k=1}^{m} C_{k-1}^{\frac{n-\alpha+2}{2}}(1) |x|^{k-1} R_{\epsilon}^{m-k+1} \int_{H_6} \frac{y_n}{(1+|y|)^{n+m-\alpha+2}} d\mu(y) \\
&\leq M R_{\epsilon_2}^{m} x_n |x|^{m-1}. \qquad (17)
\end{aligned}$$

Finally, by Lemma 2.2, we have

$$|V_7(x)| \leq M \frac{x_n}{|x|^{n-\alpha+2}}. \qquad (18)$$

Combining (12), (13), (14), (15), (16), (17) and (18), we prove the case 1.

Case 2. $\alpha = n = 2$.

In this case, $V_1(x)$, $V_2(x)$, $V_5(x)$ and $V_7(x)$ can be proved similarly in case 1. (3.7), (3.8), (3.11) and (3.13) still hold.

Moreover we have by Lemma 2.4 (iii)

$$\begin{aligned}
|V_3(x)| &\leq \int_{H_3} \sum_{k=1}^{m} \frac{k x_n y_n |x|^{k-1}}{|y|^{k+1}} \frac{2|y|^{m+2}}{y_n} \frac{y_n}{(1+|y|)^{m+2}} d\mu(y) \\
&\leq M x_n |x|^m \sum_{k=1}^{m} \frac{k}{4^{k-1}} \int_{H_3} \frac{y_n}{(1+|y|)^{m+2}} d\mu(y) \\
&\leq M \varepsilon_2 x_n |x|^m. \qquad (19)
\end{aligned}$$

By Lemma 2.4 (iv), we have

$$\begin{aligned} |V_4(x)| &\le \int_{H_4} \sum_{k=m+1}^{\infty} \frac{kx_n y_n |x|^{k-1}}{|y|^{k+1}} \frac{2|y|^{m+2}}{y_n} \frac{y_n}{(1+|y|)^{m+2}} d\mu(y) \\ &\le Mx_n|x|^m \sum_{k=m+1}^{\infty} \frac{k}{2^{k-1}} \int_{H_4} \frac{y_n}{(1+|y|)^{m+2}} d\mu(y) \\ &\le M\varepsilon_2 x_n |x|^m. \end{aligned} \tag{20}$$

Similarly as $V_3(x)$, we have

$$|V_6(x)| \le MR_{\epsilon_2}^m x_n |x|^{m-1}. \tag{21}$$

Combining (12), (13), (16), (18), (19), (20) and (21), we prove the case 2.
Hence we complete the proof of Theorem 1.3.

5. Proof of Proposition 1.6

We prove the case $0 < \alpha < n$, because the case $\alpha = n = 2$ can be proved similarly. Further, we only need prove

$$\limsup_{|x|\to\infty, x\in E} x_n^{1-\beta} |x|^{\beta-m-1} k_{\alpha,1}(\lambda, x) = \infty.$$

By Lemma 2.5, for each i we can find λ_i on $\overline{H}$ such that $\lambda_i(\overline{H}) < C_{k_{\alpha,\beta}}(E_i) + 1$ and $k_{\alpha,\beta}(\lambda_i, x) \ge 1$ on E_i. Denote by λ_i' the restriction of λ_i to the set $\{y \in \overline{H} : 2^{i-1} < |y| < 2^{i+2}\}$.

Set $\lambda = \sum_{i=1}^{\infty} a_i 2^{-i(\beta-m-1)} \lambda_i'$, where $\{a_i\}$ is a sequence of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ but $\sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta+1)} \{C_{k_{\alpha,\beta}}(E_i) + 1\} < \infty$. Then

$$\begin{aligned} \int_{\overline{H}} \frac{1}{(1+|y|)^{n+m-\alpha+2}} d\lambda(y) &= \sum_{i=1}^{\infty} a_i 2^{-i(\beta-m-1)} \int_{\overline{H}} \frac{1}{(1+|y|)^{n+m-\alpha+2}} d\lambda_i'(y) \\ &\le M \sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta+1)} \{C_{k_{\alpha,\beta}}(E_i) + 1\} < \infty. \end{aligned}$$

If $x \in E_i$, then

$$\begin{aligned} k_{\alpha,\beta}(\lambda_i', x) &\ge 1 - \int_{\{y\in\overline{H}:|y|\le 2^{i-1}\}\cap\{y\in\overline{H}:|y|\ge 2^{i+2}\}} k_{\alpha,\beta}(y,x) d\lambda_i(y) \\ &\ge 1 - M2^{-i(n-\alpha+\beta+1)} \{C_{k_{\alpha,\beta}}(E_i) + 1\}. \end{aligned}$$

We also have

$$x_n^{1-\beta} |x|^{\beta-m-1} k_{\alpha,1}(\lambda, x) = |x|^{\beta-m-1} k_{\alpha,\beta}(\lambda, x) \ge a_i k_{\alpha,\beta}(\lambda_i', x),$$

which implies that

$$\limsup_{|x|\to\infty, x\in E} x_n^{1-\beta} |x|^{\beta-m-1} k_{\alpha,1}(\lambda, x) = \infty.$$

Thus λ satisfies all the conditions in the Proposition 1.6.

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