

Invariance of green equilibrium measure on the domain

Stamatis Pouliasis^a

^aDepartment of Mathematics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece

Abstract. We prove that the Green equilibrium measure and the Green equilibrium energy of a compact set K relative to the domains D and Ω are the same if and only if D is nearly equal to Ω , for a wide class of compact sets K . Also, we prove that equality of Green equilibrium measures arises if and only if the one domain is related with a level set of the Green equilibrium potential of K relative to the other domain.

1. Introduction

Consider the following inverse problem in classical potential theory: Suppose K is a compact set in a domain D . Do the equilibrium measure and the equilibrium energy of K relative to D characterize the domain D ? More generally, if we know the energy and the measure on K , what can we conclude about the ambient domain D ? We proceed to a rigorous formulation of the problem.

Let D be a Greenian open subset of \mathbb{R}^n and denote by $G_D(x, y)$ the Green function of D . Also let K be a compact subset of D . The *Green equilibrium energy* of K relative to D is defined by

$$I(K, D) = \inf_{\mu} \iint G_D(x, y) d\mu(x) d\mu(y),$$

where the infimum is taken over all unit Borel measures μ supported on K . The *Green capacity* of K relative to D is the number

$$C_D(K) = \frac{1}{I(K, D)}.$$

When $I(K, D) < +\infty$, the unique unit Borel measure μ_K for which the above infimum is attained is the *Green equilibrium measure* and the function

$$U_{\mu_K}^D(x) = \int G_D(x, y) d\mu_K(y)$$

is the *Green equilibrium potential* of K relative to D . See e.g. [6, p. 174] or [2, p. 134]. We denote by $C_2(E)$ the logarithmic ($n = 2$) or Newtonian ($n \geq 3$) capacity of the Borel set E . When $E \subset D$, the equalities $C_D(E) = 0$ and $C_2(E) = 0$ are equivalent; [6, p. 174]. If two Borel sets $A, B \subset \mathbb{R}^n$ differ only on a set of zero capacity

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Email address: spoulias@math.auth.gr (Stamatis Pouliasis)

(namely, $C_2(A \setminus B) = C_2(B \setminus A) = 0$), then we say that A, B are *nearly everywhere equal* and write $A \stackrel{n.e.}{=} B$. Nearly everywhere equal sets have the same potential theoretic behavior.

Suppose that D is a domain and E is a compact subset of $D \setminus K$. If $C_2(E) = 0$, then the Green capacity and the Green equilibrium measure of K relative to the open sets D and $D \setminus E$ are the same. That happens because the sets D and $D \setminus E$ are nearly everywhere equal. The following question arises naturally:

Does there exist a Greenian domain Ω , not nearly everywhere equal to D , such that the Green capacity and the Green equilibrium measure of K relative to D and Ω are the same?

In general the answer is positive. However, we shall show that for a large class of compact sets (for example for compact sets with nonempty interior) the answer is negative. That is, given a compact set K with nonempty interior, the Green equilibrium measure μ_K and the positive number $C_D(K)$ completely characterizes D from the point of view of potential theory.

There is a second question:

Suppose that the Green equilibrium measures of K relative to D and Ω are the same. How are the sets D and Ω related?

In that case, the boundary of the domain relative to which K has smaller Green energy is a level set of the Green equilibrium potential of K relative to the other domain.

In our main result we give an answer to the above questions, for compact sets with nonempty interior.

Theorem 1. *Let K be a compact subset of \mathbb{R}^n with nonempty interior. Let D_1, D_2 be two Greenian subdomains of \mathbb{R}^n that contain K and let μ_1 and μ_2 be the Green equilibrium measures of K relative to D_1 and D_2 , respectively. Then*

- (i) $\mu_1 = \mu_2$ and $I(K, D_1) = I(K, D_2)$ if and only if $D_1 \stackrel{n.e.}{=} D_2$.
- (ii) If $I(K, D_1) < I(K, D_2)$ and the set

$$\tilde{D}_2 = \{x \in D_2 : U_{\mu_2}^{D_2}(x) > I(K, D_2) - I(K, D_1)\}$$

contains K , we have

$$\mu_1 = \mu_2 \text{ if and only if } D_1 \stackrel{n.e.}{=} \tilde{D}_2.$$

Moreover, we shall show that Theorem 1 is valid for a much wider class of compact sets (see Remark 4.1).

In the following section we introduce the concepts of Green potential theory that are needed for our results. Theorem 1 is proved in section 3. In section 4 we examine the case of compact sets K with empty interior, we give some counterexamples and we pose a conjecture and a question.

2. Background Material

We denote by $B(x, r)$ and $S(x, r)$ the open ball and the sphere with center x and radius r in \mathbb{R}^n , respectively. For a set $E \subset \mathbb{R}^n$, the interior, the closure and the boundary of E are denoted by E°, \bar{E} and ∂E , respectively. The surface area of the unit sphere $S(0, 1)$ of \mathbb{R}^n is denoted by σ_n .

If a property holds for all the points of a set A apart from a set of zero capacity we will say that the property holds for *nearly every (n.e.)* point of A .

The logarithmic ($n = 2$) or Newtonian ($n \geq 3$) kernel is denoted by

$$\mathcal{K}(x, y) := \begin{cases} \log \frac{1}{|x-y|}, & n = 2, \\ \frac{1}{|x-y|^{n-2}}, & n \geq 3, \end{cases}$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. If $D \subset \mathbb{R}^n$ is a Greenian open set then

$$G_D(x, y) = \mathcal{K}(x, y) + H(x, y)$$

where, for fixed y , $H(\cdot, y)$ is the greatest harmonic minorant of $\mathcal{K}(\cdot, y)$ on D . If μ is a measure with compact support on D , the Green potential of μ is the function

$$U_\mu^D(x) = \int G_D(x, y) d\mu(y), \quad x \in D.$$

Then (see [2, pp. 96-104]) U_μ^D is superharmonic on D , harmonic on $D \setminus \text{supp}(\mu)$ and the Riesz measure of U_μ^D is proportional to μ ; more precisely, $\Delta U_\mu^D = -\kappa_n \mu$, where Δ is the distributional Laplacian on D and

$$\kappa_n = \begin{cases} \sigma_2, & n = 2, \\ (n-2)\sigma_n, & n \geq 3. \end{cases}$$

We shall need the following result for the boundary behavior of a Green potential.

Theorem 2.1. [2, p. 148] *Let Ω be a Greenian open subset of \mathbb{R}^n and let μ be a Borel measure with compact support in Ω . Then*

$$\lim_{\Omega \ni x \rightarrow \xi} U_\mu^\Omega(x) = 0,$$

for nearly every point $\xi \in \partial\Omega$.

Also, if $U_{\mu_K}^D$ is the Green equilibrium potential of a compact set K relative to D , then the equality

$$U_{\mu_K}^D(x) = I(K, D) \tag{1}$$

holds for all $x \in K^\circ$ and for nearly every point $x \in K$; ([5, p. 138]).

The following theorem gives a geometric interpretation of the fact that sets of zero capacity are negligible.

Theorem 2.2. ([2, p. 125]). *Let $\Omega \subset \mathbb{R}^n$ be a domain and E a relatively closed subset of Ω with $C_2(E) = 0$. Then the set $\Omega \setminus E$ is connected.*

Remark 2.1. When the open set $D \setminus K$ is connected, it is called a *condenser* and the sets $\mathbb{R}^n \setminus D$ and K are called the *plates* of the condenser; see [1, 3]. It is well known that Green potential theory and condenser theory are equivalent; see e.g. [4, p. 700-701] or [7, p. 393]. Therefore, Theorem 1 can be restated as a theorem about condensers.

3. Dependence of Green equilibrium measure on the domain

Let D be a Greenian open subset of \mathbb{R}^n and K a compact subset of D . First we shall examine the open sets bounded by the level surfaces of the Green equilibrium potential of K relative to D . The Green equilibrium measure is the same for each of these open sets.

Lemma 3.1. *Let D be a Greenian open subset of \mathbb{R}^n and let K be a compact subset of D that has finite Green equilibrium energy relative to D . Also let μ_K be the Green equilibrium measure of K relative to D and for $0 < \alpha < I(K, D)$, let*

$$D_\alpha = \{x \in D : U_{\mu_K}^D(x) > \alpha\}.$$

If $K \subset D_\alpha$ and μ_α is the Green equilibrium measure of K relative to D_α , then $\mu_\alpha = \mu_K$ and

$$I(K, D_\alpha) = I(K, D) - \alpha.$$

Proof. The Green equilibrium potential $U_{\mu_K}^D$ is a superharmonic function on D so, in particular, it is lower semicontinuous on D . Therefore the sets

$$D_\alpha = \{x \in D : U_{\mu_K}^D(x) > \alpha\}$$

are open for all $\alpha \in (0, I(K, D))$.

Choose $\alpha \in (0, I(K, D))$ such that $K \subset D_\alpha$. Consider the function $U : \overline{D_\alpha} \mapsto \mathbb{R}$ with

$$U(x) = U_{\mu_K}^D(x) - \alpha.$$

Then U is harmonic on $D_\alpha \setminus K$. Also, by property (1) of the Green equilibrium potentials, U has boundary values $(I(K, D) - \alpha)$ n.e. on ∂K and 0 on ∂D_α . Therefore, by the extended maximum principle and the boundary behavior of the Green equilibrium potential $U_{\mu_\alpha}^{D_\alpha}$ on $D_\alpha \setminus K$,

$$U_{\mu_\alpha}^{D_\alpha}(x) = \frac{I(K, D_\alpha)}{I(K, D) - \alpha} U(x).$$

Applying the distributional Laplacian on D_α we get

$$\Delta(U_{\mu_\alpha}^{D_\alpha}) = \Delta\left(\frac{I(K, D_\alpha)}{I(K, D) - \alpha}(U_{\mu_K}^D - \alpha)\right),$$

so

$$-\kappa_n \mu_\alpha = -\kappa_n \frac{I(K, D_\alpha)}{I(K, D) - \alpha} \mu_K$$

and

$$\mu_\alpha = \frac{I(K, D_\alpha)}{I(K, D) - \alpha} \mu_K.$$

Since μ_α and μ_K are unit measures, we conclude that $\mu_\alpha = \mu_K$ and $I(K, D_\alpha) = I(K, D) - \alpha$. \square

In order to proof Theorem 1 we shall need some more lemmas. In the following lemma we show that the difference of two Green potentials of the same measure is a harmonic function.

Lemma 3.2. *Let D_1 and D_2 be two Greenian open subsets of \mathbb{R}^n such that $D_1 \cap D_2 \neq \emptyset$. Also, let μ be a Borel measure with compact support in $D_1 \cap D_2$ such that the Green potentials $U_\mu^{D_1}$ and $U_\mu^{D_2}$ are finite on $D_1 \cap D_2$. Then the difference*

$$U_\mu^{D_1} - U_\mu^{D_2}$$

is a harmonic function on $D_1 \cap D_2$.

Proof. Let

$$G_i(x, y) = \mathcal{K}(x, y) + H_i(x, y)$$

be the Green function of D_i , $i = 1, 2$, respectively. Then

$$\begin{aligned} U_\mu^{D_1}(x) - U_\mu^{D_2}(x) &= \int_K \mathcal{K}(x, y) d\mu(y) + \int_K H_1(x, y) d\mu(y) \\ &\quad - \int_K \mathcal{K}(x, y) d\mu(y) - \int_K H_2(x, y) d\mu(y) \\ &= \int_K [H_1(x, y) - H_2(x, y)] d\mu(y). \end{aligned}$$

The difference $H_1(x, y) - H_2(x, y)$ is a harmonic function on $D_1 \cap D_2$ on both variables x and y , separately. Therefore,

$$x \mapsto U_\mu^{D_1}(x) - U_\mu^{D_2}(x) = \int_K [H_1(x, y) - H_2(x, y)] d\mu(y)$$

is a harmonic function on $D_1 \cap D_2$; see ([5, Lemma 6.7, p. 103]). \square

We shall need the fact that particular parts of the boundary of an open set belong to the reduced kernel [6, p. 164] of the boundary.

Lemma 3.3. *Let Ω be an open subset of \mathbb{R}^n . Suppose that $\mathbb{R}^n \setminus \overline{\Omega} \neq \emptyset$ and let O be a connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Then for all $\xi \in \partial O$ and for all $r > 0$,*

$$C_2(B(\xi, r) \cap \partial O) > 0.$$

Proof. Let $\xi \in \partial O$ and $r > 0$. Then $B(\xi, r) \setminus \partial O$ intersects Ω and O , so it is not connected. Then, since $B(\xi, r) \cap \partial O$ is a relatively closed subset of $B(\xi, r)$, it follows from Theorem 2.2 that

$$C_2(B(\xi, r) \cap \partial O) > 0.$$

□

The next lemma gives a characterization of nearly everywhere equal sets: the Green potential of any measure with compact support is the same.

Lemma 3.4. *Let D_1 and D_2 be two Greenian domains of \mathbb{R}^n . Then $D_1 \stackrel{n.e.}{=} D_2$ if and only if there exists a Borel measure μ with compact support in $D_1 \cap D_2$ such that $U_\mu^{D_1}, U_\mu^{D_2}$ are non constant functions on each connected component of $D_1 \cap D_2$ and $U_\mu^{D_1} = U_\mu^{D_2}$ on an open ball $B \subset D_1 \cap D_2$.*

Proof. Suppose that $D_1 \stackrel{n.e.}{=} D_2$. Then $G_{D_1}(x, y) = G_{D_2}(x, y)$ for all $x, y \in D_1 \cap D_2$ ([2, Corollary 5.2.5, p. 128]). Therefore

$$U_\mu^{D_1}(x) = \int G_{D_1}(x, y) d\mu(y) = \int G_{D_2}(x, y) d\mu(y) = U_\mu^{D_2}(x),$$

for all $x \in D_1 \cap D_2$ and for all Borel measures μ with compact support in $D_1 \cap D_2$.

Conversely, suppose that μ is a Borel measure with compact support in $D_1 \cap D_2$ such that $U_\mu^{D_1}, U_\mu^{D_2}$ are non constant functions on each connected component of $D_1 \cap D_2$ and $U_\mu^{D_1} = U_\mu^{D_2}$ on a ball $B \subset D_1 \cap D_2$. By Lemma 3.2, the function

$$u = U_\mu^{D_1} - U_\mu^{D_2}$$

is harmonic on $D_1 \cap D_2$ and vanishes on the ball B . Therefore, by the identity principle ([2, Lemma 1.8.3, p. 27]), u vanishes on the connected component A of $D_1 \cap D_2$ that contains B . That is, $U_\mu^{D_1} = U_\mu^{D_2}$ on A .

Suppose that $C_2(D_1 \setminus A) > 0$. We shall show that $C_2(D_1 \cap \partial A) > 0$. Consider the decomposition

$$D_1 \setminus A = (D_1 \cap \partial A) \cup (D_1 \cap (\mathbb{R}^n \setminus \overline{A})).$$

If $D_1 \cap (\mathbb{R}^n \setminus \overline{A}) = \emptyset$, then

$$C_2(D_1 \cap \partial A) = C_2(D_1 \setminus A) > 0.$$

Suppose that $D_1 \cap (\mathbb{R}^n \setminus \overline{A}) \neq \emptyset$. Let O be a connected component of $\mathbb{R}^n \setminus \overline{A}$ that intersects D_1 . Since D_1 intersects O and $\mathbb{R}^n \setminus O$, $D_1 \cap \partial O \neq \emptyset$. Let $\xi \in D_1 \cap \partial O$ and $r > 0$ such that $B(\xi, r) \subset D_1$. By Lemma 3.3,

$$C_2(D_1 \cap \partial A) \geq C_2(B(\xi, r) \cap \partial O) > 0.$$

Therefore, in any case $C_2(D_1 \cap \partial A) > 0$. Then, since $\partial A \subset (\partial D_1 \cup \partial D_2)$ and $D_1 \cap \partial A \subset D_1$, we have that $D_1 \cap \partial A$ is a subset of ∂D_2 with positive capacity. From Theorem 2.1 we have that there exist $\xi_0 \in D_1 \cap \partial A$ such that

$$\lim_{A \ni x \rightarrow \xi_0} U_\mu^{D_2}(x) = \lim_{D_2 \ni x \rightarrow \xi_0} U_\mu^{D_2}(x) = 0.$$

Since $\text{supp}(\mu)$ is a compact subset of $D_1 \cap D_2$, $\xi_0 \in D_1$ and $\xi_0 \in \partial D_2$, $U_\mu^{D_1}$ is harmonic on an open neighborhood of ξ_0 . So

$$\lim_{A \ni x \rightarrow \xi_0} U_\mu^{D_1}(x) = \lim_{D_1 \ni x \rightarrow \xi_0} U_\mu^{D_1}(x) = U_\mu^{D_1}(\xi_0) > 0.$$

Then, since $U_\mu^{D_1} = U_\mu^{D_2}$ on A , we obtain

$$0 = \lim_{A \ni x \rightarrow \xi_0} U_\mu^{D_2}(x) = \lim_{A \ni x \rightarrow \xi_0} U_\mu^{D_1}(x) > 0,$$

which is a contradiction.

Therefore $C_2(D_1 \setminus A) = 0$. Since $A \subset D_1$, we get $D_1 \stackrel{n.e.}{=} A$. In a similar way we can show that $D_2 \stackrel{n.e.}{=} A$. Therefore, in particular, $D_1 \stackrel{n.e.}{=} D_2$. \square

We proceed to prove Theorem 1.

Proof of Theorem 1. (i) Suppose that $D_1 \stackrel{n.e.}{=} D_2$. Then the relations $\mu_1 = \mu_2$ and $I(K, D_1) = I(K, D_2)$ follow from the equality

$$G_{D_1}(x, y) = G_{D_2}(x, y)$$

for all $x, y \in K$ ([2, Corollary 5.2.5, p. 128]).

Conversely, suppose that $\mu_1 = \mu_2$ and $I(K, D_1) = I(K, D_2)$. By property (1) of the Green equilibrium potentials,

$$U_{\mu_1}^{D_1}(x) - U_{\mu_2}^{D_2}(x) = I(K, D_1) - I(K, D_2) = 0,$$

for all $x \in K^\circ$. Then, by Lemma 3.4, $D_1 \stackrel{n.e.}{=} D_2$.

(ii) Let $\tilde{\mu}_2$ be the Green equilibrium measure of K relative to \tilde{D}_2 . By Lemma 3.1, $\tilde{\mu}_2 = \mu_2$ and $I(K, \tilde{D}_2) = I(K, D_1)$. Suppose that $D_1 \stackrel{n.e.}{=} \tilde{D}_2$. Since D_1 is a domain, by Theorem 2.2, \tilde{D}_2 is also a domain. Then by (i), $\mu_1 = \tilde{\mu}_2 = \mu_2$.

Conversely, suppose that $\mu_1 = \mu_2$. Let O be a connected component of \tilde{D}_2 that intersects the interior of K . From property (1) of the Green equilibrium potentials,

$$U_{\mu_1}^{D_1}(x) - U_{\tilde{\mu}_2}^{\tilde{D}_2}(x) = I(K, D_1) - I(K, \tilde{D}_2) = 0,$$

for all $x \in K^\circ \cap O$. Also, $U_{\tilde{\mu}_2}^{\tilde{D}_2} = U_{\tilde{\mu}_2|_O}^O$ on O and $\mu_1 = \mu_2 = \tilde{\mu}_2$. By Lemma 3.4, $D_1 \stackrel{n.e.}{=} O$. Therefore

$$D_1 \cap (\tilde{D}_2 \setminus O) = \emptyset$$

and $K \subset O$, since $K \subset \tilde{D}_2$. Suppose that $\tilde{D}_2 \neq O$ and let A be a second connected component of \tilde{D}_2 . Then $K \cap A = \emptyset$, $U_{\mu_2}^{D_2}$ is harmonic on A and

$$U_{\mu_2}^{D_2} = I(K, D_2) - I(K, D_1)$$

on ∂A . By the maximum principle, $U_{\mu_2}^{D_2} = I(K, D_2) - I(K, D_1)$ on A and by the identity principle $U_{\mu_2}^{D_2} = I(K, D_2) - I(K, D_1)$ on the connected component B of $D_2 \setminus K$ that contains A . Also $\partial B \subset (\partial D_2 \cup \partial K)$. But

$$\lim_{D_2 \ni x \rightarrow \zeta} U_{\mu_2}^{D_2}(x) = 0 \neq I(K, D_2) - I(K, D_1), \text{ for n.e. } \zeta \in \partial D_2$$

and

$$U_{\mu_2}^{D_2} = I(K, D_2) \neq I(K, D_2) - I(K, D_1), \text{ n.e. on } \partial K,$$

which is a contradiction. Therefore $\tilde{D}_2 = O$ and $D_1 \stackrel{n.e.}{=} \tilde{D}_2$.

Remark 3.1. Let B be a ball that contains K . If the open set $B \setminus K$ is regular for the Dirichlet problem, the property $K \subset \tilde{D}_2$ is always true.

4. Compact sets with empty interior and counterexamples

Let K be a compact subset of \mathbb{R}^n and let D_1, D_2 be two Greenian subdomains of \mathbb{R}^n that contain K . We shall examine the case where the domains D_1, D_2 are not nearly everywhere equal and the Green equilibrium measures of K relative to D_1, D_2 are the same. It turns out that, in most cases, K must be sufficiently smooth.

Theorem 4.1. *Let K be a compact subset of \mathbb{R}^n with positive capacity and let D_1, D_2 be two Greenian subdomains of \mathbb{R}^n that are not nearly everywhere equal and contain K . Also let μ_1 and μ_2 be the Green equilibrium measures of K relative to D_1 and D_2 , respectively. If $\mu_1 = \mu_2$ and $I(K, D_1) = I(K, D_2)$, then there exists a level set L of a non constant harmonic function such that $C_2(K \setminus L) = 0$.*

Proof. By Lemma 3.2, the function

$$h(x) = U_{\mu_1}^{D_1}(x) - U_{\mu_2}^{D_2}(x)$$

is harmonic on $D_1 \cap D_2$. Since D_1, D_2 are not nearly everywhere equal, Lemma 3.4 shows that h cannot be 0 on a non-empty open subset of $D_1 \cap D_2$. From property (1) of the Green equilibrium potentials,

$$h(x) = U_{\mu_1}^{D_1}(x) - U_{\mu_2}^{D_2}(x) = I(K, D_1) - I(K, D_2) = 0,$$

for nearly every point $x \in K$. So, h is a non constant harmonic function on every connected component A of $D_1 \cap D_2$ such that $C_2(A \cap K) > 0$. Let G be the union of all the connected components A of $D_1 \cap D_2$ such that $C_2(A \cap K) > 0$ and let

$$L = \{x \in G : h(x) = 0\}.$$

Then h is a non constant harmonic function on G , L is a level set of h and $C_2(K \setminus L) = 0$. \square

Remark 4.1. It follows from Theorem 4.1 that Theorem 1 is valid for all compact subsets K that cannot be contained, except on a set of zero capacity, on a level set of a non constant harmonic function. An example of a compact set in the above class is every $(n - 1)$ -dimensional compact submanifold of \mathbb{R}^n that is not real analytic on a relatively open subset. Another example is every compact set $K \subset \mathbb{R}^2$ which has at least one connected component that is neither a singleton nor a piecewise analytic arc.

We proceed to give examples of compact sets K and pairs of domains that are answers to the first question we posed in the introduction. Keeping in mind Remark 4.1, the compact sets will lie on sufficiently smooth subsets of \mathbb{R}^n , mainly spheres and hyperplanes.

Let K be a compact subset of a sphere $S(x_0, r)$ with $C_2(K) > 0$. Let D be a Greenian open subset of \mathbb{R}^n that contains K . We denote by D^* the inverse of D with respect to $S(x_0, r)$; see e.g. [2, p. 19]. Then D^* is Greenian and (see e.g. [2, p. 95])

$$G_{D^*}(x, y) = \left(\frac{r^2}{|x - x_0| |y - x_0|} \right)^{n-2} G_D(x^*, y^*), \quad x, y \in D^*.$$

Moreover, $G_D(x, y) = G_{D^*}(x, y)$ for every $x, y \in K$. So $I(K, D) = I(K, D^*)$ and the Green equilibrium measures of K relative to D and D^* are the same. Of course, the sets D and D^* are not nearly everywhere equal in general. A similar result holds also in the case when K is a subset of a hyperplane H and D^* is the reflection of D with respect to H .

Finally we state a conjecture and a question:

Conjecture: Let K be a subset of a sphere $S(x_0, r)$ with $C_2(K) > 0$ and let D, Ω be two Greenian domains that contain K . If $I(K, D) = I(K, \Omega)$ and the Green equilibrium measures of K relative to D and Ω are the same, then $\Omega \stackrel{n.e.}{=} D$ or $\Omega \stackrel{n.e.}{=} D^*$ where D^* is the inverse of D with respect to $S(x_0, r)$.

Question: Let $n \geq 3$. Let K be a compact subset of \mathbb{R}^n and suppose that there does not exist a sphere S or a hyperplane H such that $C_2(K \setminus S) = 0$ or $C_2(K \setminus H) = 0$. Do there exist domains D, Ω which are not nearly everywhere equal and contain K such that $I(K, D) = I(K, \Omega)$ and the Green equilibrium measures of K relative to D and Ω are the same?

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