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Interpreting GPFCSP within the $Ł\Pi\frac{1}{2}$ logic framework

Aleksandar Takači[a], Aleksandar Perović[b], Aleksandar Jovanović[c]

[a] *Faculty of Technology, University of Novi Sad, Serbia*
[b] *Faculty of Transportation Engineering, University of Belgrade, Serbia*
[c] *Department(School) of Mathematics, University of Belgrade, Serbia*

Abstract. The aim of this paper is to interpret Generalized Priority Constraint Satisfaction Problem (GPFCSP) using the interpretational method. We will interpret the $Ł\Pi\frac{1}{2}$ logic into the first order theory of the reals, in order to obtain alternative, simple-complete axiomatization of $Ł\Pi\frac{1}{2}$ logic. A complete axiomatization using the interpretation method as a syntactical approach is given.

1. Introduction

In order to give a syntactical approach to GPFCSP, we will interpret the $Ł\Pi\frac{1}{2}$ logic into the first order theory of the reals. Our aim is to use the interpretation method in order to obtain alternative, simple-complete axiomatization of $Ł\Pi\frac{1}{2}$ logic in the following sense: if ϕ is an arbitrary $Ł\Pi\frac{1}{2}$-formula, then its maximal satisfaction degree is s iff $C_\phi = s$ is a theorem of $T_{Ł\Pi}\frac{1}{2}$. The reason for such approach lies in the fact that $Ł\Pi\frac{1}{2}$ logic is only complete with respect to finite theories. Namely, an $Ł\Pi\frac{1}{2}$-theory

$$T = \{\neg_\Pi(p \to \overline{0})\} \cup \{p \to \overline{10^{-n}} \mid n < \mathbb{N}\}$$

is finitely satisfiable, but not satisfiable since any evaluation of p that satisfies T must be a proper infinitesimal.

PFCSP is actually a fuzzy constraint satisfaction problem (FCSP) in which the notion of priority is introduced. Luo, Lee, Leung, Jennings [13] develop the idea and axiomatize PFSCP. Finally, Takači [20] and Takači, Škrbić [21] generalize PFCSP and obtain Generalized Priority Fuzzy Constraint Satisfaction Problem.

Our aim is to develop a logic that handles atomic symbols in the form (v_i, ρ_i). The first coordinate v_i represents local satisfaction degree of a fuzzy constraint, while the second coordinate represents its priority.

Finally, a logic whose atomic symbols are prioritized constraints may be seen as a possibilistic logic [19]. The main difference here is in the syntactical representation of the set of all valid formulas. As we have mentioned above, instead of giving a complete Hilbert-style axiomatization, we have used the interpretation method and obtain a variant of simple completeness (Theorem 3.5).

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Email addresses: stakaci@tf.uns.ac.rs (Aleksandar Takači), pera@sf.bg.ac.rs (Aleksandar Perović),
aljosha.jovanovich@gmail.com (Aleksandar Jovanović)

2. Generalized Priority Fuzzy Constraint Satisfaction Problem

Constraint satisfaction problems (CSP) deal with a set of constraints and the means to find a solution, i.e., an evaluation of variables that satisfies all the constraints and are a well developed concept. If we cannot precisely determine whether a constraint is satisfied, i.e., if there can be many levels of constraint satisfaction, we can expand CSP allowing a constraint to have a satisfaction degree from the unit interval. In practice, most of constraints have inherited fuzziness (tall, bald, strong, young, age around 24, good stamina, etc.) and they are more naturally represented as fuzzy sets. Constraints are modelled as fuzzy sets over a particular domain which leads to the fuzzy constraint satisfaction problems (FCSP). Obviously, the degree of satisfaction of a constraint is the membership degree of its domain value on the fuzzy set that represents it. In order to obtain the global satisfaction degree, we need to aggregate the values of each constraint. Operators t-norms, t-conorms and fuzzy negation are used to aggregate constraint satisfaction degrees.

Besides the satisfaction degree we add to each constraint an importance value, i.e., priority. Many concepts of priority have been studied. Priority of constraints is considered as the global importance of a constraint among other ones. The more important the constraint is, the more impact it has on the aggregated output of the PFCSP.

Priority Fuzzy Constraint Satisfaction Problem (PFCSP) make decisions that depend not only on the satisfaction degree of each constraint (which is the case in FCSP), but also on the priority that each constraint has. PFCSP are introduced by an axiomatic framework.

The problem is that FCSP only deals with the conjunction of the constraints. Obviously, PFCSP need to be generalized in order to handle disjunction and negation. The result is that PFCSP systems evolve into a GPFCSP that can handle priorities which are incorporated into each atomic formula (see [21]).

Definition 2.1. *(see [13]) A fuzzy constraint satisfaction problem (FCSP) is a triple $\langle X, D, C^f \rangle$ such that:*

- $X = \{x_1, \dots, x_n\}$ *is a set of variables.*
- $D = \{d_1, \dots, d_n\}$ *is a set of domains. Each domain* d_i *is a finite set of possible values for the corresponding variable* x_i.
- C^f *is a finite nonempty set of elements called fuzzy constraints, where each constraint* $C_i \in C^f, i \in \{1, \dots, m\}$ *has a form:*

$$C_i : d_{i_1} \times \cdots \times d_{i_k} \longrightarrow [0,1],\ 1 \leqslant k \leqslant n.$$

□

The membership degree of each constraint indicates the local degree to which the constraint is satisfied. In order to obtain the global satisfaction degree, local degrees are aggregated using a certain t–norm [18, 22]. Adding priorities to the FCSP and allowing constraints to be aggregated by any logical formula produces the GPFCSP.

Definition 2.2. *Let* (X, D, C^f) *be defined an FCSP, and let* $\rho : C^f \to [0, \infty)$ *and a compound label* v_X *of all variables in X, and ,* $g : [0, 1) \times [0, \infty] \to [0, 1]$.

Generalized PFCSP is defined as a tuple $(X, D, C^f, \rho, g, \wedge, \vee, \neg)$.

An elementary formula in generalized PFCSP is a pair $(x, \rho(C_i))$ *where* $C_i \in C^f,\ x \in Dom(C_i)$ *represents the satisfaction degree of a constraint* C_i *and* $p_i = \rho(C_i)$ *represents its priority.*

A formula in GPFCSP is defined in the following way:

(i) An elementary formula is a formula.

(ii) If f_1 *and* f_2 *are formulas then also* $\wedge(f_1, f_2)$, $\vee(f_1, f_2)$ *and* $\neg(f_1)$ *are formulas.*

For each valuation v_X *a satisfaction degree* $\alpha_F(v_X)$ *of a formula F is calculated depending on the interpretation of connectives.*

A system is a GPFCSP if

1. *Let $F = \wedge_{i\in\{1,\dots,n\}} f_i$ be a formula in GPFCSP where f_i, $i \in \{1, \dots, n\}$ are elementary formulas and let C^f be a set of constrains that appear in the formula. If for the fuzzy constraint R^f_{max} we have*

$$\rho_{max} = \rho(R^f_{max}) = \max\{\rho(R^f) \mid R^f \in C^f\},$$

   then for each formula F we have:

$$\mu_{R^f_{max}}(v_X) = 0 \Rightarrow \alpha_F(v_X) = 0.$$

2. *If $\forall R^f \in C^f$, $\rho(R^f) = \rho_0$, then for each formula F it holds:*

$$\alpha_F(v_X) = F_{\mathcal{L}}(v_X)$$

   where $F_{\mathcal{L}}$ is the interpretation of the logical formula F in fuzzy logic $\mathcal{L}(\wedge, \vee, \neg)$).
3. *For R^f_i, $R^f_j \in C^f$, assume $\rho(R^f_i) \geq \rho(R^f_j)$, $\delta > 0$ and assume that there are two different compound labels v_X and v'_X such that:*
   - *if $\forall R^f \neq R^f_i$ and $\forall R^f \neq R^f_j$, then* $\mu_{R^f}(v_X) = \mu_{R^f}(v'_X)$,
   - *if $R^f = R^f_i$, then $\mu_{R^f}(v_X) = \mu_{R^f}(v'_X) + \delta$,*
   - *if $R^f = R^f_j$, then $\mu_{R^f}(v'_X) = \mu_{R^f}(v_X) + \delta$.*

   Then the following properties hold for

$$F = \bigwedge_{k=1}^{n} (x_k, \rho(R_k)),\ x_k \in \mathrm{Dom}(R_k)$$

   or $F = \bigvee_{k=1}^{n} (x_k, \rho(R_k))$, $x_k \in \mathrm{Dom}(R_k)$:

$$\alpha_F(v_X) \geq \alpha_F(v'_X).$$

4. *Assume that two different compound labels v_X and v'_X such that $\forall R^f \in C^f$ satisfy*

$$\mu_{R^f}(v_X) \geq \mu_{R^f}(v'_X).$$

   If formula F that has no negation connective, then it holds

$$\alpha_F(v_X) \geq \alpha_F(v'_X).$$

5. *Let there be a compound label such that $\forall R^f \in C^f$, $\mu_{R^f}(v_X) = 1$.*
   If F is a formula $F = \bigwedge_{i=1}^{n} f_i$, where f_i are elementary formulas then

$$\alpha_F(v_X) = 1.$$

2.1. *GPFCSP Example*

In [13] it has been proven that if we take $\wedge(x, y) = T_L(x, y) = max(x + y - 1, 0)$, $\vee(x, y) = S_L(x, y) = min(x + y, 1)$, $\neg(x) = N(x) = 1 - x$ and finally if the elementary formulae are evaluated in the following way $\alpha_f = \alpha_{(x,\rho(C))} = S_p(x, 1 - \rho(c)) = \mu_C(x) + (1 - \rho(C)) - (\mu_C(x) * (1 - \rho(C)))$ for each constraint C (where T_L, S_L, N, S_p are the Lukasiewicz T-norm, Lukasiewicz T-conorm, standard negation and product T-norm respectively and μ_C is the membership function of the fuzzy set that represents the constraint C) we obtain a proper interpretation of GPFCSP connectives. Let us give a proper, real-life example of a GPFCSP.

For our example we will use the data needed to optimize bioethanol production in batch culture by free *Saccharomyces cerevisiae* cells from intermediates of sugar beet processing which was statistically analyzed in [5, 6] and also a neural network model was made in [2]. A GPFCSP system is based on these results will be constructed.

Our system will be based on two input variables *Initial sugar mass fraction* (x_1), *Fermentation time* (x_2) and *Ph value* (x_3) . The most important factor for evaluation of bioethanol production efficiency is Ethanol volume fraction at the end of fermentation. The GPFCSP system that is created in this example will be able to evaluate each combination of input parameters i.e. the higher the satisfaction degree obtained the better the combination is for bioethanol production. The constraints will be based on the results found in [2, 5, 6].

Let us now give the architecture of the system.

- Set of variables $X = \{x_1, x_2, x_3\}$ are:
  - x_1 - Initial sugar mass fraction,
  - x_2 - Fermentation time,
  - x_3 - Ph value.
- Set of domains $d = \{d_1, d_2, d_3\}$ are:
  - $d_1 = [5, 25]$,
  - $d_2 = [0, 48]$,
  - $d_3 = [0, 14]$.

  In order to introduce the set of constraints we need to recall the notion of triangular, trapezoidal and left(right) shoulder fuzzy numbers. A triangular fuzzy number is a fuzzy number whose membership function is first linearly increasing form point $[c - l, 0]$ to $[c, 1]$ then linearly decreasing to $[c + r, 0]$ forming a triangle. Number c is called the center of the fuzzy number and l and r are left and right tolerances respectively. The left shoulder fuzzy number has an increasing linear membership function that connects the points $[-\infty, 0], [c, 0], [d, 1], [d, \infty]$. Analogously, right shoulder fuzzy number has an decreasing membership function that connects the points $[-\infty, 1], [c, 1], [d, 0], [d, +\infty]$. Finally, the membership function of a trapezoidal fuzzy number forms a trapeze between the points $[c - l, 0], [c, 1], [d, 1], [d + r, 0]$. We denote triangular fuzzy numbers $Tri(c, l, r)$, trapezoidal fuzzy number $Trap(c, d, l, r)$, left shoulder ones $inc(c, d)$ and right shoulder ones $dec(c, d)$.
- Set of constraints $C_f = \{C_1, C_2, C_3, C_4\}$ with their according domains $d_{c_f} = \{d_{C_1}, d_{C_2}, d_{C_3}, d_{C_4}\}$ are:
  - $C_1 = Trap(4, 5, 0.5, 0.5)$, $d_{C_1} = d_3$,
  - $C_2 = Inc(5, 16)$, $d_{C_2} = d_1$,
  - $C_3 = Tri(36, 36, 12)$, $d_{C_3} = d_2$,
  - $C_4(x_1, x_2) = \frac{x_1 - 5}{20} * \frac{x_2}{48}$, $d_{C_3} = d_1 \times d_2$.

The constraint C_1 represents the fact that the optimal Ph value for bioethanol production is between 4 and 5. Constraint C_2 represents the finding that the increase of the initial sugar mass fraction to up to 16% increases the production. Constraint C_3 states that it is feasible to increase the fermentation time up to 36 hours in order to obtain optimal production. Finally, constraint C_4 states the interaction of two factors is positive to the final outcome. It has been shown that constraints C_1 and C_4 are the most important ones, and constraint C_3 is the least important. Thus we have the following constraint priorities: $\rho(C_1) = \rho(C_4) = 1, \rho(C_3) = 0.8$ and $\rho(C_2) = 0.5$. Finally, let us propose a formula whose satisfaction degree will be calculated:

$$F = \wedge(f_1, \wedge(f_4, \vee(f_2, f_3))),$$

where $f_i = (x_i, \rho(C_i)),\ x_i \in d_{c_i}, i = 1, 2, 3, 4$ is an elementary formula that represents the constraint C_i.

Suppose we want to calculate the satisfaction degree for the following evaluation: $v_X = (36, 12, 3.8)$. First, we calculate the local satisfaction degree for each constraint:

$$\begin{aligned}
\alpha_{f_1}(V_X) &= S_P(\mu_{C_1}(3.8), 1 - \rho(C_1)) = S_p(0.8, 0) = 0.8 \\
\alpha_{f_2}(V_X) &= S_P(\mu_{C_2}(12), 1 - \rho(C_2)) = S_p(0.63, 0.5) = 0.82 \\
\alpha_{f_3}(V_X) &= S_P(\mu_{C_3}(36), 1 - \rho(C_3)) = S_p(1, 0.2) = 1 \\
\alpha_{f_3}(V_X) &= S_P(\mu_{C_4}(12, 36), 1 - \rho(C_4)) = S_p(0.31, 0) = 0.31
\end{aligned}$$

Now we can calculate $\alpha_F(v_X)$:

$$\begin{aligned}
\alpha_F(v_X) &= \wedge(\alpha_{f_1}(V_X), \wedge(\alpha_{f_4}(V_X), \vee(\alpha_{f_2}(V_X), \alpha_{f_3}(V_X))) = \\
&= T_L(0, 8, T_L(0.31, S_P(1, 0.82))) = 0.11
\end{aligned}$$

3. The theory $RCF_{Ł\Pi}$

In order to give a syntactical approach to GPFCSP, we will interpret the $Ł\Pi\frac{1}{2}$ logic into the first order theory of the reals. There are Hilbert-style axiomatizations of the Łukasiewicz logic, Product logic and Gödel logics and the reader may find them, e.g., in Hajek's book [7]. Complete axiomatizations of ŁΠ and $Ł\Pi\frac{1}{2}$ logics, applications of these logics in formalization of conditional probabilities and default reasoning, characterization of definability of continuous t-norms and many other important concepts connected to our work can be found in [4, 9–12, 14–17].

Our aim is to use the interpretation method in order to obtain alternative, simple-complete axiomatization of $Ł\Pi\frac{1}{2}$ logic in the following sense: if ϕ is an arbitrary $Ł\Pi\frac{1}{2}$-formula, then its maximal satisfaction degree is s iff $C_\phi = s$ is a theorem of $T_{L\Pi}\frac{1}{2}$. The reason for such approach lies in the fact that $Ł\Pi\frac{1}{2}$ logic is only complete with respect to finite theories, provided that we have restricted the class of all models to $[0, 1]$-valued (which is a standard semantics for fuzzy logics). Namely, an $Ł\Pi\frac{1}{2}$-theory

$$T = \{\neg_\Pi(p \to \overline{0})\} \cup \{p \to \overline{10^{-n}} \mid n \in \mathbb{N}\}$$

is finitely satisfiable, but not satisfiable since any evaluation of p that satisfies T must ba a proper infinitesimal. Note that the same effect would be achieved with any sequence $\langle a_n \mid n \in \mathbb{N}\rangle$ of positive rational numbers that converges to 0 (provided that $a_n \leqslant 1$ for all n), i.e. the theory

$$\{\neg_\Pi(p \to \overline{0})\} \cup \{p \to \overline{a_n} \mid n \in \mathbb{N}\}$$

is unsatisfiable in any $[0, 1]$-valued model of $Ł\Pi\frac{1}{2}$. In fact, any theory that propagates existence of proper infinitesimal is unsatisfiable. This is a consequence of the fact that $\langle [0, 1] \leqslant\rangle$ is not ω_1-saturated. One well known way to tame this non-compactness phenomena is to extend $Ł\Pi\frac{1}{2}$ logic with certain infinitary inference rules that will provide inconsistency of theories such as the above T. For instance, one such rule is the Archimedean rule:

From the set of premises

$$\left\{\phi \to_\Pi \overline{s - \frac{1}{n}} \mid n > \frac{1}{s}\right\}$$

infer $\phi \to_\Pi \overline{s}$.

Another way is to extend semantics of the $Ł\Pi\frac{1}{2}$ logic to the hyperreal valued truth functions and use the interpretation method to show that compactness theorem will hold for the extended semantics. Since compactness can be obtain in the exactly the same way as the simple completeness, we will only show the simple completeness via interpretation method.

As the input, we have the following data:

- The satisfaction degree of each constraint.
- The priority of each constraint.

Thus, our propositional letters are pairs of the form $\langle v, \rho \rangle$, where the first coordinate refers to the satisfaction degree of the constraint, while the second coordinate refers to its priority. A query will be an $Ł\Pi\frac{1}{2}$-formula over the introduced propositional letters.

Now, we shall start with the technical details. Concerning model theoretical notions, our notation and terminology is standard and follows. Let $\mathcal{L}_{OF} = \{+, -, \cdot, \leqslant, 0, 1\}$ be the language of the ordered fields, let RCF be the first order $\mathcal{L}_{OF}$–theory of the real closed fields, and let $\mathcal{V} = \{v_n \mid n < \omega\}$, $\mathcal{P} = \{P\} \cup \{\rho_n \mid n \in \mathbb{N}\}$ and $C = \{\langle v, \rho \rangle \mid v \in \mathcal{V} \text{ and } \rho \in \mathcal{P}\}$. The letters u, v and w denote the elements of $\mathcal{V}$, while ρ, η and ζ denote the elements of $\mathcal{P}$. We define the set *For* of fuzzy propositional formulae as the set of $Ł\Pi\frac{1}{2}$-formulae over the set of propositional letters C. The elements of *For* will be denoted by ϕ, ψ and θ, indexed or primed if necessary.

Definition 3.1. Let $\mathcal{L}^* = \mathcal{L}_{OF} \cup \mathcal{V} \cup \mathcal{P} \cup \{\mathsf{C}_\alpha \mid \alpha \in For\}$. Here the elements of $\mathcal{L}^* \setminus \mathcal{L}_{OF}$ are treated as new constant symbols. We define the theory $RCF_{Ł\Pi}$ as an $\mathcal{L}^*$–theory with the following axioms:

1. All axioms of *RCF*
2. $0 \leqslant v_n \wedge v_n \leqslant 1, n \in \mathbb{N}$
3. $0 < \rho_0$
4. $\rho_n < \rho_m$, whenever $n < m$
5. $\rho_n < P, n \in \mathbb{N}$
6. $\mathsf{C}_{\langle v,\rho\rangle} = 1 - (1 - v) \cdot \rho \cdot P^{-1}$
7. $\mathsf{C}_{\neg_L \phi} = 1 - \mathsf{C}_\phi$
8. $\mathsf{C}_\phi = 0 \ \rightarrow \ \mathsf{C}_{\neg_\Pi \phi} = 1$
9. $\mathsf{C}_\phi > 0 \ \rightarrow \ \mathsf{C}_{\neg_\Pi \phi} = 0$
10. $\mathsf{C}_{\phi \wedge_L \psi} = \max(\mathsf{C}_\phi + \mathsf{C}_\psi - 1, 0)$
11. $\mathsf{C}_{\phi \vee_L \psi} = \min(\mathsf{C}_\phi + \mathsf{C}_\psi, 1)$
12. $\mathsf{C}_{\phi \wedge_\Pi \psi} = \mathsf{C}_\phi \cdot \mathsf{C}_\psi$
13. $\mathsf{C}_{\phi \vee_\Pi \psi} = \mathsf{C}_\phi + \mathsf{C}_\psi - \mathsf{C}_\phi \cdot \mathsf{C}_\psi$
14. $\mathsf{C}_\phi \leqslant \mathsf{C}_\psi \ \rightarrow \ \mathsf{C}_{\phi \rightarrow_\Pi \psi} = 1$
15. $\mathsf{C}_\phi > \mathsf{C}_\psi \ \rightarrow \ \mathsf{C}_\phi \cdot \mathsf{C}_{\phi \rightarrow_\Pi \psi} = \mathsf{C}_\psi$
16. $\mathsf{C}_{\phi \rightarrow_L \psi} = \min(1, 1 - \mathsf{C}_\phi + \mathsf{C}_\psi)$.

Let us briefly comment the above axiomatization. Pairs of the form $\langle v, p \rangle$ are typical for priority language. C_ϕ stands for prioritized satisfaction degree of the query $\phi \in For$. Axiom (2) states that each local satisfaction degree is between 0 and 1. Axioms (3), (4) and (5) state that priorities form a positive sequence[1)] whose order type is $\omega + 1$. Axiom (6) introduces priority in the calculation of the satisfaction degree. The rest of the axioms follows the usual truth functions for connectives. It is important to say that in this context, $+, -, \cdot, ^{-1}$, max and min are purely syntactical symbols.

Theorem 3.2. *$RCF_{Ł\Pi}$ is consistent.*

Proof. We use the compactness argument. That is, in order to prove consistency of $RCF_{Ł\Pi}$, it is sufficient to prove consistency of its arbitrary finite subset. Suppose that Γ is an arbitrary finite subset of $RCF_{Ł\Pi}$. Let $\phi_1, \dots, \phi_n$ be all fuzzy formulas appearing (as indices) in Γ. We construct the model $\mathcal{M}$ for Γ as follows:

- The universe M of $\mathcal{M}$ is the universe of some fixed real closed field $\mathbb{M}$. The language $\mathcal{L}_{OF}$ is interpreted as in $\mathbb{M}$. We may assume (without any loss of generality) that $\mathbb{Q} \subseteq \mathbb{M}$.

[1)]each member of the sequence is > 0

- Each v_m appearing in $\phi_1, \ldots, \phi_n$ is interpreted as $\frac{1}{n+1}$
- Each ρ_m is interpreted as $m+1$. If k is the maximum of all such interpretations, then P is interpreted as $k+1$
- $\mathsf{c}^{\mathcal{M}}_{\langle v,\rho\rangle} = 1 -^{\mathbb{M}} (1 -^{\mathbb{M}} v^{\mathcal{M}}) \cdot^{\mathbb{M}} \rho^{\mathcal{M}} \cdot P^{\mathcal{M}^{-1^{\mathbb{M}}}}$
- All constant symbols of the form c_ϕ are interpreted according to the truth tables of $\text{Ł}\Pi\frac{1}{2}$ connectives. For instance, $\mathsf{c}^{\mathcal{M}}_{\neg_L\phi} = 1 -^{\mathbb{M}} \mathsf{c}^{\mathcal{M}}_{\phi}$.

Clearly, $\langle \mathbb{M}, \mathsf{c}^{\mathcal{M}}_{\phi_1}, \ldots, \mathsf{c}^{\mathcal{M}}_{\phi_n}\rangle$ is a model of Γ, so we have our claim. □

Theorem 3.3. *For each sentence φ of $\mathcal{L}^*$ there is a sentence φ^* of $\mathcal{L}_{OF}$ such that $RCF_{\text{Ł}\Pi} \vdash \varphi$ iff $RCF \vdash \varphi^*$. In other words, $RCF_{\text{Ł}\Pi}$ is interpretable in RCF.*

Proof. Notice that we only need to equivalently eliminate constant symbols c_ϕ. Obviously, each c_ϕ has the form

$$F(\mathsf{c}_{\langle v_{i_1},\rho_{i_1}\rangle}, \mathsf{c}_{\langle v_{i_2},\rho_{i_2}\rangle}, \ldots, \mathsf{c}_{\langle v_{i_k},\rho_{i_k}\rangle}), \tag{1}$$

where F is certain composition of $+, -, \cdot, ^{-1}$, max and min. Since F is definable in *RCF*, it remains to give the elimination of $\mathsf{c}_{\langle v,\rho\rangle}$'s. It is easy to show that

$$RCF_{\text{Ł}\Pi} \vdash \varphi(F(\mathsf{c}_{\langle v_{i_1},\rho_{i_1}\rangle}, \mathsf{c}_{\langle v_{i_2},\rho_{i_2}\rangle}, \ldots, \mathsf{c}_{\langle v_{i_k},\rho_{i_k}\rangle}))$$

iff

$$RCF \vdash \forall \bar{x}, \bar{y}, \bar{z}, t(\varphi(F(\bar{z}))\wedge$$

$$\bigwedge_{i=1}^{k} z_i = 1 - \frac{(1-x_i)y_i}{t} \wedge \psi(\bar{x}) \wedge \theta(\bar{y}, t)),$$

where, $\psi(\bar{x})$ is the formula

$$0 \leqslant x_1 \leqslant 1 \wedge \cdots \wedge 0 \leqslant x_k \leqslant 1$$

and $\theta(\bar{y}, t)$ is the formula

$$0 < y_1 < t \wedge \cdots \wedge 0 < y_k < t.$$

Thus, we have established the elimination of new constants, so we have our claim. □

Corollary 3.4. *$RCF_{\text{Ł}\Pi}$ is decidable.*

Proof. By the previous theorem, for each sentence φ of $\mathcal{L}$, there is a sentence φ^* of $\mathcal{L}_{OF}$ such that $RCF_{\text{Ł}\Pi} \vdash \varphi$ iff $RCF \vdash \varphi^*$. It is well known that the latter predicate is decidable. Thus, $RCF_{\text{Ł}\Pi}$ is decidable. □

Concerning complexity, both *RCF* and $RCF_{\text{Ł}\Pi}$ are in EXPSPACE. However, prioritized queries can be modelled with Σ_0-sentences. Such sentences can be interpreted in the existential theory of the reals. In this way, using [1], we obtain a PSPACE containment for the decision procedure for Σ_0-sentences, since the translation procedure is in PTIME.

We will conclude this section with a variant of simple completeness theorem for our logic.

Theorem 3.5. *Suppose that ϕ is an arbitrary prioritized formula ($\phi \in For$). Then, the satisfaction degree of ϕ is equal to $r \in [0,1] \cap \mathbb{Q}$ iff $RCF_{\text{Ł}\Pi} \vdash \mathsf{c}_\phi = r$.*

Proof. As in the proof of Theorem 3.3, C_ϕ has a form

$$F(\mathsf{C}_{\langle v_{i_1},\rho_{i_1}\rangle}, \mathsf{C}_{\langle v_{i_2},\rho_{i_2}\rangle}, \dots, \mathsf{C}_{\langle v_{i_k},\rho_{i_k}\rangle}),$$

where F is certain composition of $+, -, \cdot,$ max and min. Now we have the following:

$$RCF_{L\Pi} \vdash F(\mathsf{C}_{\langle v_{i_1},\rho_{i_1}\rangle}, \mathsf{C}_{\langle v_{i_2},\rho_{i_2}\rangle}, \dots, \mathsf{C}_{\langle v_{i_k},\rho_{i_k}\rangle}) = r$$

if and only if

$$RCF \vdash \forall \bar{x}, \bar{y}, \bar{z}, t(F(\bar{z}) = r \wedge \bigwedge_{i=1}^{k} z_i = 1 - \frac{(1-x_i)y_i}{t} \wedge \psi(\bar{x}) \wedge \theta(\bar{y}, t)), \tag{2}$$

where $\psi(\bar{x})$ is the formula $0 \leqslant x_1 \leqslant 1 \wedge \dots \wedge 0 \leqslant x_k \leqslant 1$ and $\theta(\bar{y}, t)$ is the formula $0 < y_1 < t \wedge \dots \wedge 0 < y_k < t$. Since each two models of *RCF* satisfy the same sentences, (2) is equivalent to

$$\langle \mathbb{R}, +, \cdot, \leqslant, 0, 1\rangle \models \forall \bar{x}, \bar{y}, \bar{z}, t(F(\bar{z}) = r \wedge \bigwedge_{i=1}^{k} z_i = 1 - \frac{(1-x_i)y_i}{t} \wedge \psi(\bar{x}) \wedge \theta(\bar{y}, t)).$$

By definition, the latter statement is equivalent to the one that claims that satisfaction degree of ϕ is equal to r. Hence, we have our claim. □

4. Concluding remarks

A formalization of GPFCSP is presented in this paper. The interpretation method used here is not a technical novelty - similar ideas applied on probability are well known, see for example [3, 8]. In the future we will examine other approaches and see how they compare to the interpretation method. Also, since GPFCSP have a wide application in decision making this formalization can be used to determine the complexity of calculation in GPFCSP.

In our further research we will examine applications of GPFCSP, specially in the field of fuzzy relational databases (FRDB). Similar methodology will be used to formalize FRDB.

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