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Semi-cubically hyponormal weighted shifts with recursive type

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Abstract. In this paper, we discuss the semi-cubic hyponormality of recursively generated weighted shifts with weight $\alpha(x)$: $\sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ to give a new bridge between cubically hyponormal and quadratically hyponormal weighted shifts. Using weight sequences with first two equal weights, we show that two notions of quadratic hyponormality and semi-cubic hyponormality are different one from another. Moreover, we characterize the semi-cubic hyponormality of weighted shifts.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A bounded operator T is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for all complex polynomials p. An operator T in $\mathcal{L}(\mathcal{H})$ is *weakly k-hyponormal* if for every polynomial p of degree k or less, $p(T)$ is hyponormal ([4], [8], [9]). For a positive integer k, an operator $T \in \mathcal{L}(\mathcal{H})$ is called *semi-weakly k-hyponormal* if $T + sT^k$ is hyponormal for all $s \in \mathbb{C}$ ([10]). It is obvious that a weakly k-hyponormal operator is semi-weakly k-hyponormal. In particular, weak 2-hyponormality is equivalent to semi-weak 2-hyponormality.

For A, $B \in \mathcal{L}(\mathcal{H})$, we denote $[A, B] := AB - BA$. A k-tuple $\mathbf{T} = (T_1, ..., T_k)$ of operators on $\mathcal{H}$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^k$ is positive on the direct sum of $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ with k copies. Also an operator T is said to be (*strongly*) *k-hyponormal* for each positive integer k if $(I, T, ..., T^k)$ is hyponormal. The Bram-Halmos criterion shows that an operator T is subnormal if and only if T is k-hyponormal for all $k \geq 1$ ([1]). We note that k-hyponormality implies weak k-hyponormality for each positive integer k. The following implications provide a bridge between subnormal and hyponormal operators: subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow \cdots \Rightarrow$ weakly 3-hyponormal $\Rightarrow$ weakly 2-hyponormal $\Rightarrow$ hyponormal. However, one does not know concrete examples about converse implications for $n \geq 3$ yet; see [7], [14] and [15] for weak 2- and weak 3-hyponormalities. In particular, weakly 2-hyponormal (or weakly 3-hyponormal) is referred to as *quadratically hyponormal* (or *cubically hyponormal*, resp.). In [8] and [9], Curto-Putinar proved that there exists

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an operator that is polynomially hyponormal but not 2-hyponormal. Although the existence of a weighted shift which is polynomially hyponormal but not subnormal was established in [8] and [9], concrete example of such weighted shifts has not been found yet.

J. Stampfli ([16]) proved that a subnormal weighted shift with two equal weights $\alpha_n = \alpha_{n+1}$ for some nonnegative n has the flatness property, i.e., $\alpha_1 = \alpha_2 = \cdots$. Stampfli's result has been used to attempt the construction of nonsubnormal polynomially hyponormal weighted shifts (cf. [2], [3], [10], [14]). In [2], Choi proved that if a weighted shift W_α is polynomially hyponormal with first two equal weights, then W_α has flatness. In [3], Curto obtained a quadratically hyponormal weighted shift with first two equal weights but not satisfying flatness. Also in [14], they showed that a weighted shift W_α with weights $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$ is not cubically hyponormal. However, the flatness of cubically hyponormal weighted shifts has been not known well. Recently, in [10], it was proved that there exists a semi-cubically hyponormal weighted shift W_α with $\alpha_0 = \alpha_1 < \alpha_2$ but not 2-hyponormal.

In this paper we observe that semi-weak k-hyponormality can provide a new bridge between subnormality and hyponormality. For this study, we focus on the class of the weighted shift and study the relations of a semi-cubic hyponormality and quadratic hyponormality. In Section 2 we recall some terminology and notations concerning semi-cubically hyponormal weighted shifts. In Section 3 we characterize the semi-cubic hyponormality of weighted shifts $W_{\alpha(x)}$ with weight sequence $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ for $0 < x \leq a < b < c$. In Section 4, we characterize the semi-cubic hyponormality of weighted shift W_α with first two equal weights. Finally, using the results for the quadratic hyponormality in [6], we show that two notions of quadratic hyponormality and semi-cubic hyponormality are different one from another.

Some of the calculations in this paper were aided by using the software tool *Mathematica* ([17]).

2. Preliminaries

We recall some standard terminology and definitions about semi-cubically hyponormal weighted shifts (cf. [10]). Let $\alpha = \{\alpha_i\}_{i=0}^\infty$ be a weight sequence in the positive real number $\mathbb{R}_+$. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^\infty$, is defined by $W_\alpha(e_j) = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A weighted shift W_α is called *semi-cubically hyponormal* (or *semi-weakly* 3*-hyponormal*) if

$$[(W_\alpha + sW_\alpha^3)^*, W_\alpha + sW_\alpha^3] \geq 0,\ s \in \mathbb{C}.$$

Let P_m denote the orthogonal projection onto $\vee_{k=0}^m \{e_k\}$. For $m \in \mathbb{N}_0$, define $D_m(s)$ by

$$D_m(s) = P_m \left[\left(W_\alpha + sW_\alpha^3 \right)^*, W_\alpha + sW_\alpha^3 \right] P_m \text{ for all } s \in \mathbb{C}.$$

Then

$$D_m(s) = \begin{pmatrix} q_0 & 0 & z_0 & 0 & \cdots & 0 \\ 0 & q_1 & 0 & z_1 & \ddots & \vdots \\ \bar{z}_0 & 0 & q_2 & \ddots & \ddots & 0 \\ 0 & \bar{z}_1 & \ddots & \ddots & \ddots & z_{m-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{z}_{m-2} & 0 & q_m \end{pmatrix}, \tag{2.1}$$

where for all $k \in \mathbb{N}_0$,

$$\begin{aligned} &q_k := u_k + v_k|s|^2,\ z_k := \sqrt{w_k}\bar{s},\ u_k := \alpha_k^2 - \alpha_{k-1}^2, \\ &v_k := \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 - \alpha_{k-3}^2\alpha_{k-2}^2\alpha_{k-1}^2,\ w_k := \alpha_k^2\alpha_{k+1}^2(\alpha_{k+2}^2 - \alpha_{k-1}^2)^2 \end{aligned} \tag{2.2}$$

with $\alpha_{-3} = \alpha_{-2} = \alpha_{-1} = 0$. It is obvious that W_α is semi-cubically hyponormal if and only if $D_m(s) \geq 0$ for every $s \in \mathbb{C}$ and every $m \geq 0$.

By changing the basis of $\mathbb{C}^{m+1}$, it follows from [10, Lemma 2.1] that $D_m(t)$ in (2.1) is unitarily equivalent to $D_n^{(1)}(t) \oplus D_n^{(2)}(t)$ for $t := |s|^2$ and $n := [\frac{m}{2}]$, where

$$D_n^{(1)}(t) = \begin{pmatrix} q_0 & z_0 & 0 & & \\ z_0 & q_2 & z_2 & 0 & \\ 0 & z_2 & q_4 & z_4 & \ddots \\ & 0 & z_4 & \ddots & \ddots \\ & & \ddots & \ddots & q_{2n} \end{pmatrix}, \quad D_n^{(2)}(t) = \begin{pmatrix} q_1 & z_1 & 0 & & \\ z_1 & q_3 & z_3 & 0 & \\ 0 & z_3 & q_5 & z_5 & \ddots \\ & 0 & z_5 & \ddots & \ddots \\ & & \ddots & \ddots & q_{2n+(-1)^{m+1}} \end{pmatrix}.$$

It is clear that if two matrices $D_n^{(1)}(t)$ and $D_n^{(2)}(t)$ are positive for all $n \geq 0$, then $D_m(s) \geq 0$ for $m \geq 0$ (cf. [10]).

We now recall some terminology in [5]. Consider the following matrix in [5] below:

$$M_n(t) = \begin{pmatrix} \check{q}_0 & \check{r}_0 & 0 & & & \\ \check{r}_0 & \check{q}_1 & \check{r}_1 & 0 & & \\ 0 & \check{r}_1 & \check{q}_2 & \check{r}_2 & \ddots & \\ & 0 & \check{r}_2 & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \check{q}_{n-1} & \check{r}_{n-1} \\ & & & 0 & \check{r}_{n-1} & \check{q}_n \end{pmatrix},$$

where $\check{q}_k := \check{u}_k + \check{v}_k t, r_k := \sqrt{\check{w}_k t}\ (k \geq 0)$, and $\check{u}_k \geq 0, \check{v}_k \geq 0, \check{w}_k \geq 0, t \geq 0$. If we put $d_n(t)$ for the determinant of $M_n(t)$, then

$$d_n(t) = \sum_{i=0}^{n+1} c(n,i)\, t^i,$$

and some computations provide the following:

$$\begin{aligned} c(0,0) &= \check{u}_0, \ \ c(0,1) = \check{v}_0, \\ c(1,0) &= \check{u}_0\check{u}_1, \ c(1,1) = \check{u}_1\check{v}_0 + \check{u}_0\check{v}_1 - \check{w}_0, \ c(1,2) = \check{v}_1\check{v}_0, \\ c(n,i) &= \check{u}_n c(n-1,i) + \check{v}_n c(n-1,i-1) - \check{w}_{n-1} c(n-2,i-1), \\ c(n,n+1) &= \check{v}_0\check{v}_1\cdots\check{v}_n, \ \ \text{for all } n \geq 2, \ 0 \leq i \leq n, \end{aligned} \tag{2.3}$$

with $c(-n,-i) := 0$ for all $n, i \in \mathbb{N}$. Observe that $\check{u}_n\check{v}_{n+1} = \check{w}_n$ $(n \geq 2)$, which implies that

$$c(n,i) = \begin{cases} \check{v}_n\cdots\check{v}_2 c(1,2), & \text{if } i = n+1, \\ \check{u}_n c(n-1,n) + \check{v}_n\cdots\check{v}_3\rho, & \text{if } i = n, \\ \check{u}_n c(n-1,n-1) + \check{v}_n\cdots\check{v}_3\tau, & \text{if } i = n-1, \\ \check{u}_n c(n-1,i), & \text{if } 0 \leq i \leq n-2, \end{cases} \tag{2.4}$$

for all $n \geq 3$, where

$$\rho := \check{v}_2 c(1,1) - \check{w}_1 c(0,1) \text{ and } \tau := \check{v}_2 c(1,0) - \check{w}_1 c(0,0).$$

Recall that if $c(n,n+1) > 0$ and $c(n,i) \geq 0$ for all $n \geq 0$ with $0 \leq i \leq n$, then every matrix $M_n(t)$ is obviously positive for all $n \geq 0$ and $t > 0$. To detect the positivity of $D_m(t)$ for all $t > 0$ and $m \geq 2$, we adapt the above method to $D_n^{(\ell)}(t)$ $(\ell = 1,2)$. Denote

$$d_n^{(\ell)}(t) := \det D_n^{(\ell)}(t) = \sum_{i=0}^{n+1} c_\ell(n,i) t^i,$$

for $\ell = 1, 2$. We may see that each coefficients of $c_\ell(n, i)$ ($\ell = 1, 2$) satisfies (2.3) for all $n \geq 0$ (cf. [10]).

Now we recall a Stampfli's method ([5], [16]) for the subnormal completion. For given numbers $\alpha_0, \alpha_1, \alpha_2$ with $0 < \alpha_0 < \alpha_1 < \alpha_2$, define

$$\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2} \quad \text{for all } n \geq 3, \tag{2.5}$$

where $\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2}$ and $\Psi_1 = \frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}$. Then we obtain a recursive weight sequence $\{\alpha_n\}_{n=0}^\infty$ generated by (2.5), which is usually denoted by $(\alpha_0, \alpha_1, \alpha_2)^\wedge$; for example, see [16]. It follows from [5] that

$$\alpha_n^2 \nearrow L^2 := \frac{1}{2}\left(\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}\right) \quad \text{as} \quad n \to \infty,$$

which will be used frequently in this paper.

3. Recursive weighted shifts with Stampfli tail

In this section we characterize the semi-cubic hyponormality of weighted shifts $W_{\alpha(x)}$ with a recursive weight $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. In particular, either $a = b$ or $b = c$ forces the flatness of $W_{\alpha(x)}$ ([10]). To avoid the trivial case, we assume $x \leq a < b < c$ throughout this section.

3.1. Technical lemmas. We give several lemmas for characterizing the semi-cubic hyponormality of weighted shifts. Let x, a, b, c with $x \leq a < b < c$ be given. According to (2.5), we may produce a recursive weight sequence $\{\alpha_n\}_{n=0}^\infty$ such that $\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}$ ($n \geq 3$), where $\alpha_0^2 = x$, $\alpha_1^2 = a$, $\alpha_2^2 = b$, $\alpha_3^2 = c$, $\Psi_0 = -\frac{ab(c-b)}{b-a}$ and $\Psi_1 = \frac{b(c-a)}{b-a}$.

Lemma 3.1. *Let $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ be as above. Then*

$$u_n v_{n+2} = w_n \quad (n \geq 2). \tag{3.1}$$

Proof. The case $n = 2$ in (3.1) follows easily from a direct computation. So we assume $n \geq 3$. Observe that

$$\alpha_{n+1}^2\alpha_n^2 = \Psi_1\alpha_n^2 + \Psi_0. \tag{3.2}$$

Using (3.2), we have

$$\alpha_{n+2}^2\alpha_{n+1}^2\alpha_n^2 = \left(\Psi_1\alpha_{n+1}^2 + \Psi_0\right)\alpha_n^2 = \left(\Psi_1^2 + \Psi_0\right)\alpha_n^2 + \Psi_1\Psi_0, \tag{3.3}$$

which implies that

$$v_{n+2} = \alpha_{n+2}^2\alpha_{n+3}^2\alpha_{n+4}^2 - \alpha_{n-1}^2\alpha_n^2\alpha_{n+1}^2 = \left(\Psi_1^2 + \Psi_0\right)\left(\alpha_{n+2}^2 - \alpha_{n-1}^2\right) = \left(\Psi_1^2 + \Psi_0\right)(u_n + u_{n+1} + u_{n+2}).$$

Since $u_n = \alpha_n^2 - \alpha_{n-1}^2 = -\frac{\Psi_0}{\alpha_{n-2}^2\alpha_{n-1}^2}u_{n-1}$, we obtain

$$u_n + u_{n+1} + u_{n+2} = u_n - \frac{\Psi_0}{\alpha_{n-1}^2\alpha_n^2}u_n + \frac{\Psi_0^2}{\alpha_{n-1}^2\alpha_n^4\alpha_{n+1}^2}u_n = \left(1 - \frac{\Psi_0}{\alpha_{n-1}^2\alpha_n^2} + \frac{\Psi_0^2}{\alpha_{n-1}^2\alpha_n^4\alpha_{n+1}^2}\right)u_n,$$

which implies that

$$u_n v_{n+2} = \left(\Psi_1^2 + \Psi_0\right)\left(1 - \frac{\Psi_0}{\alpha_{n-1}^2\alpha_n^2} + \frac{\Psi_0^2}{\alpha_{n-1}^2\alpha_n^4\alpha_{n+1}^2}\right)u_n^2. \tag{3.4}$$

On the other hand, for $n \geq 3$, since

$$w_n = \alpha_n^2\alpha_{n+1}^2\left(\alpha_{n+2}^2 - \alpha_{n-1}^2\right)^2 = \alpha_n^2\alpha_{n+1}^2\left(u_n + u_{n+1} + u_{n+2}\right)^2, \tag{3.5}$$

by (3.4) and (3.5), we can obtain

$$u_n v_{n+2} - w_n = \Xi\left(1 - \frac{\Psi_0}{\alpha_{n-1}^2\alpha_n^2} + \frac{\Psi_0^2}{\alpha_{n-1}^2\alpha_n^4\alpha_{n+1}^2}\right)u_n^2,$$

where

$$\Xi := \Psi_1^2 + \Psi_0 - \alpha_n^2\alpha_{n+1}^2\left(1 - \frac{\Psi_0}{\alpha_{n-1}^2\alpha_n^2} + \frac{\Psi_0^2}{\alpha_{n-1}^2\alpha_n^4\alpha_{n+1}^2}\right).$$

It follows from (3.3) that $\left(\Psi_1^2 + \Psi_0\right)\alpha_{n-1}^2 = \alpha_{n+1}^2\alpha_n^2\alpha_{n-1}^2 - \Psi_1\Psi_0$ ($n \geq 3$), which induces that

$$\begin{aligned}\Xi &= \frac{1}{\alpha_{n-1}^2\alpha_n^2}\left((\Psi_1^2 + \Psi_0)\alpha_{n-1}^2\alpha_n^2 - \alpha_{n-1}^2\alpha_n^4\alpha_{n+1}^2 + \Psi_0\alpha_n^2\alpha_{n+1}^2 - \Psi_0^2\right)\\ &= \frac{1}{\alpha_{n-1}^2\alpha_n^2}\left(-\Psi_1\Psi_0\alpha_n^2 + \Psi_0\alpha_n^2\alpha_{n+1}^2 - \Psi_0^2\right).\end{aligned}$$

By (3.2), obviously $\Xi = 0$. Thus $u_n v_{n+2} = w_n$ ($n \geq 3$). Hence the proof is complete. □

We note from Lemma 3.1 that every coefficient $c_\ell(n, i)$ $(0 \leq i \leq n+1)$ of $d_n^{(\ell)}(t)$ $(\ell = 1, 2)$ satisfies (2.4) for all $n \geq 3$. If $\eta_n := \frac{v_n}{u_n}$ $(n \geq 0)$, where the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are given in (2.2), then by Lemma 3.1, the following result can be provided.

Lemma 3.2. *Let $\alpha(x)$: $\sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ be as above. Then for $\ell \geq 2$,*

$$q_\ell - \cfrac{z_\ell^2}{q_{\ell+2} - \cfrac{z_{\ell+2}^2}{\ddots \cfrac{\ddots}{q_{\ell+2k-2} - \frac{z_{\ell+2k-2}^2}{q_{\ell+2k}}}}} = v_\ell t + \frac{u_\ell}{1 + \eta_{\ell+2}t + \eta_{\ell+2}\eta_{\ell+4}t^2 + \cdots + \eta_{\ell+2}\eta_{\ell+4}\cdots\eta_{\ell+2k}t^k},$$

where $t = |s|^2$ and $k \geq 1$.

Proof. Using Lemma 3.1 and $z_n^2 = w_n|s|^2$, we obtain that for $n \geq 2$,

$$q_n - \frac{z_n^2}{q_{n+2}} = u_n + v_n t - \frac{u_n v_{n+2} t}{u_{n+2} + v_{n+2} t} = v_n t + \frac{u_n}{1 + \eta_{n+2} t}. \tag{3.6}$$

For a large number n, it follows from (3.6) that

$$q_{n-2} - \frac{z_{n-2}^2}{q_n - \frac{z_n^2}{q_{n+2}}} = v_{n-2}t + \frac{u_{n-2}}{1 + \eta_n t + \eta_n\eta_{n+2}t^2}.$$

Similarly, we have

$$\begin{aligned}q_{n-4} - \cfrac{z_{n-4}^2}{q_{n-2} - \cfrac{z_{n-2}^2}{q_n - \frac{z_n^2}{q_{n+2}}}} &= u_{n-4} + v_{n-4}t - \frac{u_{n-4}v_{n-2}t}{v_{n-2}t + \frac{u_{n-2}}{1+\eta_n t+\eta_n\eta_{n+2}t^2}}\\ &= v_{n-4}t + \frac{u_{n-4}}{1 + \eta_{n-2}t + \eta_{n-2}\eta_n t^2 + \eta_{n-2}\eta_n\eta_{n+2}t^3}.\end{aligned}$$

Continuing this process in the mathematical induction with $\ell = n - 2(k-1)$ $(k \geq 1)$, we have this lemma. □

Lemma 3.3. *Let $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ be as above. Then*
(i) $\eta_{n+1} \geq \eta_n$ *for all* $n \geq 4$,
(ii) $\lim_{n\to\infty} \eta_n = Q := \frac{(\Psi_1^2+\Psi_0)^2}{\Psi_0^2} L^4$.

Proof. It follows by the definition of η_n and (3.3) that for all $n \geq 4$,

$$\eta_n = \frac{\left(\Psi_1^2 + \Psi_0\right)\left(\alpha_n^2 - \alpha_{n-3}^2\right)}{\alpha_n^2 - \alpha_{n-1}^2} = \left(\Psi_1^2 + \Psi_0\right)\left(1 + \frac{\alpha_{n-1}^2 - \alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2} + \frac{\alpha_{n-2}^2 - \alpha_{n-3}^2}{\alpha_n^2 - \alpha_{n-1}^2}\right).$$

Using (2.5), we get

$$\frac{\alpha_{n-1}^2 - \alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2} = -\frac{\alpha_{n-1}^2\alpha_{n-2}^2}{\Psi_0} \quad \text{and} \quad \frac{\alpha_{n-2}^2 - \alpha_{n-3}^2}{\alpha_n^2 - \alpha_{n-1}^2} = \frac{\alpha_{n-1}^2\alpha_{n-2}^4\alpha_{n-3}^2}{\Psi_0^2},$$

which implies that

$$\eta_n = \left(\Psi_1^2 + \Psi_0\right)\left(1 - \frac{\alpha_{n-1}^2\alpha_{n-2}^2}{\Psi_0} + \frac{\alpha_{n-1}^2\alpha_{n-2}^4\alpha_{n-3}^2}{\Psi_0^2}\right).$$

Since $\{\alpha_n\}_{n=1}^\infty$ is non-decreasing, $\Psi_0 < 0$ and $\Psi_1^2 + \Psi_0 > 0$, we have $\eta_n \leq \eta_{n+1}$ for all $n \geq 4$. Also, since $\alpha_n^2 \to L^2$ $(n \to \infty)$ and $L^4 = L^2\Psi_1 + \Psi_0$, we can obtain

$$\lim_{n\to\infty} \eta_n = \left(\Psi_1^2 + \Psi_0\right)\left(1 - \frac{L^4}{\Psi_0} + \frac{L^8}{\Psi_0^2}\right) = \frac{\left(\Psi_1^2 + \Psi_0\right)^2}{\Psi_0^2} L^4.$$

Hence the proof is complete. □

For $t(= |s|^2) \geq 0$ and for each $n \in \mathbb{N}$, we define

$$A_n(t) = \begin{pmatrix} q_0 & 0 & -\sqrt{w_0 t} & 0 & \cdots & 0 \\ 0 & q_1 & 0 & -\sqrt{w_1 t} & \ddots & \vdots \\ -\sqrt{w_0 t} & 0 & q_2 & 0 & \ddots & 0 \\ 0 & -\sqrt{w_1 t} & 0 & q_3 & \ddots & -\sqrt{w_{n-2} t} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\sqrt{w_{n-2} t} & 0 & q_n \end{pmatrix}, \tag{3.7}$$

where the q_i and w_i are as in (2.2). Applying [15, Lemma 3.1] to the matrix $D_n(s)$ as in (2.1), we can see that W_α is semi-cubically hyponormal if and only if $A_n(t) \geq 0$ for all $t \geq 0$ and $n \geq 0$.

We now consider the quadratic form for $A_n(t)$ at a vector $\mathbf{x} = (x_0, \ldots, x_n)$ in $\mathbb{R}_+^{n+1}$ as follows:

$$F_n(x_0, x_1, ..., x_n, t) := \langle A_n(t)\mathbf{x}, \mathbf{x}\rangle.$$

For $n \geq 4$, it follows from Lemma 3.1 that

$$F_n(x_0, x_1, ..., x_n, t) = \sum_{i=0}^{1} (u_i + tv_i)\, x_i^2 + t\sum_{i=2}^{3} v_i x_i^2 - 2\sqrt{t}\sum_{i=0}^{1} \sqrt{w_i} x_i x_{i+2} + u_{n-1}x_{n-1}^2 + u_n x_n^2 + \sum_{i=2}^{n-2}\left(\sqrt{u_i}x_i - \sqrt{tv_{i+2}}x_{i+2}\right)^2.$$

For our convenience, we denote $f_2(x_0, x_1, x_2, x_3, t)$ as follows:

$$f_2(x_0, ..., x_3, t) := \sum_{i=0}^{1} (u_i + tv_i)\, x_i^2 + t\sum_{i=2}^{3} v_i x_i^2 - 2\sqrt{t}\sum_{i=0}^{1} \sqrt{w_i} x_i x_{i+2}.$$

Lemma 3.4. *Let* $\alpha(x):\ \sqrt{x},(\sqrt{a},\ \sqrt{b},\ \sqrt{c})^\wedge$ *and* $n \geq 4$. *Then the following conditions are equivalent:*
(i) $F_n(x_0,...,x_n,t) \geq 0$ *for any* $x_i,\ t \in \mathbb{R}_+\ (i = 0,1,\ldots,n)$;
(ii) *for any* $x_i, t \in \mathbb{R}_+\ (i = 0,1,2,3)$,

$$f_2(x_0,x_1,x_2,x_3,t) + P(2;n)x_2^2 + P(3;n)x_3^2 \geq 0,$$

where for $\ell \geq 2$,

$$P(\ell;n) = \frac{u_\ell}{1 + \eta_{\ell+2}t + \eta_{\ell+2}\eta_{\ell+4}t^2 + \cdots + \eta_{\ell+2}\eta_{\ell+4}\cdots\eta_{\ell+2[(n-\ell)/2]}t^{[(n-\ell)/2]}}.$$

Proof. For brevity, we write $F_n := F(x_0,...,x_n,t)$ and $f_2 := f_2(x_0,...,x_3,t)$. Using definitions of q_n and z_n in (2.2), we have

$$\begin{aligned} F_4 &= f_2 + u_3x_3^2 + \left(\sqrt{u_4+tv_4}x_4 - \frac{\sqrt{tw_2}}{\sqrt{u_4+tv_4}}x_2\right)^2 + \left(u_2 - \frac{tw_2}{u_4+tv_4}\right)x_2^2 \\ &= f_2 + u_3x_3^2 + \left(\sqrt{q_4}x_4 - \frac{z_2}{\sqrt{q_4}}x_2\right)^2 + \left(u_2 - \frac{z_2^2}{q_4}\right)x_2^2. \end{aligned} \tag{3.8}$$

Substituting $x_4 = \frac{z_2}{q_4}x_2$ in (3.8), we get

$$F_4(x_0,...,x_4,t) \geq 0 \Longrightarrow f_2 + u_3x_3^2 + \left(u_2 - \frac{z_2^2}{q_4}\right)x_2^2 \geq 0.$$

A similar method proves that

$$F_5 = f_2 + \sum_{i=4}^{5}\left(\sqrt{q_i}x_i - \frac{z_{i-2}}{\sqrt{q_i}}x_{i-2}\right)^2 + \left(u_2 - \frac{z_2^2}{q_4}\right)x_2^2 + \left(u_3 - \frac{z_3^2}{q_5}\right)x_3^2. \tag{3.9}$$

If we take $x_4 = \frac{z_2}{q_4}x_2$ and $x_5 = \frac{z_3}{q_5}x_3$ in (3.9) again, then we have

$$F_5 \geq 0 \Longrightarrow f_2 + \left(u_2 - \frac{z_2^2}{q_4}\right)x_2^2 + \left(u_3 - \frac{z_3^2}{q_5}\right)x_3^2 \geq 0.$$

Also, similarly we obtain

$$F_6 = f_2 + \sum_{i=5}^{6}\left(\sqrt{q_i}x_i - \frac{z_{i-2}}{\sqrt{q_i}}x_{i-2}\right)^2 + \left(u_3 - \frac{z_3^2}{q_5}\right)x_3^2 + \left(\sqrt{q_4 - \frac{z_4^2}{q_6}}x_4 - \sqrt{\frac{z_2^2}{q_4 - \frac{z_4^2}{q_6}}}x_2\right)^2 + \left(u_2 - \frac{z_2^2}{q_4 - \frac{z_4^2}{q_6}}\right)x_2^2. \tag{3.10}$$

Since x_4, x_5 and x_6 in (3.10) are arbitrary, we may take

$$x_4 = \frac{z_2}{q_4 - \frac{z_4^2}{q_6}}x_2,\ \ x_5 = \frac{z_3}{q_5}x_3 \ \text{ and }\ x_6 = \frac{z_2z_4}{q_6\left(q_4 - \frac{z_4^2}{q_6}\right)}x_2\ \left(= \frac{z_4}{q_6}x_4\right),$$

so that we may have the following implication

$$F_6(x_0,...,x_6,t) \geq 0 \Longrightarrow f_2 + \left(u_2 - \frac{z_2^2}{q_4 - \frac{z_4^2}{q_6}}\right)x_2^2 + \left(u_3 - \frac{z_3^2}{q_5}\right)x_3^2 \geq 0.$$

For $n \geq 4$ and $\ell = 2, 3$, if we continue the above processes $\left[\frac{n-\ell}{2}\right]$ times, then we may take the coefficients $P(\ell; n)$ of x_ℓ^2 such that

$$F_n(x_0, ..., x_n, t) \geq 0 \Longrightarrow f_2(x_0, ..., x_3, t) + P(2; n)x_2^2 + P(3; n)x_3^2 \geq 0,$$

where

$$P(\ell; n) = u_\ell - \cfrac{z_\ell^2}{q_{\ell+2} - \cfrac{z_{\ell+2}^2}{q_{\ell+4} - \cfrac{z_{\ell+4}^2}{\ddots \, \cfrac{\ddots}{q_{\ell+2[(n-\ell)/2]-2} - \cfrac{z_{\ell+2[(n-\ell)/2]-2}^2}{q_{\ell+2[(n-\ell)/2]}}}}}}.$$

Applying Lemma 3.2, it follows at once that

$$P(\ell; n) = \frac{u_\ell}{1 + \eta_{\ell+2}t + \eta_{\ell+2}\eta_{\ell+4}t^2 + \cdots + \eta_{\ell+2}\eta_{\ell+4}\cdots\eta_{\ell+2[(n-\ell)/2]}t^{[(n-\ell)/2]}}.$$

For the converse implication, we note that $F_n(x_0, ..., x_n, t)$ for $n \geq 4$ can be expressed by sum of $f_2(x_0, ..., x_3, t) + P(2; n)x_2^2 + P(3; n)x_3^2$ and other quadratic forms. Hence $F_n(x_0, ..., x_n, t) \geq 0$ for all $x_i, t \in \mathbb{R}_+$ $(i = 0, ..., n)$ and $n \geq 4$. □

Applying the argument in [15] to the quadratic form of F_n in Lemma 3.4 and using Lemma 3.3, we obtain easily the following lemma.

Lemma 3.5. *Suppose $n \geq 4$. Then $F_n(x_0, x_1, ..., x_n, t) \geq 0$ for any $x_i \in \mathbb{R}_+$ and $t > \frac{1}{Q}$ if and only if $f_2(x_0, ..., x_3, t) \geq 0$ for any $x_0, ..., x_3 \in \mathbb{R}_+$ and $t > \frac{1}{Q}$.*

To obtain the dominating number of x, we let

$$\widehat{h_3} = \min\{\sqrt{a}, \sqrt{\Theta}\}, \text{ where } \Theta := \frac{ab\left\{(c-a)^2(c-b)Q + c\left(a^2bc - a^2b^2 + a^2c^2 + 2ab^2c - 4abc^2 + bc^3\right)\right\}}{a^2bc(b-a)^2 + (c-a)^2(a^2 + bc - 2ab)Q},$$

for Q as in Lemma 3.3 (ii).

Lemma 3.6. $\sup\{x : f_2(x_0, x_1, x_2, x_3, t) \geq 0, \ t > 1/Q\} \leq (\widehat{h_3})^2.$

Proof. Since the function $f_2(x_0, ..., x_3, t)$ is the quadratic form, the corresponding symmetric matrix $\Omega(t)$ to $f_2(x_0, ..., x_3, t)$ can be represented by

$$\Omega(t) = \begin{pmatrix} x + abxt & 0 & -\sqrt{txab^2} & 0 \\ 0 & a - x + abct & 0 & -\sqrt{tab(c-x)^2} \\ -\sqrt{txab^2} & 0 & tbc\alpha_4^2 & 0 \\ 0 & -\sqrt{tab(c-x)^2} & 0 & t\left(c\alpha_4^2\alpha_5^2 - xab\right) \end{pmatrix},$$

where

$$\alpha_4^2 = \frac{b(c^2 - 2ac + ab)}{c(b-a)}, \quad \alpha_5^2 = \frac{a(2c-a)b^2 + c(c^2 - 4ac + a^2)b + a^2c^2}{(b-a)(c^2 - 2ac + ab)}.$$

We can easily see that $\det \Omega(t) = d_1(t) \cdot d_2(t)$, where

$$d_1(t) = \det \begin{pmatrix} x + abxt & -\sqrt{txab^2} \\ -\sqrt{txab^2} & tbc\alpha_4^2 \end{pmatrix}, \qquad d_2(t) = \det \begin{pmatrix} a - x + abct & -\sqrt{tab\,(c-x)^2} \\ -\sqrt{tab\,(c-x)^2} & t\left(c\alpha_4^2\alpha_5^2 - xab\right) \end{pmatrix}.$$

A straightforward computation shows that

$$d_1(t) = \frac{b^2xt}{b-a}\left(abt(c^2 - 2ac + ab) + (a-c)^2\right) > 0 \text{ for all } t > 0.$$

By a simple calculation, we have

$$d_2(t) = \frac{bt\,(A(x)t + B(x))}{(b-a)^2},$$

where

$$A(x) = abc\left(-a\,(b-a)^2\,x - a^2b^2 + a^2bc + a^2c^2 + 2ab^2c - 4abc^2 + bc^3\right),$$
$$B(x) = (c-a)^2\left(ab(c-b) + x(2ab - bc - a^2)\right).$$

Note that $d_2(t) \geq 0$ for all $t > \frac{1}{Q} \Leftrightarrow A(x) \geq 0$ and $-\frac{B(x)}{A(x)} \leq \frac{1}{Q}$. From the assumption $a < b < c$, a direct computation shows that $A(x) \geq 0$ for all $x \leq a$. Therefore, $d_2(t) \geq 0$ for $t > \frac{1}{Q} \Leftrightarrow x \leq \Theta$. Since $0 < x \leq a$, we have $x \leq (\widehat{h_3})^2$. □

3.2. Characterization. The following is contained in main results of this paper.

Theorem 3.7. *Let $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $x \leq a < b < c$ and let $W_{\alpha(x)}$ be the associated weighted shift. Then the following assertions are equivalent:*
(i) *$W_{\alpha(x)}$ is semi-cubically hyponormal;*
(ii) *$F_n(x_0, x_1, ..., x_n, t) \geq 0$ for all $x_0, x_1, ..., x_n, t$ in $\mathbb{R}_+$ and all $n \geq 2$;*
(iii) *$F_n(x_0, x_1, ..., x_n, t) \geq 0$ for all $x_0, x_1, ..., x_n$ in $\mathbb{R}_+$, $t > 1/Q$ and all $n \geq 4$;*
(iv) *$f_2(x_0, x_1, x_2, x_3, t) \geq 0$ for all x_0, x_1, x_2, x_3 in $\mathbb{R}_+$ and all $t > 1/Q$;*
(v) *$\sqrt{x} \leq \widehat{h_3}$.*

Proof. (i) ⇔ (ii) and (ii) ⇒ (iii) It is obvious.
(iii) ⇔ (iv) and (iv) ⇒ (v) Use Lemma 3.5 and Lemma 3.6, respectively.
(v) ⇒ (i) Let $0 < \sqrt{x} \leq \widehat{h_3}$. To show the semi-cubic hyponormality of $W_{\alpha(x)}$, we will use the methods in Section 2 under the same notation, i.e., we will prove that every coefficient $c_\ell(n, i)$ is nonnegative ($n \geq 0; 0 \leq i \leq n+1$) in polynomials of determinants $D_n^{(\ell)} \equiv D_n^{(\ell)}(t)$ ($\ell = 1, 2$) for $t := |s|^2$ ($s \in \mathbb{C}$).

First of all, we can see from (2.3) that

$$c_1(0,1) = \check{v}_0 = v_0 = xab > 0 \text{ and } c_2(0,1) = \check{v}_1 = v_1 = abc > 0,$$

for $0 < x \leq a < b$. Also we have $c_1(n, 0) = \check{u}_n \cdots \check{u}_0 = u_{2n}u_{2(n-1)} \cdots u_0 > 0$ and $c_2(n, 0) = u_{2n+1}u_{2n-1} \cdots u_1 \geq 0$ for $0 < x \leq a$.

Claim 1. For $n \geq 1$, $c_1(n, i) \geq 0$ with $1 \leq i \leq n + 1$.
A straightforward computation shows that for all $x > 0$,

$$c_1(1,1) = \frac{bx\left(a(b-c)^2 + (c-a)(b-a)(a+c)\right)}{b-a} > 0, \qquad c_1(1,2) = \frac{ab^3(ab - 2ac + c^2)x}{b-a} > 0.$$

Since $\breve{u}_2 = u_4$, $\breve{v}_2 = v_4$ and $\breve{w}_1 = w_2$ in the matrix $D_n^{(1)}$, we get

$$\rho_1 = \breve{v}_2 c_1(1,1) - \breve{w}_1 c_1(0,1) = \frac{b^3(a-c)^2\left(ab^2 + a^2c + bc^2 - 3abc\right)^2 x}{c\,(a-b)^4} > 0,$$
$$\tau_1 = \breve{v}_2 c_1(1,0) - \breve{w}_1 c_1(0,0) = 0.$$

From (2.3), we have the following:

$$c_1(2,1) = \breve{u}_2 c_1(1,1) > 0, \quad c_1(2,3) = \breve{v}_2 c_1(1,2) > 0,$$
$$c_1(2,2) = \breve{u}_2 c_1(1,2) + \breve{v}_2 c_1(1,1) - \breve{w}_1 c_1(0,1) = \breve{u}_2 c_1(1,2) + \rho_1 > 0.$$

Using (2.4) and the mathematical induction, we have $c_1(n,i) \geq 0$ for $x \leq a$ and all $n \geq 3$ with $1 \leq i \leq n+1$.

Claim 2. For $n \geq 1$, $c_2(n,i) \geq 0$ with $1 \leq i \leq n+1$.

From standard computations, it follows that

$$c_2(1,2) = ab^2c\left(\frac{bc^3 - a^2b^2 + a^2c^2 - 4abc^2 + 2ab^2c + a^2bc}{(a-b)^2} - ax\right).$$

Write $\hat{x} = \frac{bc^3 - a^2b^2 + a^2c^2 - 4abc^2 + 2ab^2c + a^2bc}{a(a-b)^2}$. For simple computations, we sometimes substitute $b = a + h$ and $c = a + h + k$ for any h, $k > 0$. A straightforward calculation shows that $\hat{x} > a$ and $\hat{x} > \Theta$, which implies that $c_2(1,2) > 0$ for $0 < x \leq (\widehat{h_3})^2$. So $c_2(2,3) = \breve{v}_2 c_2(1,2) > 0$ and thus $c_2(n,n+1) = \breve{v}_n \cdots \breve{v}_3 c_2(1,2) > 0$ $(n \geq 3)$. Denote

$$x_i := \sup\{x > 0 : c_2(i,i) \geq 0\} \text{ for } i = 1,2,3.$$

By some computations, we can obtain $\Theta < x_3 < x_i$ $(i = 1,2)$ (see Appendix for the detail), i.e., $x_i \geq (\widehat{h_3})^2$. Hence $c_2(i,i) \geq 0$ for $i = 1,2,3$ and $x \in (0, (\widehat{h_3})^2]$. Since $\tau_2 = \breve{v}_2 c_2(1,0) - \breve{w}_1 c_2(0,0) = 0$, using (2.3) and (2.4), we obtain that three coefficients $c_2(2,1)$, $c_2(3,1)$ and $c_2(3,2)$ are positive. Since $\breve{u}_2 = u_5$, $\breve{v}_2 = v_5$ and $\breve{w}_1 = w_3$ in the matrix $D_n^{(2)}$, we have

$$\begin{aligned}\rho_2 &= \breve{v}_2 c_2(1,1) - \breve{w}_1 c_2(0,1) \\ &= \frac{b^2(a-c)^2(c-b)\left(ab^2 + a^2c + bc^2 - 3abc\right)^2}{(b-a)^5(ab - 2ac + c^2)}\left(ab(c-b) - \left(a^2 - 2ba + bc\right)x\right).\end{aligned}$$

Write

$$\hat{s} := \frac{ab(c-b)}{a^2 - 2ab + bc}.$$

By substitution $b = a + h$ and $c = a + h + k$ $(h,k > 0)$, we can have $\hat{s} < a$ and $\hat{s} < \Theta$, which induces $\rho_2 \geq 0$ for $x \in (0, \hat{s}]$ and $\rho_2 < 0$ for $x \in (\hat{s}, (\widehat{h_3})^2]$. So we consider two cases below.

Now we consider $0 < x \leq \hat{s}$, i.e., $\rho_2 \geq 0$. Using (2.4) and $c_2(n,n+1) \geq 0$ $(n \geq 2)$, we have

$$c_2(n,n) = \breve{u}_n c_2(n-1,n) + \breve{v}_n \cdots \breve{v}_3 \rho_2 \geq 0 \ (n \geq 3).$$

Since $\tau_2 = 0$ in (2.4), obviously $c_2(n,n-1) \geq 0$ $(n \geq 3)$. Using the mathematical induction in (2.4), we have $c_2(n,i) \geq 0$ $(n \geq 3; 1 \leq i \leq n-2)$ for all $x \in (0, \hat{s}]$.

Next we suppose that $\hat{s} < x \leq (\widehat{h_3})^2$, i.e., $\rho_2 < 0$. We already obtained $c_2(n,i) \geq 0$ for $n = 1,2,3$ with $1 \leq i \leq n+1$, which is independent to the sign of ρ_2. For all $n \geq 4$, using (2.4), we can see that

$$c_2(n,n) = \breve{v}_{n-1} \cdots \breve{v}_3 \breve{u}_n \left(\breve{v}_2 c_2(1,2) + \frac{\breve{v}_n}{\breve{u}_n}\rho_2\right).$$

It follows from Lemma 3.3 that $\frac{\check{v}_n}{\check{u}_n} \nearrow Q$ $(n \to \infty)$. Hence, if $\Omega_2 := \check{v}_2 c_2(1,2) + Q\rho_2 \geq 0$ for $x \in (\hat{s}, (\widehat{h_3})^2]$, since $\rho_2 < 0$, we have $c_2(n,n) \geq 0$ $(n \geq 4)$. Observe that

$$\Omega_2 = b^2(c-b)(ab^2 + a^2c + bc^2 - 3abc)^2 \frac{N_1 - N_2 x}{(b-a)^5(ab - 2ac + c^2)},$$

where

$$N_1 := ab\left((a-c)^2(c-b)Q + c(bc^3 - a^2b^2 + a^2c^2 - 4abc^2 + 2ab^2c + a^2bc)\right),$$
$$N_2 := (c-a)^2(a^2 - 2ab + bc)Q + a^2bc(b-a)^2.$$

Since $a < b < c$, we have $N_1 > 0$ and $N_2 > 0$. Observe that $\Theta = \frac{N_1}{N_2}$. This implies that $\Omega_2 \geq 0$ for all $x \in (\hat{s}, (\widehat{h_3})^2]$. Furthermore, since $c_2(n,n) \geq 0$ $(n \geq 3)$ and $\tau_2 = 0$, we get $c_2(n, n-1) = \check{u}_n c_2(n-1, n-1) \geq 0$ $(n \geq 3)$. And, by the mathematical induction we have $c_2(n,i) \geq 0$ $(n \geq 3; 2 \leq i \leq n-1)$. Hence the proof is complete. □

For $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $0 < x \leq a < b < c$, we denote

$$h_2^+ := \left(\sup\left\{x | W_{\alpha(x)} \text{ is positively quadratically hyponormal}\right\}\right)^{\frac{1}{2}},$$
$$h_2 := \left(\sup\left\{x | W_{\alpha(x)} \text{ is quadratically hyponormal}\right\}\right)^{\frac{1}{2}},$$

as in [5, Theorem 4.3]. Recall that the weighted shift $W_{\alpha(x)}$ is positively quadratically hyponormal if and only if it is quadratically hyponormal ([15, Theorem 4.1]). Then it follows from [5, Theorem 4.3] that

$$h_2^+ = h_2 = \min\left\{\sqrt{a}, \left(\frac{a^2b^2c + ab^2(c-a)K + ab(c-b)K^2}{a^3b + ab(c-a)K + (a^2 - 2ab + bc)K^2}\right)^{\frac{1}{2}}\right\}, \tag{3.11}$$

where $K := -\frac{\Psi_1^2}{\Psi_0} L^2$.

We now give an example of a weighted shift with quadratic hyponormality but not semi-cubic hyponormality.

Example 3.8. Let $W_{\alpha(x)}$ be a weighted shift with weight sequence $\alpha(x) : \sqrt{x}, \left(1, \sqrt{2}, \sqrt{3}\right)^\wedge$, where $0 < x \leq 1$. A straightforward computation shows that $\Psi_0 = -2, \Psi_1 = 4$, $K = 8\sqrt{2} + 16$ and $Q = 49(\sqrt{2} + 2)^2$. So by (3.11), we obtain $h_2 = \left(\frac{2}{50881}(23043 - 3104\sqrt{2})\right)^{\frac{1}{2}} \approx 0.85628$ (cf. [5, Example 4.5]) and $\widehat{h_3} = \frac{1}{17}\left(\frac{1}{411}(108047 - 19208\sqrt{2})\right)^{\frac{1}{2}} \approx 0.82520$. The interval $(\widehat{h_3}, h_2]$ is the range of $\sqrt{x}$ such that $W_{\alpha(x)}$ is quadratically hyponormal but not semi-cubically hyponormal. In fact, we will prove that two notions of quadratic hyponormality and semi-cubic hyponormality are different one from another (see Theorem 4.2 below).

4. Recursive weighted shifts with two equal weights

We first recall from [10] that there exists a nontrivial semi-cubically hyponormal weighted shift. So it is worthwhile to discuss semi-cubically hyponormal weighted shifts with first two equal weights (cf. [3]). For this purpose we consider a recursively generated weighted shift W_α with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$. To avoid the trivial case we assume $1 < x < y$ in this section. Let us consider $x = 1 + h$ and $y = 1 + h + k$ for all h, $k > 0$. Then

$$Q = \frac{1}{4h^4k^2}(h^3 + h^2 + hk + 2h^2k + k^2 + hk^2)^2((h+1)(h+k) + S_{h,k})^2,$$

where Q as in Lemma 3.3 (ii) and $S_{h,k} = \left((h+1)(h(h+k)^2 + (h-k)^2)\right)^{1/2}$.

The following theorem comes immediately from Theorem 3.7.

Theorem 4.1. *Let $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ and let W_α be the associated weighted shift. Put $x = 1 + h$ and $y = 1 + h + k$ $(h, k > 0)$. Then W_α is semi-cubically hyponormal if and only if $\widehat{p_3}(h,k) = \sum_{i=0}^{9} \xi_i k^i \le 0$, where*

$$\xi_0 = 2h^9 (h+1)^4, \quad \xi_1 = h^8 (16h+7)(h+1)^3,$$
$$\xi_2 = 4h^6 \left(3h + 14h^2 + 14h^3 - 1\right)(h+1)^2,$$
$$\xi_3 = h^5 (h+1)\left(3h + 98h^2 + 190h^3 + 112h^4 - 4\right),$$
$$\xi_4 = h^4 \left(2h + 109h^2 + 322h^3 + 356h^4 + 140h^5 - 5\right),$$
$$\xi_5 = 2h^3 (h+1)\left(5h + 46h^2 + 88h^3 + 56h^4 - 1\right),$$
$$\xi_6 = h^2 (h+1)\left(13h + 64h^2 + 104h^3 + 56h^4 - 1\right),$$
$$\xi_7 = h^2 (h+1)\left(34h + 42h^2 + 16h^3 + 9\right),$$
$$\xi_8 = 2h\left(4h + h^2 + 2\right)(h+1)^2, \text{ and } \xi_9 = (h+1)^3.$$

We exhibit the relationship between semi-cubic hyponormality and quadratic hyponormality of weighted shift W_α with a weight $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$. Recall from [6] that W_α is quadratically hyponormal if and only if $p_2(h,k) = \sum_{i=0}^{7} \rho_i k^i \le 0$, where

$$\rho_0 = h^7 (h+2)(h+1)^3, \quad \rho_1 = h^6 \left(16h + 6h^2 + 7\right)(h+1)^2,$$
$$\rho_2 = h^4 \left(5h + 53h^2 + 96h^3 + 66h^4 + 15h^5 - 4\right),$$
$$\rho_3 = h^3 (h+1)\left(5h + 52h^2 + 65h^3 + 20h^4 - 4\right),$$
$$\rho_4 = h^2 \left(8h + 35h^2 + 15h^3 - 1\right)(h+1)^2,$$
$$\rho_5 = 3h^2 \left(6h + 2h^2 + 3\right)(h+1)^2, \ \rho_6 = h(h+5)(h+1)^3, \ \rho_7 = (h+1)^3.$$

We now denote

$$\mathcal{R}_2 = \{(h,k) : W_\alpha \text{ is quadratically hyponormal}\},$$
$$\widehat{\mathcal{R}}_3 = \{(h,k) : W_\alpha \text{ is semi-cubically hyponormal}\},$$

and will see that the quadratic hyponormality and semi-cubic hyponormality of W_α are different one from another in the following theorem.

Theorem 4.2. *Let W_α be a weighted shift with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$. Then $\mathcal{R}_2 \backslash \widehat{\mathcal{R}}_3$, $\widehat{\mathcal{R}}_3 \backslash \mathcal{R}_2$ and $\mathcal{R}_2 \cap \widehat{\mathcal{R}}_3$ are all nonempty sets.*

Proof. Let $x = 1 + h$ and $y = 1 + h + k$ with $h, k > 0$. To show that the sets in the conclusion of this theorem are nonempty, we take a proper number $h = \frac{1}{100}$ (in fact, we may find a proper number using the Mathematica computer program). We denote $f(k) = p_2(1/100, k)$ and $g(k) = \widehat{p_3}(1/100, k)$ for $k > 0$. Then

$$f(k) = \sum_{i=0}^{7} c_i k^i \ \text{ and } \ g(k) = \sum_{j=0}^{9} d_j k^j,$$

where $c_i \equiv \rho_i|_{h=1/100}$ $(0 \le i \le 7)$ and $d_j \equiv \xi_j|_{h=1/100}$ $(0 \le j \le 9)$ are the coefficients of polynomials $f(k)$ and $g(k)$, respectively. Observe that

$$c_0, c_1, c_5, c_6, c_7 > 0 \text{ and } c_i < 0 \ (i = 2, 3, 4),$$
$$d_0, d_1, d_7, d_8, d_9 > 0 \text{ and } d_j < 0 \ (j = 2, ..., 6).$$

Then it follows from Descartes' rule of signs in calculus that each of polynomials $f(k)$ and $g(k)$ has two sign changes. Hence each of $f(k)$ and $g(k)$ has at most two positive roots. We now consider the following sets

$$\mathcal{K}_2 = \{k > 0 : f(k) \le 0\} \text{ and } \widehat{\mathcal{K}}_3 = \{k > 0 : g(k) \le 0\},$$

which are the projections of $\mathcal{R}_2$ and $\widehat{\mathcal{R}}_3$, respectively. Since

$$f(0) > 0,\ f(1/100) < 0,\ f(1) > 0 \text{ and } f'(k) > 0 \text{ for all } k \ge 1,$$

by a simple computation, $f(k)$ has only two positive roots α_1 and α_2 in $\mathbb{R}_+$ such that $\mathcal{K}_2 = [\alpha_1, \alpha_2]$; in fact, $\alpha_1 = 0.000787776068\cdots$ and $\alpha_2 = 0.0422764016\cdots$. Observe that

$$g(0) > 0,\ g(1/100) < 0,\ g(1) > 0 \text{ and } g'(k) > 0 \text{ for all } k \ge 1,$$

which implies that $g(k)$ has only two positive roots β_1 and β_2 in $\mathbb{R}_+$ such that $\widehat{\mathcal{K}}_3 = [\beta_1, \beta_2]$; in fact, $\beta_1 = 0.000786885627\cdots$ and $\beta_2 = 0.0402782805\cdots$. Hence we obtain $[\alpha_1, \beta_2] = \mathcal{K}_2 \cap \widehat{\mathcal{K}}_3$, $(\beta_2, \alpha_2] = \mathcal{K}_2 \backslash \widehat{\mathcal{K}}_3$ and $[\beta_1, \alpha_1) = \widehat{\mathcal{K}}_3 \backslash \mathcal{K}_2$, which prove that $\mathcal{R}_2 \cap \widehat{\mathcal{R}}_3$, $\mathcal{R}_2 \backslash \widehat{\mathcal{R}}_3$ and $\widehat{\mathcal{R}}_3 \backslash \mathcal{R}_2$ are nonempty. Hence the proof is complete. □

Appendix

I. Proof of $x_3 > \Theta$ in the proof of Theorem 3.7.

To simplify the computations, it is convenient to make the substitutions $b = a + h$, $c = a + h + k$ with $h, k > 0$. Then

$$Q = \frac{(a+h)(h(h+k)^2 + a(h^2+hk+k^2))^2(a(h^2+k^2) + (h+k)(h^2+hk+S))}{2a^2h^4k^2},$$

where $S := \sqrt{(a+h)(ah^2 + h^3 - 2ahk + 2h^2k + ak^2 + hk^2)}$, which implies that

$$x_3 - \Theta = \frac{\psi_{1,a,h,k}\left((a+h+k)S + a^2(k-h) - 2ah^2 - h^3 - ahk - 2h^2k - ak^2 - hk^2\right)}{\sum_{i=0}^{10} \phi_{1,i}k^i \left(\psi_{2,a,h,k}S + \sum_{i=0}^{9} \phi_{2,i}k^i\right)},$$

where

$$\psi_{1,a,h,k} := a^3h^4k^2(h+k)^3(a+h+k)(ah^2 + h^3 + ahk + 2h^2k + ak^2 + hk^2)^4,$$
$$\psi_{2,a,h,k} := (h+k)^3(h^2 + ak + hk)(h(h+k)^2 + a(h^2+hk+k^2))^2,$$
$$\phi_{1,0} := h^{10}(a+h)^4,\ \ \phi_{1,1} := h^8(a+h)^3(a^2 + 6ah + 10h^2),$$
$$\phi_{1,2} := h^6(a+h)^2(a^4 + 4a^3h + 25a^2h^2 + 58ah^3 + 45h^4),$$
$$\phi_{1,3} := h^6(a+h)(10a^4 + 62a^3h + 187a^2h^2 + 248ah^3 + 120h^4),$$
$$\phi_{1,4} := 2h^5(5a^5 + 50a^4h + 190a^3h^2 + 349a^2h^3 + 308ah^4 + 105h^5),$$
$$\phi_{1,5} := h^4(a+2h)(8a^4 + 84a^3h + 253a^2h^2 + 301ah^3 + 126h^4),$$
$$\phi_{1,6} := h^3(4a^5 + 75a^4h + 348a^3h^2 + 684a^2h^3 + 616ah^4 + 210h^5),$$
$$\phi_{1,7} := h^2(a+h)(a^4 + 42a^3h + 173a^2h^2 + 248ah^3 + 120h^4),$$
$$\phi_{1,8} := h^2(a+h)^2(19a^2 + 58ah + 45h^2),$$
$$\phi_{1,9} := 2h(a+h)^3(3a+5h),\ \ \phi_{1,10} := (a+h)^4,$$

and

$$\phi_{2,0} := h^{10}(a+h)^3,\ \ \phi_{2,1} := h^8(a+h)^2(a+3h)^2,$$
$$\phi_{2,2} := h^6(a+h)(2a^4+4a^3h+25a^2h^2+52ah^3+36h^4),$$
$$\phi_{2,3} := h^6(11a^4+64a^3h+169a^2h^2+196ah^3+84h^4),$$
$$\phi_{2,4} := 2h^5(7a^4+49a^3h+128a^2h^2+147ah^3+63h^4),$$
$$\phi_{2,5} := 2h^4(8a^4+55a^3h+139a^2h^2+154ah^3+63h^4),$$
$$\phi_{2,6} := 7h^3(a+h)(a+2h)^2(2a+3h),\ \ \phi_{2,7} := 9h^2(a+h)^2(a+2h)^2,$$
$$\phi_{2,8} := h(a+h)^3(4a+9h),\ \ \phi_{2,9} := (a+h)^4.$$

Since

$$S^2 - \frac{(a^2(h-k)+2ah^2+h^3+ahk+2h^2k+ak^2+hk^2)^2}{(a+h+k)^2} = \frac{4a^2(a+h)k^3}{(a+h+k)^2} > 0,$$

we can obtain that $x_3 - \Theta > 0$ for all $h,\ k > 0$.

II. Proof of $x_1 > x_3$ and $x_2 > x_3$ in the proof of Theorem 3.7.
If we put $b = a+h$, $c = a+h+k$ with $h, k > 0$ and follow the similar method in Appendix I, we may obtain the following expressions

$$x_1 - x_3 = \frac{ah^2k(a+h+k)\sum_{i=0}^{10} g_{1,i}k^i}{(h+k)^2(h^2+ak+hk)\sum_{i=0}^{10} g_{2,i}k^i},$$
$$x_2 - x_3 = \frac{a^3h^2k^3(a+h)(h+k)^4(ah^2+h^3+ahk+2h^2k+ak^2+hk^2)^4}{\sum_{i=0}^{10} g_{2,i}k^i \sum_{i=0}^{7} g_{3,i}k^i},$$

where $g_{i,j}$ are some polynomials with a and h such that all of the coefficients of g_{ij} are strictly positive. Hence $x_1 > x_3$ and $x_2 > x_3$.

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