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Existence of covering topological R-modules

Nazmiye Alemdar[a], Osman Mucuk[b]

[a] *Department of Mathematics, Erciyes University Kayseri 38039, Turkey*
[b] *Department of Mathematics, Erciyes University Kayseri 38039, Turkey*

Abstract. Let R be a topological ring with identity and M a topological (left) R-module such that the underlying topology of M is path connected and has a universal cover. Let $0 \in M$ be the identity element of the additive group structure of M, and N a submodule of the R-module $\pi_1(M, 0)$. In this paper we prove that if R is discrete, then there exists a covering morphism $p\colon (\widetilde{M}_N, \tilde{0}) \to (M, 0)$ of topological R-modules with characteristic group N and such that the structure of R-module on M lifts to $\widetilde{M}_N$. In particular, if N is a singleton group, then this cover becomes a universal cover.

Introduction

The theory of covering spaces is one of the most interesting theories in algebraic topology. It is well known that the group structure of a topological group X lifts to its simply connected covering space, i.e., if X is an additive topological group, $p\colon \widetilde{X} \to X$ is a simply connected covering map, $0 \in X$ is the identity element and $\tilde{0} \in \widetilde{X}$ is such that $p(\tilde{0}) = 0$, then $\widetilde{X}$ becomes a topological group with identity $\tilde{0}$ such that p is a morphism of topological groups (see for example [4]). Related to this a monodromy principle for topological ring is proved in [7] and for topological R-modules in [8].

Let X be a topological space which has a universal cover, $x_0 \in X$ and G a subgroup of the fundamental group $\pi_1(X, x_0)$ of X at the point x_0. We know from [9, Theorem 10.42] that there is a covering map $p\colon (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ of pointed spaces. Here note that if G is singleton this becomes the universal covering map. Further, if X is a topological group, then $\widetilde{X}_G$ becomes a topological group such that p is a morphism of topological groups.

In this paper we apply this method to topological R-modules in the case where the topological ring R is discrete and obtain a more general result than the one for the topological group case. So this generalized result guarantees that the R-module structure of a topological R-module lifts to the universal cover. Recently, in [1], some results on the covering morphisms of R-module objects in the category of groupoids have been given (see also [5]). The problem of universal covers of non-connected topological groups was first studied in [10], where it is proved that a topological group X determines an obstruction class k_X in $H^3(\pi_0(X), \pi_1(X, e))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure to a universal cover. In [6] an analogous algebraic result is given in terms of crossed modules and group objects in the category of groupoids (see also [3] for a revised version, which generalizes these results and shows the relation with the theory of obstructions to extension for groups).

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Email addresses: nakari@erciyes.edu.tr (Nazmiye Alemdar), mucuk@erciyes.edu.tr (Osman Mucuk)

1. Preliminaries on covering spaces

We assume the usual theory of covering maps. All spaces X are assumed to be locally path connected and semi-locally 1-connected, so that each path component of X admits a simply connected cover.

Recall that a covering map $p\colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every cover of X in the sense that if $q\colon \widetilde{Y} \to X$ is another cover of X then there exists a map $r\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a cover). A covering map $p\colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. Note that a simply connected cover is a universal cover.

We call a subset V of X *liftable* if it is open, path connected and V lifts to each cover of X, that is, if $p\colon \widetilde{X} \to X$ is a covering map, $\imath\colon V \to X$ is the inclusion map, and $\tilde{x} \in \widetilde{X}$ satisfies $p(\tilde{x}) = x \in V$, then there exists a map (necessarily unique) $\hat{\imath}\colon V \to \widetilde{X}$ such that $p\hat{\imath} = \imath$ and $\hat{\imath}(x) = \tilde{x}$.

It is easy to see that V is liftable if and only if it is open, path connected and for all $x \in V$, $\imath_\star\pi_1(V, x) = \{e\}$, where $\pi_1(V, x)$ is the fundamental group of V at the base point x, $\imath_\star$ is the map induced by the inclusion map $\imath\colon V \to X$ and e is the identity element of the fundamental group $\pi_1(X, x)$. Remark that if X is a semi-locally simply connected topological space, then each point $x \in X$ has a liftable neighbourhood.

For a covering map $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$, the subgroup $p_\star(\pi_1(\widetilde{X}, \tilde{x}_0))$ of $\pi_1(X, x_0)$ is called *characteristic group* of p, where $p_\star$ is the morphism induced by p. Two covering maps $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ and $q\colon (\widetilde{Y}, \tilde{y}_0) \to (X, x_0)$ are called *equivalent* if there is a homeomorphism $f\colon (\widetilde{X}, \tilde{x}_0) \to (\widetilde{Y}, \tilde{y}_0)$ such that $qf = p$.

Let X be a topological space which has a universal cover and $x_0 \in X$ and G a subgroup of the fundamental group $\pi_1(X, x_0)$ of X at the point x_0. We recall a construction from [9, p.295] as follows: Let $P(X, x_0)$ be the set of all paths of α in X with $\alpha(0) = x_0$. Define a relation on $P(X, x_0)$ by $\alpha \simeq \beta$ if and only if $\alpha(1) = \beta(1)$ and $[\alpha\beta^{-1}] \in G$. The relation defined in this way is an equivalence relation. Denote the equivalence relation of α by $\langle\alpha\rangle_G$ and define $\widetilde{X}_G$ as the set of all such equivalence classes. Define a function $p\colon \widetilde{X}_G \to X$ by $p(\langle\alpha\rangle_G) = \alpha(1)$. Let α_0 be the constant path at x_0 and $\tilde{x}_0 = \langle\alpha_0\rangle_G \in \widetilde{X}_G$. If $\alpha \in P(X, x_0)$ and U is an open neighbourhood of $\alpha(1)$, then a path of the form $\alpha\lambda$, where λ is a path in U with $\lambda(0) = \alpha(1)$, is called a *continuation* of α. For an $\langle\alpha\rangle_G \in \widetilde{X}_G$ and an open neighbourhood U of $\alpha(1)$, let

$$(\langle\alpha\rangle_G, U) = \{\langle\alpha\lambda\rangle_G \colon \lambda(I) \subseteq U\}.$$

Then the subsets $(\langle\alpha\rangle_G, U)$ form a basis for a topology on $\widetilde{X}_G$ such that the map $p\colon (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ is continuous.

Theorem 1.1. ([9, Theorem 10.34]) *Let (X, x_0) be a pointed topological space and G a subgroup of $\pi_1(X, x_0)$. If X is connected, locally path connected and semi-locally simply connected, then $p\colon (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ is a covering map with characteristic group G .*

Remark 1.2. Let X be a connected, locally path connected and semi-locally simply connected topological space and $q\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map. Let G be the characteristic group of q. Then the covering map q is equivalent to the covering map $p\colon (\widetilde{X}_G, \tilde{x}_0) \to (X, x_0)$ corresponding to G.

So from Theorem 1.1 the following result is obtained.

Theorem 1.3. ([9, Theorem 10.42]) *Suppose that X is a connected, locally path connected and semi-locally simply connected topological group. Let $e \in X$ be the identity element and $q\colon (\widetilde{X}, \tilde{e}) \to (X, e)$ a covering map. Then the group structure of X lifts to $\widetilde{X}$, i.e $\widetilde{X}$ becomes a topological group such that $\tilde{e}$ is identity and $q\colon (\widetilde{X}, \tilde{e}) \to (X, e)$ is a morphism of topological groups.*

2. Existence of covering topological modules

In this section we apply the preliminaries of Section 1 to topological R-modules and obtain covering R-modules of some topological R-modules.

A *topological ring* is a ring R with a topology on the underlying set such that the structure maps $(x, y) \mapsto x + y$, $x \mapsto -x$ and $(x, y) \mapsto xy$ are all continuous. A topological ring R is called *discrete* if the underlying space of R is discrete.

Definition 2.1. Let R be a topological ring with identity 1_R. A *topological (left) R-module* is an additive abelian topological group M together with a continuous function $\delta\colon R \times M \to M, (r, a) \mapsto ra$ called *scalar multiplication* of R on M such that for $r, s \in R$ and $a, b \in M$

(i) $r(a + b) = ra + rb$;

(ii) $(r + s)a = ra + sa$;

(iii) $(rs)a = r(sa)$;

(iv) $1_R a = a$.

Let R be a topological ring with identity 1_R and M, M' be topological R-modules. A *morphism* of topological R-modules is a continuous group morphism $f\colon M' \to M$ such that $f(ra) = rf(a)$ for $a \in M'$ and $r \in R$. A morphism $f\colon M' \to M$ of topological R-modules is called *cover* if f is a covering map on the underlying topological spaces.

In [2, Theorem 3.1] the following theorem is proved.

Theorem 2.2. *If R is a countable, Noetherian ring and M is any R-module, then the underlying abelian group of M is isomorphic to the fundamental group $\pi_1(T(M))$ for some path connected topological R-module $T(M)$.*

This result enables us to find examples of topological R-modules which are not simply connected and so they have non-trivial covering spaces.

As a consequence of Theorem 2.2, taking $R = \mathbb{Z}$ the following corollary holds.

Corollary 2.3. *Every abelian group is isomorphic to the fundamental group of some topological group.*

We now generalize Theorem 1.1 to the topological R-modules. Let M be a topological R-module with a continuous scalar multiplication $\delta\colon R \times M \to M$ and $0 \in M$ the identity element of the additive group. Then the scalar multiplication δ induces a map

$$\tilde{\delta}\colon R \times \pi_1(M, 0) \to \pi_1(M, 0), (r, [\alpha]) \mapsto r[\alpha] = [r\alpha]$$

which makes $\pi_1(M, 0)$ an R-module. Let N be a submodule of $\pi_1(M, 0)$, i.e., N is a subgroup of the additive group M and $rn \in N$ whenever $r \in R$ and $n \in N$. Hence by Theorem 1.1 we have a covering map $p\colon (\widetilde{M}_N, \widetilde{0}) \to (M, 0)$ corresponding to N as a subgroup of the additive group $\pi_1(M, 0)$.

We now prove a general result for topological R-modules. If the topological ring R is chosen as the ring $\mathbb{Z}$ of integers endowed with the discrete topology, then this theorem reduces to Theorem 1.1.

Theorem 2.4. *Let M be a topological R-module such that R is a discrete topological ring. Let $0 \in M$ be the identity element of the additive group and let N be a submodule of $\pi_1(M, 0)$. Suppose that the underlying space of M is connected, locally path connected and semi-locally simply connected. Let $p\colon (\widetilde{M}_N, \tilde{0}) \to (M, 0)$ be the covering map corresponding to N as a subgroup of the additive group $\pi_1(M, 0)$ as in Theorem 1.1. Then the R-module structure of M lifts to $\widetilde{M}_N$, i.e., $\widetilde{M}_N$ is a topological R-module and $p\colon \widetilde{M}_N \to M$ is a morphism of topological R-modules.*

Proof. As in the proof of Theorem 1.1 let $P(M, 0)$ be the set of all paths α in M with initial point 0 and $\widetilde{M}_N$ the set of equivalence classes defined via N as in Section 1. The addition defined on $P(M, 0)$ by

$$(\alpha + \beta)(t) = \alpha(t) + \beta(t)$$

for $\alpha, \beta \in P(M, 0)$ and $0 \leq t \leq 1$ induces an addition on $\widetilde{M}_N$ defined by

$$\langle\alpha\rangle_N + \langle\beta\rangle_N = \langle\alpha + \beta\rangle_N.$$

By Theorem 1.1 this addition is well-defined, $\widetilde{M}_N$ is a topological group with this addition and $p\colon (\widetilde{M}_N, \tilde{0}) \to (M, 0), \langle\alpha\rangle_N \mapsto p(1)$ is a morphism of topological groups.

In addition to this we define a scalar multiplication on $\widetilde{M}_N$ by

$$\tilde{\delta}\colon R \times \widetilde{M}_N \to \widetilde{M}_N, (r, \langle\alpha\rangle_N) \mapsto r\langle\alpha\rangle_N = \langle r\alpha\rangle_N$$

where the path $r\alpha$ is defined by $(r\alpha)(t) = r(\alpha(t))$ for $t \in [0, 1]$. Here note that if $r \in R$ and $\langle\alpha\rangle_N \in \widetilde{M}_N$, then the initial point of the path $r\alpha$ is 0 and so $\langle r\alpha\rangle_N \in \widetilde{M}_N$. For $\beta \in P(M, 0)$ and $r \in R$ we have that

$$(r\beta)^{-1}(t) = (r\beta)(1 - t) = r\beta(1 - t) = r\beta^{-1}(t)$$

and

$$(r\beta^{-1})(r\alpha) = r(\beta^{-1}\alpha).$$

So if $\alpha \simeq_N \beta$ and $r \in R$, then $[\alpha\beta^{-1}] \in N$ and

$$[(r\alpha)(r\beta)^{-1})] = [(r\alpha)(r\beta^{-1})] = [r(\alpha\beta^{-1})].$$

Since N is an R-submodule of $\pi_1(M, 0)$, we have that $[r(\alpha\beta^{-1})] \in N$ and so $\langle r\alpha\rangle_N = \langle r\beta\rangle_N$, i.e., $\tilde{\delta}$ is well-defined. For $\langle\alpha\rangle_N, \langle\beta\rangle_N \in \widetilde{M}_N$ and $r, s \in R$ we have the following

(i) $(r + s)\langle\alpha\rangle_N = \langle(r + s)\alpha\rangle_N = r\langle\alpha\rangle_N + s\langle\alpha\rangle_N$;

(ii) $r(\langle\alpha\rangle_N + \langle\beta\rangle_N) = r\langle\alpha\rangle_N + r\langle\beta\rangle_N$;

(iii) $(rs)\langle\alpha\rangle_N = \langle(rs)\alpha\rangle_N = r(s\langle\alpha\rangle_N)$;

(iv) $1\langle\alpha\rangle_N = \langle 1\alpha\rangle_N = \langle\alpha\rangle_N$.

So $\widetilde{M}_N$ becomes an R-module. We now prove that the scalar multiplication

$$\tilde{\delta}\colon R \times \widetilde{M}_N \to \widetilde{M}_N, (r, \langle\alpha\rangle_N) \mapsto r\langle\alpha\rangle_N = \langle r\alpha\rangle_N$$

is continuous. Let $(\langle r\alpha\rangle_N, W)$ be a basic open neighbourhood of $\langle r\alpha\rangle_N$. So W is an open neighbourhood of $(r\alpha)(1) = r\alpha(1)$. Since the map $\delta\colon R \times M \to M$ is continuous there are open neighbourhoods V and U of r and $\alpha(1)$ respectively in R and M such that $\delta(V \times U) = VU \subseteq W$. Since R is discrete $\{r\} \times (\langle\alpha\rangle_N, U)$ is an open neighbourhood of $(r, \langle\alpha\rangle_N)$ in $R \times \widetilde{M}_N$ and

$$\tilde{\delta}(\{r\} \times (\langle\alpha\rangle_N, U)) = r(\langle\alpha\rangle_N, U) = (\langle r\alpha\rangle_N, rU) \subseteq (\langle r\alpha\rangle_N, W)$$

since $VU \subseteq W$ and therefore $\tilde{\delta}$ is continuous.

Further, by the definition of $p\colon \widetilde{M}_N \to M$, for $\langle\alpha\rangle_N$ and $r \in R$ we have that

$$p(r\langle\alpha\rangle_N) = p(\langle r\alpha\rangle_N) = r\alpha(1) = rp(\langle\alpha\rangle_N).$$

□

In Theorem 2.4 if the R-submodule N is chosen to be the singleton, then the following result is obtained.

Corollary 2.5. *Let R be a discrete topological ring and M a topological R-module such that the underlying space of M is connected, locally path connected and semi-locally simply connected. Let $0 \in M$ be the identity element of the additive group. Suppose that $p\colon (\widetilde{M}, \tilde{0}) \to (M, 0)$ is a universal covering map. Then the R-module structure of M lifts to $\widetilde{M}$.*

We now give a result which generalizes Theorem 1.3 in the case $R = \mathbb{Z}$.

Theorem 2.6. *Let R be a discrete topological ring with identity. Suppose that M is a topological R-module whose underlying space is connected, locally path connected and semi-locally simply connected. Let $0 \in M$ be the identity element and $p\colon (\widetilde{M}, \tilde{0}) \to (M, 0)$ a covering map in which $\widetilde{M}$ is path connected and the characteristic group $p_\star(\pi_1(\widetilde{M}, \tilde{0})) = N$ of p is an R-submodule of $\pi_1(M, 0)$. Then the R-module structure of M lifts to $\widetilde{M}$.*

Proof. Let $p_\star\colon \pi_1(\widetilde{M}, \tilde{0}) \to \pi_1(M, 0)$ be the morphism induced by $p\colon \widetilde{M} \to M$. By assumption $N = p_\star(\pi_1(\widetilde{M}, \tilde{0}))$ is an R-submodule of $\pi_1(M, 0)$. By Remark 1.2, we can assume that $\widetilde{M} = \widetilde{M}_N$ and, by Theorem 2.4, the module structure lifts to $\widetilde{M}$, as required. □

Theorem 2.7. *Let R be a simply connected topological ring and M a topological R-module such that the underlying space of M is path connected. Let $0 \in M$ be the identity element of additive group and $p\colon (\widetilde{M}, \tilde{0}) \to (M, 0)$ a covering map in which $\widetilde{M}$ is path connected. Then the characteristic group N of p is an R-submodule of $\pi_1(M, 0)$.*

Proof. Let M be a path connected topological R-module given by a scalar multiplication $\delta\colon R \times M \to M$ and 0 the identity element of the additive group of M. Consider the map of pointed topological spaces

$$f\colon R \times \widetilde{M}, (1_R, \tilde{0}) \to (M, 0)$$

defined by $f(r, \tilde{x}) = rp(\tilde{x})$. Since $f = \delta(1 \times p)$ as a composite of the continuous maps is continuous and since R is simply connected, we have that

$$f_\star(\pi_1(R \times \widetilde{M}, (1_R, \tilde{0})) \subseteq p_\star(\pi_1(\widetilde{M}, \tilde{0})),$$

because for a path $\tilde{a}$ at $\tilde{0}$ in $\widetilde{M}$ and a path ρ in R at 1_R, we have

$$\begin{aligned} f_*([\rho, \tilde{a}]) &= \delta_*(1 \times p)_*([\rho, \tilde{a}]) \\ &= \delta_*([\rho, p(\tilde{a})]) \\ &= [\rho p(\tilde{a})] \\ &= [\rho][p(\tilde{a})] \end{aligned}$$

and since R is simply connected, $[\rho] = [1]$ and so that

$$f_*([\rho, \tilde{a}]) = [p(\tilde{a})] \in p_*(\pi_1(\widetilde{M}, \tilde{0})).$$

Hence there exists a unique continuous map

$$\tilde{\delta}\colon R \times \widetilde{M}, (1_R, \tilde{0}) \to (\widetilde{M}, \tilde{0})$$

such that $p\tilde{\delta} = f$. So the map $\tilde{\delta}$ defines an induced map $R \times \pi_1(\widetilde{M}, \tilde{0}) \to \pi_1(\widetilde{M}, \tilde{0})$. Let $p_\star\colon \pi_1(\widetilde{M}, \tilde{0}) \to \pi_1(M, 0)$ be the morphism induced by $p\colon \widetilde{M} \to M$. We now prove that the characteristic group $p_\star(\pi_1(\widetilde{M}, \tilde{0})) = N$ of p is an R-submodule of $\pi_1(M, 0)$. For if $r \in R$ and $[\alpha] \in N$, then $p_\star([\alpha_1]) = [\alpha]$ for some $[\alpha_1] \in \pi_1(\widetilde{M}, \tilde{0})$. Hence $[r\alpha_1] \in \pi_1(\widetilde{M}, \tilde{0})$ and $p_\star([r\alpha_1]) = [p(r\alpha_1)] = [rp(\alpha_1)] = [r\alpha] \in N$. □

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References

[1] N. Alemdar, O. Mucuk, The liftings of R-modules to covering groupoids, Hacettepe Journal of Mathematics and Statistics 41 (6) (2012) 813–822.

[2] A. Bateson , Fundamental groups of topological R-modules, Trans. Amer. Math. Soc. V.270 No.2 (1982) 525–536.

[3] R. Brown, O. Mucuk, Covering groups of non-connected topological groups revisited, Math. Proc. Camb. Phil. Soc. 115 (1994) 97–110.
[4] C. Chevalley, Theory of Lie groups, Princeton University Press, 1946.
[5] O. Mucuk , Coverings and ring-groupoids, Geor. Math. J. 5 (1998) 475–482.
[6] O. Mucuk, Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid, PhD Thesis, University of Wales, 1993.
[7] O. Mucuk, M. Özdemir , A monodromy principle for the universal covers of topological rings, Ind. Jour. Pure and App. Math 31 (12) (2000) 1531–1535.
[8] O. Mucuk, M. Özdemir , A monodromy principle for the universal coverings of topological modules, Tamkang J. Math. 34 (2003) 299–307.
[9] J.J. Rotman, An Introduction to Algebraic Topology, Graduate Texts in Mathematics; 119, Springer-Verlag, New York, 1988.
[10] R.L. Taylor, Covering groups of non-connected topological groups, Proc. Amer. Math. Soc. 5 (1954) 753–768.