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On β-paracompact spaces

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Abstract. In this paper, as a weak form of paracompact and expandable spaces, β-paracompact spaces and β-expandable spaces are defined, respectively. Some fundamental properties of these spaces are given. Also, it is proved that every β-paracompact space is a β-expandable space and the relations between these spaces and some previously studied spaces are investigated.

1. Introduction

In 2006, Al-Zoubi [5] used semi-open sets to define S-paracompact spaces which are a generalization of paracompact spaces. In 2008, Li and Song [13] introduced and studied S-expandable spaces which are a weaker form than S-paracompact spaces. S-expandable spaces are also defined in terms of semi-open sets.

In this paper, we introduce and study some new classes of spaces. In Section 3, we define and investigate β-locally finite collections by using β-open sets defined by Abd El-Monsef, El-Deeb and Mahmoud [1]. In Section 4, we introduce β-paracompact spaces and investigate their properties and their relations with other types of spaces. We obtain some examples and counterexamples. Also, we examine some functions on β-paracompact spaces. In Section 5, we introduce β-expandable spaces by using β-open sets and β-locally finiteness. In particular, we explore the relationship between β-paracompact spaces and β-expandable spaces.

2. Preliminaries

In the course of this work a space means always a topological space, and no separation axioms are assumed unless clearly indicated. Let (X,τ) and (Y,σ) (or X and Y) be spaces and A a subset of X. The closure of A, the interior of A and the relative topology on A in (X,τ) will be denoted by $cl(A)$, $int(A)$ and τ_A, respectively. Let ω denote the first infinite cardinal. A subset A is called semi-open [12] if there exists an open set U of X such that $U \subseteq A \subseteq cl(U)$. This is equivalent to say that $A \subseteq cl(int(A))$. A is called regular open [18] (resp. regular closed [18], preopen [15]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$, $A \subseteq int(cl(A))$). The family of all subsets of a space (X,τ) which are semi-open (resp. regular open, regular closed, preopen) is denoted by $SO(X,\tau)$ (resp. $RO(X,\tau)$, $RC(X,\tau)$, $PO(X,\tau)$). The collection $RO(X,\tau)$ is a base for a topology τ_s on X and $\tau_s \subseteq \tau$. The space (X,τ_s) is called semiregular [18]. A subset A is called semi-preopen [7] if there exists a preopen set U of X such that $U \subseteq A \subseteq cl(U)$. A subset A of X is semi-preopen if and only

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if $A \subseteq cl(int(cl(A)))$ [7]. Abd El-Monsef et al. [1] have introduced a weak form of open sets called β-open sets. The notion of β-open sets is equivalent to that of semi-preopen sets. A subset A of X is called β-open set [1] if $A \subseteq cl(int(cl(A)))$. The family of all β-open (or semi-preopen) sets of X is denoted by $\beta O(X,\tau)$ (or $SPO(X,\tau)$). In this paper, we will use the term " β-open ". The complement of a β-open (or semi-preopen) set is said to be β-closed [1] (or semi-preclosed [7]). The β-closure [3] (or semi-preclosure [7]) of a subset A of X, denoted by $\beta cl(A)$ (or spcl(A)), is the intersection of all the β-closed (or semi-preclosed) subsets of X containing A . A space (X,τ) is said to be β-compact [2] if every cover of (X,τ) by β-open sets has a finite subcover.

Lemma 2.1. (see [3] and [7]) *For a subset A of a topological space (X,τ), the following conditions hold:*
(a) $\beta cl(A) = A \cup int(cl(int(A)))$,
(b) $x \in \beta cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \beta O(X,\tau)$ *containing* x,
(c) A *is* β*-closed if and only if* $A = \beta cl(A)$.

Definition 2.2. A space (X,τ) is called:
(a) *extremally disconnected* (briefly e.d.) [18] if the closure of every open set in (X,τ) is open.
(b) *submaximal* [8] if each dense subset of X is open in X.

Lemma 2.3. ([17]) (X,τ) *is submaximal if and only if every preopen set is open.*

Lemma 2.4. ([10]) *In a submaximal, extremally disconnected space* (X,τ) *all semi-open sets are open.*

Lemma 2.5. ([7]) *If V is open and A is semi-preopen (or β-open) then $V \cap A$ is semi-preopen (or β-open).*

3. β-locally finite collections

Recall that a collection $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ of subsets of a space (X,τ) is said to be *locally finite* [18] if for each $x \in X$, there exists an open set U in (X,τ) containing x and U intersects F_α at most for finitely many α.

Definition 3.1. A collection $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ of subsets of a space (X,τ) is said to be *β-locally finite* if for each $x \in X$, there exists a β-open set U in (X,τ) containing x and U intersects F_α at most for finitely many α.

It is clear that every locally finite collection of subsets of a space (X,τ) is β-locally finite but the converse is not true in general.

Example 3.2. Consider the real numbers $\mathbb{R}$ with the usual topology τ_u. Let be $\mathcal{F} = \{\, [\frac{1}{n+1}, \frac{1}{n}] : n = 1,2,...\}$. Then $\mathcal{F}$ is β-locally finite in $(\mathbb{R},\tau_u)$ since $(-1,0] \in \beta O(\mathbb{R},\tau_u)$ but $\mathcal{F}$ is not locally finite. Hence, locally finiteness is not provided at the point 0.

Theorem 3.3. *Let (X,τ) be an e.d. submaximal space. Then every β-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ of subsets of X is locally finite.*

Proof. It is clear from Lemma 2.3 and Lemma 2.4. □

Lemma 3.4. *Let $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ be a collection of subsets of a space (X,τ). Then, the following conditions hold:*
(a) *If $\mathcal{F} = \{\, F_\alpha : \alpha \in I \,\}$ is a β-locally finite collection of subsets of X and $G_\alpha \subseteq F_\alpha$, for each $\alpha \in I$, then $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ is also β-locally finite.*
(b) *$\mathcal{F}$ is β-locally finite if and only if $\{\beta cl(F_\alpha) : \alpha \in I\}$ is β-locally finite.*

Proof. Obvious. □

4. β-paracompact spaces

Recall that a space (X, τ) is said to be *S-paracompact* [5] if every open cover of X has a locally finite semi-open refinement.

Definition 4.1. A space (X, τ) is said to be *β-paracompact* if every open cover of X has a β-locally finite β-open refinement.

It is clear that every compact space is β-paracompact space but the converse is not true in general.

Example 4.2. For any infinite set X, (X, τ_{dis}) is a β-paracompact space. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of X. Then $\mathcal{U}$ has a β-locally finite β-open refinement, say $\mathcal{V} = \left\{\{x\} : x \in X\right\}$ but every open cover hasn't a finite subcover since X is infinite set. So, X is not compact space.

It is clear that every S-paracompact space is β-paracompact but the following example shows that the converse is not true in general.

Example 4.3. Let the space (X, τ) be as in Application 3.2 of [6]. Namely, let $X = N \cup N^-$ with the topology $\tau = \{U \subseteq X : N \subseteq U\} \bigcup \{\emptyset\}$ such that N is the set of all positive integers and N^- is the set of all negative integers. Then (X, τ) is a β-paracompact space since $\beta O(X, \tau) = \{A \subseteq X : A \cap N \neq \emptyset\}$ and every open cover of X has a β-locally finite β-open refinement, say $\left\{\{x\} : x \in N\right\} \bigcup \left\{\{x, -x\} : x \in N^-\right\}$. On the other hand, (X, τ) is not S-paracompact space since $SO(X, \tau) = \tau$ and $\left\{N \cup \{x\} : x \in N^-\right\}$ is an open cover of X which admits no locally finite semi-open refinement.

Theorem 4.4. *Let (X, τ) be an e.d. submaximal space. If (X, τ) is a β-paracompact space then (X, τ) is paracompact space.*

Proof. This follows directly from the fact that if (X, τ) is an e.d. submaximal space, then $\tau = \tau^\alpha = SO(X, \tau) = PO(X, \tau) = \beta O(X, \tau)$. □

Recall that a subset A of a space (X, τ) is called an *α-set* if $A \subseteq int(cl(int(A)))$ [16]. The family of all α-sets of a space (X, τ) is denoted by τ^α and it is a topology on X such that $\tau \subseteq \tau^\alpha$ [16]. Also, $SPO(X, \tau) = SPO(X, \tau^\alpha)$ (or $\beta O(X, \tau) = \beta O(X, \tau^\alpha)$) [7].

Theorem 4.5. *If (X, τ^α) is β-paracompact, then (X, τ) is β-paracompact.*

Proof. The proof follows immediately from $\tau \subseteq \tau^\alpha$ and $\beta O(X, \tau) = \beta O(X, \tau^\alpha)$. □

The following example shows that the converse of the Theorem 4.5 is not true in general.

Example 4.6. Let (X, τ) be as in Example 2.11 of [5]. Namely, let $X = \mathbb{R}$ be the set of the real numbers with the topology $\tau = \left\{\emptyset, X, \{1\}\right\}$. Then (X, τ) is β-paracompact but (X, τ^α) is not β-paracompact since $\tau^\alpha = \beta O(X, \tau^\alpha) = \{\emptyset\} \cup \{U \subseteq X : 1 \in U\}$ and the collection $\left\{\{1, x\} : x \in X\right\}$ is an open cover of (X, τ^α) which admits no β-locally finite β-open refinement in (X, τ^α).

Recall that a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be *pre β-open* [14] (resp. *pre β-closed* [14]) if for every β-open (resp. β-closed) set B of (X, τ), $f(B)$ is β-open (resp. β-closed) in (Y, σ), and $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be *β-irresolute* [14] if for every β-open set A of (Y, σ), $f^{-1}(A)$ is β-open in (X, τ). It is clear that if $f : (X, \tau) \longrightarrow (Y, \sigma)$ is continuous and open, then f is β-irresolute and pre β-open.

Lemma 4.7. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a surjective function. Then f is pre β-closed if and only if for every $y \in Y$ and every β-open set U in (X, τ) which contains $f^{-1}(y)$, there exists a $V \in \beta O(Y, \sigma)$ such that $y \in V$ and $f^{-1}(V) \subset U$.*

Proof. To prove necessity, let $y \in Y$ and U be a β-open set in (X, τ) such that $f^{-1}(y) \subset U$. Set $V = Y - f(X - U)$. Then by hypothesis, $V \in \beta O(Y, \sigma)$ such that $y \in V$ and $f^{-1}(V) \subset U$.

To prove sufficiency, let K be a β-closed subset of (X, τ) and $y \in Y - f(K)$. Then, $f^{-1}(y) \subset X - K$. By hypothesis, there exists a $V_y \in \beta O(Y, \sigma)$ containing y such that $f^{-1}(V_y) \subset X - K$. Therefore, $y \in V_y \subset Y - f(K)$. Hence, $Y - f(K) = \bigcup\{V_y : y \in Y - f(K)\}$. Thus, $Y - f(K) \in \beta O(Y, \sigma)$ and so $f(K)$ is β-closed in (Y, σ). □

Theorem 4.8. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a continuous, open and pre β-closed surjection function such that $f^{-1}(y)$ is β-compact for each $y \in Y$. If (X, τ) is β-paracompact, then (Y, σ) is β-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of (Y, σ). Then, $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha) : \alpha \in I\}$ is an open cover of the β-paracompact space (X, τ) and so it has a β-locally finite β-open refinement, say $\mathcal{V} = \{V_\beta : \beta \in B\}$. Since f is pre β-open, the collection $f(\mathcal{V}) = \{f(V_\beta) : \beta \in B\}$ is a β-open refinement of $\mathcal{U}$. Finally, we shall show that the collection $f(\mathcal{V})$ is β-locally finite in (Y, σ). Let $y \in Y$. For each $x \in f^{-1}(y)$ there exists a β-open set U_x containing x such that U_x intersects at most finitely many members of $\mathcal{V}$. The collection $\{U_x : x \in f^{-1}(y)\}$ is a β-open cover of $f^{-1}(y)$ and therefore there exists a finite subset K of $f^{-1}(y)$ such that $f^{-1}(y) \subset \bigcup_{x \in K} U_x$. Because f is pre β-closed, by Lemma 4.7, there exists a β-open set P_y containing y in (Y, σ) such that $f^{-1}(P_y) \subset \bigcup_{x \in K} U_x$. Then, $f^{-1}(P_y)$ intersects at most finitely many members of $\mathcal{V}$. Therefore P_y intersects at most finitely many members of $f(\mathcal{V})$. Thus, $f(\mathcal{V})$ is β-locally finite in (Y, σ). □

Theorem 4.9. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a β-irresolute, closed surjection function such that $f^{-1}(y)$ is compact for each $y \in Y$. If (Y, σ) is β-paracompact, then (X, τ) is β-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of (X, τ). Since $f^{-1}(y)$ is compact for each $y \in Y$, there exists a finite subset $I(y)$ of I such that $f^{-1}(y) \subseteq \bigcup_{\alpha \in I(y)} U_\alpha$. Because f is closed, there exists an open set V_y containing y such that $f^{-1}(V_y) \subseteq \bigcup_{\alpha \in I(y)} U_\alpha$. Therefore, $\{V_y : y \in Y\}$ is an open cover of the β-paracompact space (Y, σ) and it has a β-locally finite β-open refinement $\mathcal{W} = \{W_\beta : \beta \in B\}$. Since f is β-irresolute, $\{f^{-1}(W_\beta) : \beta \in B\}$ is a β-locally finite β-open cover of (X, τ). For each $\beta \in B$, there exists a $y(\beta) \in Y$ such that $W_\beta \subseteq V_{y(\beta)}$. Therefore, $f^{-1}(W_\beta) \subseteq f^{-1}(V_{y(\beta)}) \subseteq \bigcup_{\alpha \in I(y(\beta))} U_\alpha = F_{y(\beta)}$. Put $\mathcal{F} = \{f^{-1}(W_\beta) \bigwedge F_{y(\beta)} : \beta \in B \text{ and } y(\beta) \in Y\}$, where $f^{-1}(W_\beta) \bigwedge F_{y(\beta)} = \{f^{-1}(W_\beta) \cap U_\alpha : \beta \in B \text{ and } \alpha \in I(y(\beta))\}$. By Lemma 2.5, each set of $\mathcal{F}$ is a β-open subset of X. Then, the family $\mathcal{F}$ is a β-locally finite β-open refinement of $\mathcal{U}$. Thus (X, τ) is β-paracompact. □

Theorem 4.10. *Let (X, τ) be a compact space and let (Y, σ) be an β-paracompact space. Then, $(X, \tau) \times (Y, \sigma)$ is β-paracompact.*

Proof. Let $f : (X, \tau) \times (Y, \sigma) \longrightarrow (Y, \sigma)$ be the projection. It is known that f is open, surjective and continuous. Then f is β-irresolute. Also, f is the closed function since X is compact. For each $y \in Y$, $f^{-1}(y) = X \times \{y\}$ is a compact subset of $X \times Y$ because $X \times \{y\}$ is homeomorphic to X. Since (Y, σ) is β-paracompact, by Theorem 4.9, $(X, \tau) \times (Y, \sigma)$ is β-paracompact. □

5. β-expandable spaces

Recall that a space X is said to be *expandable* [11] (resp. *s-expandable* [4]) if for every locally finite (resp. s-locally finite) collection $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of subsets of X there exists a locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of open subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Lambda$. A space X is said to be *S-expandable* [13] if for every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of subsets of X there exists a locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of semi-open subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Lambda$.

Definition 5.1. A space X is said to be *β-expandable* (resp. *ω-β-expandable*) if for every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ (resp. $|\Lambda| \leq \omega$) of subsets of X there exists a β-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of β-open subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Lambda$.

Theorem 5.2. *Let X be a space. Then the following conditions are equivalent:*
(a) *X is β-expandable.*
(b) *Every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of closed subsets of X, there exists a β-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of β-open subsets of X such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Lambda$.*

Proof. (a) $\Rightarrow$ (b) It is clear from Definition 5.1.
(b) $\Rightarrow$ (a) Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite collection. We know that $\{cl(F_\alpha) : \alpha \in \Lambda\}$ is also a locally finite collection. By the hypothesis, there exists a β-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ of β-open subsets of X such that $cl(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in \Lambda$. Because of $F_\alpha \subseteq cl(F_\alpha) \subseteq G_\alpha$ for each $\alpha \in \Lambda$, X is β-expandable. □

Theorem 5.3. *Every β-paracompact space is β-expandable.*

Proof. Let X be a β-paracompact space and $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a locally finite collection of closed subsets of X. Let Γ be the collection of all finite subsets of Λ. For each $\gamma \in \Gamma$, let $V_\gamma = X \backslash \bigcup\{F_\alpha : \alpha \notin \gamma\}$. Because $\mathcal{F}$ is the locally finite collection, V_γ is open. Also, V_γ meets only finitely many elements of $\mathcal{F}$. Let $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$. Then $\mathcal{V}$ is an open cover of X. Since X is β-paracompact, $\mathcal{V}$ has a β-locally finite β-open refinement, say $\mathcal{W} = \{W_\delta : \delta \in \Delta\}$. Set

$$U_\alpha = \bigcup\{W_\delta \in \mathcal{W} : W_\delta \cap F_\alpha \neq \emptyset\} \quad for\ each\ \alpha \in \Lambda.$$

Because arbitrary unions of β-open sets are β-open set, U_α is β-open and $F_\alpha \subseteq U_\alpha$ for each $\alpha \in \Lambda$. Now, we shall try to show that $\{U_\alpha : \alpha \in \Lambda\}$ is β-locally finite. Since $\mathcal{W}$ is β-locally finite, for each $x \in X$, there exists a β-open set U_x in (X, τ) containing x and U_x intersects at most finitely many members of $\mathcal{W}$. Also, by the definition of U_α, we say that

$$U_x \cap U_\alpha \neq \emptyset\ if\ and\ only\ if\ U_x \cap W_\delta \neq \emptyset\ and\ W_\delta \cap F_\alpha \neq \emptyset \quad for\ some\ \delta \in \Delta.$$

Since $\mathcal{W}$ is refinement of $\mathcal{V}$, there is an member V_γ of $\mathcal{V}$ containing W_δ for each member W_δ of $\mathcal{W}$. Then, W_δ meets only finitely many F_α for each $\delta \in \Delta$. Thus, $\{U_\alpha : \alpha \in \Lambda\}$ is β-locally finite. □

It is clear that every S-expandable space is β-expandable but the following example shows that the converse is not true in general.

Example 5.4. Let (X, τ) be as in Example 4.3. By Theorem 5.3, (X, τ) is β-expandable space but (X, τ) is not S-expandable. Indeed, for the locally finite collection $\left\{\{x\} : x \in N^-\right\}$ of (X, τ), there exists no locally finite collection $\left\{U_x : x \in N^-\right\}$ of semi-open subsets of X such that $x \in U_x$ for each $x \in N^-$.

Lemma 5.5. *For an extremally disconnected submaximal space X, the following conditions are equivalent:*
(a) *X is β-expandable,*
(b) *X is S-expandable,*
(c) *X is expandable,*
(d) *X is s-expandable.*

Proof. It is clear from Lemma 2.3, Lemma 2.4 and Theorem 3.3. □

Theorem 5.6. *Let X be a space. Then X is ω-β-expandable if and only if every countable open cover of X has a β-locally finite β-open refinement.*

Proof. Sufficiency is similar to the proof of Theorem 5.3.
To prove necessity, let $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ be a countable open cover of X. Put $A_i = \bigcup\{U_j : j \leqslant i\}$ for each $i \in \mathbb{N}$. Let $B_1 = A_1$ and $B_i = A_i \backslash A_{i-1}$ such that $i = 2, 3, \ldots$. Therefore, $B_i \subseteq U_i$ for each $i \in \mathbb{N}$. For $x \in X$, let $i(x) = min\{i \in \mathbb{N} : x \in U_i\}$. Then, $x \in B_{i(x)}$. Let $\mathcal{A} = \{B_i : i \in \mathbb{N}\}$. Therefore, $\mathcal{A}$ is a refinement of $\mathcal{U}$ and $\mathcal{A}$ is locally finite since $U_i \cap B_j = \emptyset$ for $j > i$. Because X is ω-β-expandable, there exists a β-locally finite collection

$\{G_i : i \in \mathbb{N}\}$ of β-open subsets of X such that $B_i \subseteq G_i$ for each $i \in \mathbb{N}$. Let $V_i = U_i \cap G_i$ for each $i \in \mathbb{N}$. By Lemma 2.5, $V_i \in \beta O(X, \tau)$ for each $i \in \mathbb{N}$. Let $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$. Since $\{G_i : i \in \mathbb{N}\}$ is β-locally finite, by Lemma 3.4(a), $\mathcal{V}$ is β-locally finite. Because $\mathcal{A}$ is a cover of X, there exists some $i \in \mathbb{N}$ such that $x \in B_i$ for each $x \in X$. Since $B_i \subseteq V_i$, $x \in V_i$. Thus, $\mathcal{V}$ is a β-locally finite β-open refinement of $\mathcal{U}$. □

Theorem 5.7. *If (X, τ^α) is β-expandable, then (X, τ) is β-expandable.*

Proof. This follows immediately from $\tau \subseteq \tau^\alpha$ and $\beta O(X, \tau) = \beta O(X, \tau^\alpha)$. □

Let X be a space and U a clopen subset of X. Then it is clear that every locally finite family of U is locally finite in X. Moreover, the intersection of a β-open subset and an open subset is β-open subset. Thus, we have the following theorem.

Theorem 5.8. *Let (X, τ) be a β-expandable space. Then every clopen subset of (X, τ) is β-expandable.*

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