

## Nonunique fixed point theorems in partial metric spaces

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**Abstract.** In this paper we prove the existence of fixed points of certain self-maps in the context of partial metric spaces. In fact, the fixed point theorems presented here can be considered as a continuation, in part, of the works of L.B. Ćirić on the existence of fixed points but not uniqueness in the realm of metric spaces. Our results generalize, enrich and improve earlier results on the topic in the literature.

### 1. Introduction and preliminaries

Fixed point theory has been a subject of great interest since its wide application potential in nonlinear functional analysis. The existence and uniqueness of fixed points of operators in metric spaces have been studied intensively by many authors since the report of Banach [9] on the topic appeared in 1922. In this celebrated paper, Banach proved that every contraction in a complete metric space has a unique fixed point. Following this initial paper, a number of fixed point theorems proved in various types of abstract spaces such as metric spaces, quasi-metric spaces, cone metric spaces, Menger spaces, fuzzy metric spaces,  $b$ -metric spaces,  $G$ -metric spaces (see e.g. [1, 10, 12–15, 18, 19]). Following this trend, Matthews [30] introduced a new abstract space called partial metric space. In this distinguished paper, the author proved a fixed point theorem which is an analog of the Banach contraction mapping principle. Later some interested authors showed that partial metric spaces have many applications both in mathematics and computer science (see e.g. [28, 29, 33, 37–39]). Recently, some more results on fixed point theory on partial metric spaces appeared in [2, 3, 6, 7, 17, 20–27, 32, 35, 36, 41], etc.

In the sequel, the letters  $\mathbb{R}^+$ ,  $\mathbb{N}$  and  $\omega$  will denote the set of non-negative real numbers, the set of positive integers and the set of non-negative integer numbers, respectively.

The definition of partial metric space is given by Matthews [30] as follows:

**Definition 1.1.** A partial metric on a (non-empty) set  $X$  is a function  $p : X \times X \rightarrow \mathbb{R}^+$  satisfying the following conditions for all  $x, y, z \in X$ :

$$(P1) \quad x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y),$$

$$(P2) \quad p(x, x) \leq p(x, y),$$

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(P3)  $p(x, y) = p(y, x)$ ,

(P4)  $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ .

Then, the pair  $(X, p)$  is called a partial metric space.

Observe that from (P1) and (P2) it follows that if  $p(x, y) = 0$  then  $x = y$ .

**Example 1.2.** (See e.g. [30, 31]) Let  $X = \mathbb{R}^+$  and  $p$  on  $X$  defined by  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Then  $(X, p)$  is a partial metric space.

**Example 1.3.** (See e.g. [26, 40]) Let  $(X, d)$  and  $(X, p)$  be a metric space and a partial metric space, respectively. Functions  $\rho_i : X \times X \rightarrow \mathbb{R}^+$  ( $i \in \{1, 2, 3\}$ ) given by

$$\begin{aligned} \rho_1(x, y) &= d(x, y) + p(x, y), \\ \rho_2(x, y) &= d(x, y) + \max\{u(x), u(y)\}, \\ \rho_3(x, y) &= d(x, y) + a, \end{aligned}$$

define partial metrics on  $X$ , where  $u : X \rightarrow \mathbb{R}^+$  is an arbitrary function and  $a \geq 0$ .

**Example 1.4.** (See [30, 31]) Let  $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$  and define  $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ . Then  $(X, p)$  is a partial metric space.

**Example 1.5.** (See [30]) Let  $X = [0, 1] \cup [2, 3]$  and define  $p : X \times X \rightarrow \mathbb{R}^+$  by  $p(x, y) = \begin{cases} \max\{x, y\}, & \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y|, & \{x, y\} \subset [0, 1]. \end{cases}$

Then  $(X, p)$  is a partial metric space.

Each partial metric  $p$  on a set  $X$  induces a  $T_0$  topology  $\tau_p$  on  $X$ , which has as a base the family of open  $p$ -balls  $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$  where  $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$  for all  $x \in X$  and  $\epsilon > 0$ .

From the definition of the topology  $\tau_p$  it immediately follows that a sequence  $\{x_n\}_{n \in \mathbb{N}}$  in a partial metric space  $(X, p)$  converges to a point  $x \in X$  ( $x_n \rightarrow x$ , in short) with respect to  $\tau_p$  if and only if  $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$ .

Notice that the limit of a sequence in partial metric space is not necessarily unique. For example, consider the sequence  $\{\frac{1}{n+1}\}_{n \in \mathbb{N}}$  in the partial metric space defined in Example 1.2. It is clear that

$$p(1, 1) = \lim_{n \rightarrow \infty} p\left(1, \frac{1}{n+1}\right) \quad \text{and} \quad p(2, 2) = \lim_{n \rightarrow \infty} p\left(2, \frac{1}{n+1}\right).$$

However, under certain conditions, the limit of a sequence is unique as the following lemma shows.

**Lemma 1.6.** (See e.g [26, 40]) Let  $(X, p)$  be a partial metric space and let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence in  $X$  such that  $x_n \rightarrow x$  and  $x_n \rightarrow y$  with respect to  $\tau_p$ . If

$$\lim_{n \rightarrow \infty} p(x_n, x_n) = p(x, x) = p(y, y),$$

then  $x = y$ .

The usual metric spaces are closely connected to partial metric spaces. One can easily show (see e.g. [31]) that the function  $d_p : X \times X \rightarrow \mathbb{R}^+$  defined as

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{1}$$

is a metric on  $X$ .

The functions  $d_m^p$  and  $p_0$  defined on  $X \times X$  by

$$\begin{aligned} d_m^p(x, y) &= \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\} \\ &= p(x, y) - \min\{p(x, x), p(y, y)\}, \end{aligned} \quad (2)$$

and by  $p_0(x, x) = 0$  for all  $x \in X$  and  $p_0(x, y) = p(x, y)$  for  $x \neq y$ , are also metrics on  $X$  (see [5] and [16], respectively).

Observe that if  $p$  is a metric on  $X$  then  $p = d_m^p$ .

The following topological inclusions are well-known and easy to check:  $\tau_p \subseteq \tau_{d_p} = \tau_{d_p^m} \subseteq \tau_{p_0}$ .

Furthermore, the following equivalence will be useful later on:

$$\lim_{n \rightarrow \infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n) = \lim_{n, m \rightarrow \infty} p(x_n, x_m). \quad (3)$$

Note that in the partial metric space  $(X, p)$  of Example 1.2 both  $d_p$  and  $d_p^m$  are the Euclidean metric on  $X$ .

Some fundamental concepts like Cauchy sequence and completeness in a partial metric space are defined in the next.

**Definition 1.7.** (See e.g.[25, 30, 31]) Let  $(X, p)$  be a partial metric space.

1. A sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  is called a Cauchy sequence in  $(X, p)$  if  $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$  exists and is finite.
2.  $(X, p)$  is called complete if every Cauchy sequence  $\{x_n\}_{n \in \mathbb{N}}$  converges with respect to  $\tau_p$  to a point  $x \in X$  such that  $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$ .

The following lemma provides nice characterizations of Cauchyness and completeness for partial metric spaces. Its proof is easily accessible in the literature or can be derived by elementary means.

**Lemma 1.8.** (See [31]) Let  $(X, p)$  be a partial metric space.

1. A sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  is a Cauchy sequence in  $(X, p)$  if and only if it is a Cauchy sequence in the metric space  $(X, d_p)$ .
2.  $(X, p)$  is complete if and only if the metric space  $(X, d_p)$  is complete.

The partial metric spaces of Example 1.2, Example 1.4 and Example 1.5 are complete.

In our context the following characterization will be useful.

**Lemma 1.9.** Let  $(X, p)$  be a partial metric space. A sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  is a Cauchy sequence in  $(X, p)$  if and only if it satisfies the following condition:

(\*) for each  $\varepsilon > 0$  there is  $n_0 \in \mathbb{N}$  such that  $p(x_n, x_m) - p(x_n, x_n) < \varepsilon$  whenever  $n_0 \leq n \leq m$ .

*Proof.* We first prove the “if” part. Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence in  $(X, p)$  satisfying (\*). We shall show that then the sequence  $\{p(x_n, x_n)\}_{n \in \mathbb{N}}$  converges for the Euclidean metric on  $\mathbb{R}^+$ . Indeed, let  $\varepsilon = 1$ . Then, by (\*), there is  $n_0 \in \mathbb{N}$  such that  $p(x_n, x_n) \leq p(x_{n_0}, x_n) < 1 + p(x_{n_0}, x_{n_0})$  whenever  $n \geq n_0$ . Thus, the sequence  $\{p(x_n, x_n)\}_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}^+$ , so it has a subsequence  $\{p(x_{n_k}, x_{n_k})\}_{k \in \mathbb{N}}$  that converges to an  $a \in \mathbb{R}^+$  for the Euclidean metric. Now choose an  $\varepsilon > 0$ . Then, there is  $k_0 \in \mathbb{N}$  such that condition (\*) is satisfied whenever  $m \geq n \geq n_{k_0}$ , and condition  $|p(x_{n_k}, x_{n_k}) - a| < \varepsilon$  also holds for all  $k \geq k_0$ . Take any  $n \geq n_{k_0}$ . Then, we have

$$p(x_n, x_n) - a \leq p(x_n, x_{n_{k_0}}) - a < \varepsilon + p(x_{n_{k_0}}, x_{n_{k_0}}) - a < 2\varepsilon,$$

and for  $k \in \mathbb{N}$  with  $n_k \geq n$ , we deduce that

$$a - p(x_n, x_n) < \varepsilon + p(x_{n_k}, x_{n_k}) - p(x_n, x_n) < 2\varepsilon.$$

Consequently  $\lim_{n \rightarrow \infty} p(x_n, x_n) = a$ . Then, by (\*), it immediately follows that  $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = a$ . We conclude that  $\{x_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(X, p)$ .

The converse follows from Lemma 1.8.

**Lemma 1.10.** Let  $(X, p)$  be a partial metric space and let  $\{x_n\}_{n \in \mathbb{N}}$  and  $\{y_n\}_{n \in \mathbb{N}}$  be sequences in  $X$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  with respect to  $\tau_{d_p}$ . Then

$$\lim_{n \rightarrow \infty} p(x_n, y_n) = p(x, y).$$

For our purposes, we need to recall the following notion

**Definition 1.11.** (cf. [11]) Let  $(X, p)$  be a partial metric space and  $T$  a self-map of  $X$ .

1.  $T$  is called orbitally continuous if

$$\lim_{i, j \rightarrow \infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i \rightarrow \infty} p(T^{n_i}x, z) = p(z, z), \quad (4)$$

implies

$$\lim_{i, j \rightarrow \infty} p(TT^{n_i}x, TT^{n_j}x) = \lim_{i \rightarrow \infty} p(TT^{n_i}x, Tz) = p(Tz, Tz), \quad (5)$$

for each  $x \in X$ .

Equivalently,  $T$  is orbitally continuous provided that if  $T^{n_i}x \rightarrow z$  with respect to  $\tau_{d_p}$ , then  $T^{n_i+1}x \rightarrow Tz$  with respect to  $\tau_{d_p}$ , for each  $x \in X$ .

2.  $(X, p)$  is called orbitally complete if every Cauchy sequence of type  $\{T^{n_i}x\}_{i \in \mathbb{N}}$  converges with respect to  $\tau_{d_p}$ , that is, if there is  $z \in X$  such that

$$\lim_{i, j \rightarrow \infty} p(T^{n_i}x, T^{n_j}x) = \lim_{i \rightarrow \infty} p(T^{n_i}x, z) = p(z, z). \quad (6)$$

In this paper, we prove some non-unique fixed point theorems for certain type of self-maps in the context of partial metric spaces. In fact, the fixed point theorems presented here can be considered as a continuation, in part, of the work of Ćirić [11], that is, the given theorems investigate conditions only for the existence of fixed points but not uniqueness. Our results generalize, enrich and improve some earlier results on the topic in the literature (see e.g. [4, 11, 22, 34]). We also give examples that show the advantages of using partial metric spaces instead of metric spaces in this context.

## 2. The results

In this section we give some non-unique fixed point theorems for partial metric spaces and present some examples illustrating our results.

**Theorem 2.1.** Let  $T$  be an orbitally continuous self-map of a  $T$ -orbitally complete partial metric space  $(X, p)$ . If there is  $k \in (0, 1)$  such that

$$\min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} \leq k(p(x, y) - p(x, x)) + p(y, y), \quad (7)$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \omega}$  converges with respect to  $\tau_{d_p}$  to a fixed point of  $T$ .

*Proof.* Take an arbitrary point  $x_0 \in X$ . We define the iterative sequence  $\{x_n\}_{n \in \omega}$  as follows:

$$x_{n+1} = Tx_n, \quad n \in \omega.$$

If there exists  $n_0 \in \omega$  such that  $x_{n_0} = x_{n_0+1}$ , then  $x_{n_0}$  is a fixed point of  $T$ . Assume then that  $x_n \neq x_{n+1}$  for each  $n \in \omega$ .

Substituting  $x = x_n$  and  $y = x_{n+1}$  in (7) we find the inequality

$$\begin{aligned} \min\{p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} - \min\{d_m^p(x_n, x_{n+2}), d_m^p(x_{n+1}, x_{n+1})\} \\ \leq k(p(x_n, x_{n+1}) - p(x_n, x_n)) + p(x_{n+1}, x_{n+1}). \end{aligned}$$

Substituting now  $x = x_{n+1}$  and  $y = x_n$  in (7), we obtain

$$\begin{aligned} \min\{p(x_{n+2}, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})\} - \min\{d_m^p(x_{n+1}, x_{n+1}), d_m^p(x_{n+2}, x_n)\} \\ \leq k(p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1})) + p(x_n, x_n), \end{aligned}$$

which imply that

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \leq k(p(x_n, x_{n+1}) - p(x_n, x_n)) + p(x_{n+1}, x_{n+1}), \tag{8}$$

and

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \leq k(p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1})) + p(x_n, x_n). \tag{9}$$

Suppose  $p(x_{n_0}, x_{n_0+1}) \leq p(x_{n_0+1}, x_{n_0+2})$  for some  $n_0 \in \omega$ . Then, from the preceding two inequalities we deduce that

$$(1 - k)p(x_{n_0}, x_{n_0+1}) \leq \min\{p(x_{n_0+1}, x_{n_0+1}) - kp(x_{n_0}, x_{n_0}), p(x_{n_0}, x_{n_0}) - kp(x_{n_0+1}, x_{n_0+1})\}.$$

If, for instance,  $p(x_{n_0+1}, x_{n_0+1}) \leq p(x_{n_0}, x_{n_0})$ , we have

$$\begin{aligned} (1 - k)p(x_{n_0}, x_{n_0+1}) &\leq p(x_{n_0+1}, x_{n_0+1}) - kp(x_{n_0}, x_{n_0}) \\ &\leq (1 - k)p(x_{n_0+1}, x_{n_0+1}) \\ &\leq (1 - k)p(x_{n_0}, x_{n_0}), \end{aligned}$$

so, by using (P2),  $p(x_{n_0}, x_{n_0+1}) = p(x_{n_0}, x_{n_0}) = p(x_{n_0+1}, x_{n_0+1})$ , and hence  $x_{n_0} = x_{n_0+1}$ , a contradiction.

Therefore  $p(x_n, x_{n+1}) > p(x_{n+1}, x_{n+2})$  for all  $n \in \omega$ .

Hence, by (8) we get

$$\begin{aligned} p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) &\leq k(p(x_n, x_{n+1}) - p(x_n, x_n)) \\ &\leq k^2(p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})) \\ &\leq \dots \leq k^{n+1}(p(x_0, x_1) - p(x_0, x_0)), \end{aligned} \tag{10}$$

for all  $n \in \omega$ .

We shall show that  $\{x_n\}_{n \in \omega}$  is a Cauchy sequence in  $(X, p)$ . Indeed, let  $n, m \in \omega$  with  $n < m$ . Then, by using (10) and (P4), we derive that

$$\begin{aligned} p(x_n, x_m) - p(x_n, x_n) &\leq p(x_n, x_{n+1}) + \dots + p(x_{m-1}, x_m) - \sum_{k=n}^{m-1} p(x_k, x_k) \\ &\leq (k^n + \dots + k^{m-1})p(x_0, x_1). \end{aligned}$$

Therefore, the sequence  $\{x_n\}_{n \in \omega}$  satisfies condition (\*) of Lemma 1.9, so it is a Cauchy sequence in  $(X, p)$ . Since  $x_n = T^n x_0$  for all  $n$ , and  $(X, p)$  is  $T$ -orbitally complete, there is  $z \in X$  such that  $x_n \rightarrow z$  with respect to  $\tau_{d_p}$ . By the orbital continuity of  $T$ , we deduce that  $x_n \rightarrow Tz$  with respect to  $\tau_{d_p}$ . Hence  $z = Tz$  which concludes the proof.

**Corollary 2.2.** [11, Theorem 1]. *Let  $T$  be an orbitally continuous self-map of a  $T$ -orbitally complete metric space  $(X, d)$ . If there is  $k \in (0, 1)$  such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y), \tag{11}$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \omega}$  converges to a fixed point of  $T$ .

The following are examples where Theorem 2.1 can be applied but not Corollary 2.2 for the metrics  $d_p$  and  $d_m^p$ , and  $p_0$ , respectively.

**Example 2.3.** Let  $X = \{0, 1, 2\}$  and let  $p$  be the partial metric on  $X$  given by  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Define  $T : X \rightarrow X$  by  $T0 = T1 = 0$  and  $T2 = 1$ . Since  $(X, p)$  is complete, then it is  $T$ -orbitally complete. Moreover, it is obvious that  $T$  is orbitally continuous. An easy computation shows that

$$\min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} \leq \frac{1}{2}(p(x, y) - p(x, x)) + p(y, y),$$

for all  $x, y \in X$ . So the conditions of Theorem 2.1 are satisfied. However,

$$\min\{d_p(T1, T2), d_p(1, T1), d_p(2, T2)\} - \min\{d_p(1, T2), d_p(T1, 2)\} = 1 - 0 = 1 > k = kd_p(1, 2),$$

for any  $k \in (0, 1)$ , so Corollary 2.2 cannot be applied to the complete metric space  $(X, d_p)$ . In fact, it cannot be applied to  $(X, d_m^p)$ , because  $d_m^p = d_p$ , in this case.

**Example 2.4.** Let  $X = [1, \infty)$  and let  $p$  be the partial metric on  $X$  given by  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Define  $T : X \rightarrow X$  by  $Tx = (x + 1)/2$  for all  $x \in X$ . Since  $(X, p)$  is complete, then it is  $T$ -orbitally complete. Obviously  $T$  is continuous with respect to  $\tau_{d_p}$ , so it is orbitally continuous.

Next we show that  $T$  satisfies the contraction condition (7) for any  $k \in (0, 1)$ . We distinguish two cases for  $x, y \in X$ :

Case 1.  $x = y$ . Then

$$\begin{aligned} & \min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} \\ &= \min\left\{\frac{x+1}{2}, x, x\right\} - \left(x - \frac{x+1}{2}\right) = 1 \\ &\leq x = p(x, x) = k((p(x, y) - p(x, x)) + p(y, y)). \end{aligned}$$

Case 2.  $x \neq y$ . We assume without loss of generality that  $x > y$ .

If  $Tx \geq y$ , we have

$$\begin{aligned} & \min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} \\ &= \min\left\{\frac{x+1}{2}, x, y\right\} - \min\left\{x - \frac{y+1}{2}, \frac{x+1}{2} - y\right\} \\ &= y - \left(\frac{x+1}{2} - y\right) = 2y - \frac{x+1}{2} \\ &\leq y = p(y, y) = k((p(x, y) - p(x, x)) + p(y, y)). \end{aligned}$$

If  $Tx < y$ , we have

$$\begin{aligned} & \min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} \\ &= \min\left\{\frac{x+1}{2}, x, y\right\} - \min\left\{x - \frac{y+1}{2}, y - \frac{x+1}{2}\right\} \\ &= \frac{x+1}{2} - \left(y - \frac{x+1}{2}\right) = x + 1 - y \\ &< y = p(y, y) = k((p(x, y) - p(x, x)) + p(y, y)). \end{aligned}$$

Therefore, the conditions of Theorem 2.1 are satisfied. In fact  $T$  has a (unique) fixed point,  $x = 1$ .

Finally, we show that Corollary 2.2 cannot be applied to the self-map  $T$  and the complete metric space  $(X, p_0)$ . Indeed, given  $k \in (0, 1)$ , choose  $x > 1$  such that  $x + 1 > 2kx$ , and let  $y = Tx$ . Then

$$\begin{aligned} & \min\{p_0(Tx, Ty), p_0(x, Tx), p_0(y, Ty)\} - \min\{p_0(x, Ty), p_0(Tx, y)\} \\ &= \min\left\{\frac{x+1}{2}, x\right\} - \min\{x, 0\} = \frac{x+1}{2} > kx = kp_0(x, y). \end{aligned}$$

Hence, the contraction condition (11) is not satisfied.

Our next result extends [11, Theorem 3] to partial metric spaces.

**Theorem 2.5.** *Let  $T$  be an orbitally continuous self-map of a partial metric space  $(X, p)$ . Suppose that  $T$  satisfies the inequality*

$$\min\{p(Tx, Ty), p(x, Tx), p(y, Ty)\} - \min\{d_m^p(x, Ty), d_m^p(Tx, y)\} < p(x, y) - p(x, x) + p(y, y), \quad (12)$$

for all  $x, y \in X$  with  $x \neq y$ . If for some  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \omega}$  has a cluster point  $z \in X$  with respect to  $\tau_{d_p}$ , then  $z$  is a fixed point of  $T$ .

*Proof.* Let  $x_0 \in X$  be such that the sequence  $\{T^n x_0\}_{n \in \omega}$  has a cluster point  $z \in X$  with respect to  $\tau_{d_p}$ . Define the iterative sequence  $\{x_n\}_{n \in \omega}$  as  $x_{n+1} = Tx_n$ ,  $n \in \omega$ .

If there exists  $n_0 \in \omega$  such that  $x_{n_0} = x_{n_0+1}$ , then  $x_{n_0}$  is a fixed point of  $T$ . Assume then that  $x_n \neq x_{n+1}$  for each  $n \in \omega$ .

As in the proof of Theorem 2.1, substituting  $x = x_n$  and  $y = x_{n+1}$  in (12) we find the inequality

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} < p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_{n+1}, x_{n+1}),$$

and substituting  $x = x_{n+1}$  and  $y = x_n$  in (12), we obtain

$$\min\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} < p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1}) + p(x_n, x_n).$$

If  $p(x_{n_0}, x_{n_0+1}) \leq p(x_{n_0+1}, x_{n_0+2})$  for some  $n_0 \in \omega$ , we deduce from the preceding two inequalities that  $p(x_{n_0}, x_{n_0}) < p(x_{n_0+1}, x_{n_0+1})$  and  $p(x_{n_0+1}, x_{n_0+1}) < p(x_{n_0}, x_{n_0})$ , respectively, a contradiction.

Consequently  $p(x_n, x_{n+1}) > p(x_{n+1}, x_{n+2})$  for all  $n \in \omega$ , and thus the sequence  $\{p(T^n x_0, T^{n+1} x_0)\}_{n \in \omega}$  is convergent. Since  $\{T^n x_0\}_{n \in \omega}$  has a cluster point  $z \in X$  with respect to  $\tau_{d_p}$ , then there is a subsequence  $\{T^{n_i} x_0\}_{i \in \omega}$  of  $\{T^n x_0\}_{n \in \omega}$  which converges to  $z$  with respect to  $\tau_{d_p}$ . By the orbital continuity of  $T$  we have  $T^{n_i+1} x_0 \rightarrow Tz$  with respect to  $\tau_{d_p}$ , so by Lemma 1.10,

$$\lim_{i \rightarrow \infty} p(T^{n_i} x_0, T^{n_i+1} x_0) = p(z, Tz). \quad (13)$$

Therefore

$$\lim_{n \rightarrow \infty} p(T^n x_0, T^{n+1} x_0) = p(z, Tz). \quad (14)$$

Again, by the orbital continuity of  $T$  we have  $T^{n_i+2} x_0 \rightarrow T^2 z$  with respect to  $\tau_{d_p}$  and hence

$$\lim_{n \rightarrow \infty} p(T^{n_i+1} x_0, T^{n_i+2} x_0) = p(Tz, T^2 z),$$

so

$$p(Tz, T^2 z) = p(z, Tz). \quad (15)$$

Assume  $Tz \neq z$ , that is,  $p(z, Tz) > 0$ . So, one can replace  $x$  and  $y$  with  $z$  and  $Tz$ , respectively, in (12) to deduce that

$$\min\{p(z, Tz), p(Tz, T^2 z)\} < p(z, Tz),$$

which yields that  $p(Tz, T^2 z) < p(z, Tz)$ . This contradicts the equality (15). Thus,  $Tz = z$ . The proof is complete.

Motivated by Ćirić's theorems [11], Pachpatte proved in [34, Theorem 1] that if  $T$  is an orbitally continuous self-map of a  $T$ -orbitally complete metric space  $(X, d)$  such that there is  $k \in (0, 1)$  with

$$\min\{[d(Tx, Tx)]^2, d(x, y)d(Tx, Ty), [d(Ty, y)]^2\} - \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\} \leq kd(x, Tx)d(Ty, y)$$

(16)

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \omega}$  converges to a fixed point of  $T$ .

However, Pachpatte's theorem has a very limited field of application since under its conditions, if we denote by  $z$  any fixed point of  $T$ , it follows that for each  $y \in X$ ,  $Ty = z$  or  $Ty = y$ . Indeed, let  $y \neq z$  and suppose  $Ty \neq z$ . Then from

$$\min\{[d(Tz, Ty)]^2, d(z, y)d(Tz, Ty), [d(y, Ty)]^2\} - \min\{d(z, Tz)d(y, Ty), d(z, Ty)d(y, Tz)\} \leq kd(z, Tz)d(y, Ty),$$

it follows

$$\min\{[d(z, Ty)]^2, d(z, y)d(z, Ty), [d(y, Ty)]^2\} = 0.$$

Hence  $d(y, Ty) = 0$ , i.e.,  $y = Ty$ .

In our next result we modify the contraction condition (16) and thus obtain a new fixed point theorem that avoids the inconvenience indicated above. In fact, this will be done in the more general setting of partial metric spaces and, to this end, the following notation will be used: If  $p$  is a partial metric on a set  $X$  we denote by  $p'$  the function defined on  $X \times X$  by  $p'(x, y) = p(x, y) - p(x, x)$  for all  $x, y \in X$ . (Of course,  $p' = p$  whenever  $p$  is a metric on  $X$ .)

**Theorem 2.6.** *Let  $T$  be an orbitally continuous self-map of a  $T$ -orbitally complete partial metric space  $(X, p)$ . If there is  $k \in (0, 1)$  such that*

$$\min\{[p'(x, Tx)]^2, p'(x, y)p'(Tx, Ty), [p'(y, Ty)]^2\} - \min\{d_m^p(x, Tx)d_m^p(y, Ty), d_m^p(x, Ty)d_m^p(y, Tx)\} \leq k \min\{p'(x, Tx)p'(y, Ty), [p'(x, y)]^2\}, \quad (17)$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \omega}$  converges with respect to  $\tau_{d_p}$  to a fixed point of  $T$ .

*Proof.* As in the proof of Theorem 2.1, take an arbitrary point  $x_0 \in X$  and define the iterative sequence  $\{x_n\}_{n \in \omega}$  as  $x_{n+1} = Tx_n$ ,  $n \in \omega$ .

If there exists  $n_0 \in \omega$  such that  $x_{n_0} = x_{n_0+1}$ , then  $x_{n_0}$  is a fixed point of  $T$ . Assume then that  $x_n \neq x_{n+1}$  for each  $n \in \omega$ .

Substituting  $x = x_n$  and  $y = x_{n+1}$  in (17) we find the inequality

$$\min\{[p'(x_n, x_{n+1})]^2, p'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [p'(x_{n+1}, x_{n+2})]^2\} \leq k \min\{p'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [p'(x_n, x_{n+1})]^2\}. \quad (18)$$

By (18) we deduce that

$$\begin{aligned} & \min\{[p'(x_n, x_{n+1})]^2, p'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [p'(x_{n+1}, x_{n+2})]^2\} \\ & = [p'(x_{n+1}, x_{n+2})]^2, \end{aligned}$$

and hence

$$p'(x_{n+1}, x_{n+2}) \leq kp'(x_n, x_{n+1}),$$

for all  $n \in \omega$ . Therefore

$$p(x_n, x_{n+1}) - p(x_n, x_n) \leq k^n(p(x_0, x_1) - p(x_0, x_0)),$$

for all  $n \in \mathbb{N}$ . As in the proof of Theorem 2.1, we deduce that the sequence  $\{x_n\}_{n \in \omega}$  is Cauchy in  $(X, p)$ . Since  $x_n = T^n x_0$  for all  $n$ , and  $(X, p)$  is  $T$ -orbitally complete, there is  $z \in X$  such that  $x_n \rightarrow z$  with respect to  $\tau_{d_p}$ . By the orbital continuity of  $T$ , we deduce that  $x_n \rightarrow Tz$  with respect to  $\tau_{d_p}$ . Hence  $z = Tz$  which concludes the proof.



**Corollary 2.7.** Let  $T$  be an orbitally continuous self-map of a  $T$ -orbitally complete metric space  $(X, d)$ . If there is  $k \in (0, 1)$  such that

$$\min\{[d(x, Tx)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\} - \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\} \leq k \min\{d(x, Tx)d(y, Ty), [d(x, y)]^2\}, \quad (19)$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \omega}$  converges to a fixed point of  $T$ .

**Remark 2.8.** Note that if  $(X, p)$  is the complete partial metric space of Example 1.2, then each orbitally continuous self-map  $T$  of  $X$  such that  $Tx \leq x$  for all  $x \in X$  has a fixed point. Indeed, for such a  $T$  we have  $p'(x, Tx) = 0$  for all  $x \in X$ , so condition (17) in Theorem 2.6, is trivially satisfied.

The following is an example where Theorem 2.6 can be applied but not Corollary 2.7 for any of the metrics  $d_p$ ,  $d_m^p$  and  $p_0$ .

**Example 2.9.** Let  $(X, p)$  be the partial metric space of Example 1.2. Define  $T : X \rightarrow X$  by  $Tx = x - 1$  if  $x \geq 2$  and  $Tx = 0$  if  $x < 1$ . Then  $T$  is orbitally continuous because for each  $x \in X$  one has  $T^n x \rightarrow 0$  with respect to  $\tau_{d_p}$ , and  $T0 = 0$ . Moreover, by Remark 2.8 the contraction condition (17) is also satisfied, and thus all the conditions of Theorem 2.6 hold.

Now take  $x \geq 3$  and  $y = Tx$ . Then  $x - y = 1$ , and  $y \geq 2$ . Hence

$$\begin{aligned} & \min\{[d_p(x, Tx)]^2, d_p(x, y)d_p(Tx, Ty), [d_p(y, Ty)]^2\} - \min\{d_p(x, Tx)d_p(y, Ty), d_p(x, Ty)d_p(y, Tx)\} \\ &= \min\{1, (x - y)^2, 1\} - 0 = 1 \\ &= \min\{d_p(x, Tx)d_p(y, Ty), [d_p(x, y)]^2\}. \end{aligned}$$

Therefore, condition (19) is not satisfied for any  $k \in (0, 1)$ , so we cannot apply Corollary 2.7 to  $(X, d_p)$  (and thus to  $(X, d_m^p)$ ) and the self-map  $T$ .

Finally, given  $k \in (0, 1)$ , choose  $x \geq 3$  with  $x > 1/(1 - k)$ , and  $y = Tx$ . Then

$$\begin{aligned} & \min\{[p_0(x, Tx)]^2, p_0(x, y)p_0(Tx, Ty), [p_0(y, Ty)]^2\} - \min\{p_0(x, Tx)p_0(y, Ty), p_0(x, Ty)p_0(y, Tx)\} \\ &= \min\{x^2, x(x - 1), (x - 1)^2\} - 0 = (x - 1)^2 \\ &> kx(x - 1) \\ &= k \min\{p_0(x, Tx)p_0(y, Ty), [p_0(x, y)]^2\}. \end{aligned}$$

Therefore, we cannot apply Corollary 2.7 to  $(X, p_0)$  and the self-map  $T$  (note that, in fact,  $T$  is orbitally continuous for  $(X, p_0)$ ).

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