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On controllability for Sturm-Liouville type differential inclusions

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Abstract. We consider a second-order differential inclusion and we obtain sufficient conditions for h-local controllability along a reference trajectory.

1. Introduction

In this paper we are concerned with the following second-order differential inclusion

$$(p(t)x'(t))' \in F(t,x(t)) \quad a.e.\ ([0,T]), \quad x(0) \in X_0, \quad x'(0) \in X_1, \tag{1.1}$$

where $F : [0,T] \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is a set-valued map, $X_0, X_1 \subset \mathbf{R}^n$ are closed sets and $p(.) : [0,T] \to (0,\infty)$ is continuous. Let S_F be the set of all solutions of (1.1) and let $R_F(T)$ be the reachable set of (1.1). For a solution $z(.) \in S_F$ and for a locally Lipschitz function $h : \mathbf{R}^n \to \mathbf{R}^m$ we say that the differential inclusion (1.1) is *h-locally controllable* around $z(.)$ if $h(z(T)) \in \mathrm{int}(h(R_F(T)))$. In particular, if h is the identity mapping the above definitions reduces to the usual concept of local controllability of systems around a solution.

The aim of the present paper is to obtain a sufficient condition for h-local controllability of inclusion (1.1). This result is derived using a technique developed by Tuan for differential inclusions ([11]). More exactly, we show that inclusion (1.1) is h-locally controlable around the solution $z(.)$ if a certain linearized inclusion is λ-locally controlable around the null solution for every $\lambda \in \partial h(z(T))$, where $\partial h(.)$ denotes Clarke's generalized Jacobian of the locally Lipschitz function h. The key tools in the proof of our result is a continuous version of Filippov's theorem for solutions of problem (1.1) obtained in [2] and a certain generalization of the classical open mapping principle in [12].

Our result may be interpreted as an extension of the controllability results in [7] to h-controllability.

We note that existence results and qualitative properties of the solutions of problem (1.1) may be found in [2-8] etc.

The paper is organized as follows: in Section 2 we present some preliminary results to be used in the sequel and in Section 3 we present our main results.

2. Preliminaries

Let us denote by I the interval $[0,T]$ and let X be a real separable Banach space with the norm $\|.\|$ and with the corresponding metric $d(.,.)$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I,

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by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $\|x(.)\|_C = \sup_{t\in I}\|x(t)\|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $\|x(.)\|_1 = \int_I \|x(t)\|dt$.

Consider $F : I \times X \to \mathcal{P}(X)$ a set-valued map, $x_0, x_1 \in X$ and $p(.) : I \to (0,\infty)$ a continuous mapping that defines the Cauchy problem

$$(p(t)x'(t))' \in F(t, x(t)) \quad a.e. \ ([0,T]), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{2.1}$$

A continuous mapping $x(.) \in C(I,X)$ is called a solution of problem (2.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that:

$$f(t) \in F(t, x(t)) \quad a.e. \ (I), \tag{2.2}$$

$$x(t) = x_0 + p(0)x_1 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \int_0^s f(u)duds \quad \forall t \in I. \tag{2.3}$$

Note that, if we denote $S(t,u) := \int_u^t \frac{1}{p(s)}$, $t \in I$, then (2.3) may be rewrite as

$$x(t) = x_0 + p(0)x_1 S(t,0) + \int_0^t S(t,u)f(u)du \quad \forall t \in I, \tag{2.4}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (2.1) if (2.2) and (2.3) are satisfied.

Hypothesis 2.1. i) *$F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.*
ii) *There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that, for any $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1), F(t,x_2)) \le L(t)\|x_1 - x_2\| \quad \forall\, x_1, x_2 \in X.$$

Hypothesis 2.2. *Let S be a separable metric space, $X_0, X_1 \subset X$ are closed sets, $a_0(.) : S \to X_0$, $a_1(.) : S \to X_1$ and $c(.) : S \to (0,\infty)$ are given continuous mappings.*
The continuous mappings $g(.) : S \to L^1(I,X)$, $y(.) : S \to C(I,X)$ are given such that

$$(p(t)(y(s))'(t))' = g(s)(t), \quad y(s)(0) \in X_0, \quad (y(s))'(0) \in X_1.$$

There exists a continuous function $q(.) : S \to L^1(I, \mathbf{R}_+)$ such that

$$d(g(s)(t), F(t, y(s)(t))) \le q(s)(t) \quad a.e. \ (I), \ \forall\, s \in S.$$

Theorem 2.3 ([2]). *Assume that Hypotheses 2.1 and 2.2 are satisfied.*
Then there exist $M > 0$ and the continuous functions $x(.) : S \to L^1(I,X)$, $h(.) : S \to C(I,X)$ such that for any $s \in S$ $(x(s)(.), h(s)(.))$ is a trajectory-selection of (1.1) satisfying for any $(t,s) \in I \times S$

$$x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s),$$

$$\|x(s)(t) - y(s)(t)\| \le M[c(s) + \|a_0(s) - y(s)(0)\| + \|a_1(s) - (y(s))'(0)\| + \int_0^t q(s)(u)du]. \tag{2.5}$$

The proof of Theorem 2.3 may be found in [2].

In what follows we assume that $X = \mathbf{R}^n$.

A closed convex cone $C \subset \mathbf{R}^n$ is said to be *regular tangent cone* to the set X at $x \in X$ ([10]) if there exists continuous mappings $q_\lambda : C \cap B \to \mathbf{R}^n$, $\forall \lambda > 0$ satisfying

$$\lim_{\lambda \to 0+} \max_{v \in C \cap B} \frac{\|q_\lambda(v)\|}{\lambda} = 0,$$

$$x + \lambda v + q_\lambda(v) \in X \quad \forall \lambda > 0, v \in C \cap B,$$

where B is the closed unit ball in $\mathbf{R}^n$.

From the multitude of the intrinsic tangent cones in the literature (e.g. [1]) the *contingent*, the *quasitangent* and *Clarke's tangent cones*, defines, respectively, by

$$\begin{aligned} K_x X &= \{v \in \mathbf{R}^n; \quad \exists s_m \to 0+,\ x_m \in X : \tfrac{x_m - x}{s_m} \to v\} \\ Q_x X &= \{v \in \mathbf{R}^n; \quad \forall s_m \to 0+, \exists x_m \in X : \tfrac{x_m - x}{s_m} \to v\} \\ C_x X &= \{v \in \mathbf{R}^n; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X : \tfrac{y_m - x_m}{s_m} \to v\} \end{aligned}$$

seem to be among the most oftenly used in the study of various problems involving nonsmooth sets and mappings. We recall that, in contrast with K_xX, Q_xX, the cone C_xX is convex and one has $C_xX \subset Q_xX \subset K_xX$.

The results in the next section will be expressed, in the case when the mapping $g(.) : X \subset \mathbf{R}^n \to \mathbf{R}^m$ is locally Lipschitz at x, in terms of the Clarke generalized Jacobian, defined by ([9])

$$\partial g(x) = \mathrm{co}\{\lim_{i \to \infty} g'(x_i); \quad x_i \to x, \quad x_i \in X \backslash \Omega_g\},$$

where Ω_g is the set of points at which g is not differentiable.

Corresponding to each type of tangent cone, say $\tau_x X$ one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \mathrm{graph}(G)$ as follows

$$\tau_y G(x; v) = \{w \in \mathbf{R}^n;\ (v, w) \in \tau_{(x,y)}\mathrm{graph}(G)\}, \quad \in \tau_x X.$$

We recall that a set-valued map, $A(.) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $\mathrm{graph}(A(.)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

Hypothesis 2.4. i) *Hypothesis 2.1 is satisfied and* $X_0, X_1 \subset \mathbf{R}^n$ *are closed sets.*

ii) $(z(.), f(.)) \in C(I, \mathbf{R}^n) \times L^1(I, \mathbf{R}^n)$ *is a trajectory-selection pair of (1.1) and a family* $P(t, .) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$, $t \in I$ *of convex processes satisfying the condition*

$$P(t, u) \subset Q_{f(t)}F(t, .)(z(t); u) \quad \forall u \in dom(P(t, .)),\ a.e.\ t \in I \tag{2.6}$$

is assumed to be given and defines the variational inclusion

$$(p(t)v'(t))' \in P(t, v(t)). \tag{2.7}$$

Remark 2.5. We note that for any set-valued map $F(., .)$, one may find an infinite number of families of convex processes $P(t, .)$, $t \in I$, satisfying condition (2.6); in fact any family of closed convex subcones of the quasitangent cones, $\overline{P}(t) \subset Q_{(z(t), f(t))}graph(F(t, .))$, defines the family of closed convex processes

$$P(t, u) = \{v \in \mathbf{R}^n;\ (u, v) \in \overline{P}(t)\}, \quad u, v \in \mathbf{R}^n,\ t \in I$$

that satisfy condition (2.6). One is tempted, of course, to take as an "intrinsic" family of such closed convex process, for example Clarke's convex-valued directional derivatives $C_{f(t)}F(t, .)(z(t); .)$.

We recall (e.g. [1]) that, since $F(t,.)$ is assumed to be Lipschitz a.e. on I, the quasitangent directional derivative is given by

$$Q_{f(t)}F(t,.)((z(t);u)) = \{w \in \mathbf{R}^n;\ \lim_{\theta\to 0+} \frac{1}{\theta} d(f(t)+\theta w, F(t,z(t)+\theta u)) = 0\}. \tag{2.8}$$

In what follows B or $B_{\mathbf{R}^n}$ denotes the closed unit ball in $\mathbf{R}^n$ and 0_n denotes the null element in $\mathbf{R}^n$.
Consider $h : \mathbf{R}^n \to \mathbf{R}^m$ an arbitrary given function.

Definition 2.6. Inclusion (1.1) is said to be *h-locally controllable* around $z(.)$ if $h(z(T)) \in \mathrm{int}(h(R_F(T)))$. Inclusion (1.1) is said to be *locally controllable* around the solution $z(.)$ if $z(T) \in \mathrm{int}(R_F(T))$.

Finally a key tool in the proof of our results is the following generalization of the classical open mapping principle due to Warga ([12]).

For $k \in N$ we define

$$\Sigma_k := \{\gamma = (\gamma_1, ..., \gamma_k);\quad \sum_{i=1}^{k} \gamma_i \le 1, \quad \gamma_i \ge 0,\ i = 1,2,...,k\}.$$

Lemma 2.7. ([12]) *Let $\delta \le 1$, let $g(.) : \mathbf{R}^n \to \mathbf{R}^m$ be a mapping that is C^1 in a neighborhood of 0_n containing $\delta B_{\mathbf{R}^n}$. Assume that there exists $\beta > 0$ such that for every $\theta \in \delta\Sigma_n$, $\beta B_{\mathbf{R}^m} \subset g'(\theta)\Sigma_n$. Then, for any continuous mapping $\psi : \delta\Sigma_n \to \mathbf{R}^m$ that satisfies $\sup_{\theta\in\delta\Sigma_n} \|g(\theta) - \psi(\theta)\| \le \frac{\delta\beta}{32}$ we have $\psi(0_n) + \frac{\delta\beta}{16} B_{\mathbf{R}^m} \subset \psi(\delta\Sigma_n)$.*

3. The main result

In what follows we assume that Hypothesis 2.4 is satisfied, C_0 is a regular tangent cone to X_0 at $z(0)$, C_1 is a regular tangent cone to X_1 at $z'(0)$, denote by S_P the set of all solutions of the differential inclusion

$$(p(t)v'(t))' \in P(t, v(t)), \quad v(0) \in C_0, \quad v'(0) \in C_1$$

and by $R_P(T) = \{x(T);\ x(.) \in S_P\}$ its reachable set at time T.

Theorem 3.1 *Assume that Hypothesis 2.4 is satisfied and let $h : \mathbf{R}^n \to \mathbf{R}^m$ be a Lipschitz function with Lipschitz constant $l > 0$.*
Then inclusion (1.1) is h-locally controllable around the solution $z(.)$ if

$$0_m \in \mathrm{int}(\lambda R_P(T)) \quad \forall \lambda \in \partial h(z(T)). \tag{3.1}$$

Proof. By (3.1), since $\lambda R_P(T)$ is a convex cone, it follows that $\lambda R_P(T) = \mathbf{R}^m\ \forall\lambda \in \partial h(z(T))$. Therefore using the compactness of $\partial h(z(T))$ (e.g. [9]), we have that for every $\beta > 0$ there exist $k \in N$ and $u_j \in R_P(T)$ $j = 1,2,...,k$ such that

$$\beta B_{\mathbf{R}^m} \subset \lambda(u(\Sigma_k)) \quad \forall\lambda \in \partial h(z(T)), \tag{3.2}$$

where

$$u(\Sigma_k) = \{u(\gamma) := \sum_{j=1}^{k} \gamma_j u_j, \quad \gamma = (\gamma_1, ..., \gamma_k) \in \Sigma_k\}.$$

Using an usual separation theorem we deduce the existence of $\beta_1, \rho_1 > 0$ such that for all $\lambda \in L(\mathbf{R}^n, \mathbf{R}^m)$ with $d(\lambda, \partial h(z(T))) \le \rho_1$ we have

$$\beta_1 B_{\mathbf{R}^m} \subset \lambda(u(\Sigma_k)). \tag{3.3}$$

Since $u_j \in R_P(T)$, $j = 1,...,k$, there exist $(w_j(.), g_j(.))$, $j = 1,...,k$ trajectory-selection pairs of (2.7) such that $u_j = w_j(T)$, $j = 1,...,k$. We note that $\beta > 0$ can be taken small enough to provide $\|w_j(0)\| \le 1$, $j = 1,...,k$.

Define

$$w(t,s)=\sum_{j=1}^{k}s_jw_j(t),\quad \overline{g}(t,s)=\sum_{j=1}^{k}s_jg_j(t),\quad \forall s=(s_1,...,s_k)\in \mathbf{R}^k.$$

Obviously, $w(.,s)\in S_P$, $\forall s\in\Sigma_k$.

Taking into account the definition of C_0 and C_1 we conclude that for every $\varepsilon>0$ there exists a continuous mapping $o_\varepsilon:\Sigma_k\to\mathbf{R}^n$ such that

$$z(0)+\varepsilon w(0,s)+o_\varepsilon(s)\in X_0,\quad z'(0)+\varepsilon\frac{\partial w}{\partial t}(0,s)+o_\varepsilon(s)\in X_1 \tag{3.4}$$

$$\lim_{\varepsilon\to 0+}\max_{s\in\Sigma_k}\frac{||o_\varepsilon(s)||}{\varepsilon}=0. \tag{3.5}$$

We recall that $(z(.),f(.))$ is a trajectory-selection pair of (1.1). Define

$$p_\varepsilon(s)(t):=\frac{1}{\varepsilon}\mathrm{d}(\overline{g}(t,s),F(t,z(t)+\varepsilon w(t,s))-f(t)),$$

$$q(t):=\sum_{j=1}^{k}[||g_j(t)||+L(t)||w_j(t)||],\quad t\in I.$$

Then, for every $s\in\Sigma_k$ one has

$$\begin{aligned}p_\varepsilon(s)(t)&\leq||\overline{g}(t,s)||+\tfrac{1}{\varepsilon}\mathrm{d}_H(0_n,F(t,z(t)+\varepsilon w(t,s))-f(t))\leq||\overline{g}(t,s)||+\\ &\tfrac{1}{\varepsilon}\mathrm{d}_H(F(t,z(t)),F(t,z(t)+\varepsilon w(t,s)))\leq||\overline{g}(t,s)||+L(t)||w(t,s)||\leq q(t).\end{aligned} \tag{3.6}$$

Next, if $s_1,s_2\in\Sigma_k$ one has

$$\begin{aligned}&|p_\varepsilon(s_1)(t)-p_\varepsilon(s_2)(t)|\leq||\overline{g}(t,s_1)-\overline{g}(t,s_2)||+\tfrac{1}{\varepsilon}\mathrm{d}_H(F(t,z(t)+\varepsilon w(t,s_1)),\\ &F(t,z(t)+\varepsilon w(t,s_2)))\leq||s_1-s_2||.\max_{j=\overline{1,k}}[||g_j(t)||+L(t)||w_j(t)||],\end{aligned}$$

thus $p_\varepsilon(.)(t)$ is Lipschitz with a Lipschitz constant not depending on ε.

On the other hand, from (2.8) it follows that

$$\lim_{\varepsilon\to 0}p_\varepsilon(s)(t)=0\quad a.e.\,(I),\quad\forall s\in\Sigma_k$$

and hence

$$\lim_{\varepsilon\to 0+}\max_{s\in\Sigma_k}p_\varepsilon(s)(t)=0\quad a.e.\ (I). \tag{3.7}$$

Therefore, from (3.6), (3.7) and the Lebesgue dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0+}\int_0^T\max_{s\in\Sigma_k}p_\varepsilon(s)(t)\mathrm{d}t=0. \tag{3.8}$$

By (3.4), (3.5), (3.8) and the upper semicontinuity of the Clarke generalized Jacobian we can find $\varepsilon_0,e_0>0$ such that

$$\max_{s\in\Sigma_k}\frac{||o_{\varepsilon_0}(s)||}{\varepsilon_0}+\int_0^T\max_{s\in\Sigma_k}p_{\varepsilon_0}(s)(t)\mathrm{d}t\leq\frac{\beta_1}{2^8l^2}, \tag{3.9}$$

$$\varepsilon_0 w(T,s)\leq\frac{e_0}{2}\quad\forall s\in\Sigma_k. \tag{3.10}$$

If we define

$$y(s)(t):=z(t)+\varepsilon_0w(t,s),\quad g(s)(t):=f(t)+\varepsilon_0\overline{g}(t,s)\quad s\in\mathbf{R}^k,$$

$$a_0(s) := z(0) + \varepsilon_0 w(0,s) + o_{\varepsilon_0}(s), \quad a_1(s) := z'(0) + \varepsilon_0 \frac{\partial w}{\partial t}(0,s) + o_{\varepsilon_0}(s),\ s \in \mathbf{R}^k,$$

then we apply Theorem 2.3 and we find that there exists a continuous function $x(.) : \Sigma_k \to C(I, \mathbf{R}^n)$ such that for any $s \in \Sigma_k$ the function $x(s)(.)$ is a solution of the differential inclusion $(p(t)x'(t))' \in F(t, x(t)), \quad x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s)\ \forall s \in \Sigma_k$ and one has

$$||x(s)(T) - y(s)(T)|| \leq \frac{\varepsilon_0 \beta_1}{2^6 l} \quad \forall s \in \Sigma_k. \tag{3.11}$$

We define

$$h_0(x) := \int_{\mathbf{R}^n} h(x - by)\chi(y)\mathrm{d}y, \quad x \in \mathbf{R}^n,$$
$$\phi(s) := h_0(z(T) + \varepsilon_0 w(T, s)),$$

where $\chi(.) : \mathbf{R}^n \to [0,1]$ is a C^∞ function with the support contained in $B_{\mathbf{R}^n}$ that satisfies $\int_{\mathbf{R}^n} \chi(y)\mathrm{d}y = 1$ and $b = \min\{\frac{e_0}{2}, \frac{\varepsilon_0 \beta_1}{2^6 l}\}$.

Therefore $h_0(.)$ is of class C^∞ and verifies

$$||h(x) - h_0(x)|| \leq lb, \tag{3.12}$$

$$h_0'(x) = \int_{\mathbf{R}^n} h'(x - by)\chi(y)\mathrm{d}y. \tag{3.13}$$

In particular,

$$h_0'(x) \in \overline{\mathrm{co}}\{h'(u); \quad ||u - x|| \leq b, \quad h'(u) \text{ exists}\},$$
$$\phi'(s)\mu = h_0'(z(T) + \varepsilon_0 w(T, s))\varepsilon_0 w(T, \mu) \quad \forall \mu \in \Sigma_k.$$

Let us denote

$$\lambda(s) := h_0'(z(T) + \varepsilon_0 w(T, s)).$$

Therefore, $\phi'(s)\mu = \lambda(s)\varepsilon_0 w(T, \mu)\ \forall \mu \in \Sigma_k$.

Using again the upper semicontinuity of the Clarke generalized Jacobian we obtain

$$\mathrm{d}(\lambda(s), \partial h(z(T))) = \mathrm{d}(h_0'(z(T) + \varepsilon_0 w(T, s)), \partial h(z(T))) \leq \sup\{\mathrm{d}(h_0'(u), \partial h(z(T)));$$
$$||u - z(t)|| \leq ||u - (z(T) + \varepsilon_0 w(T, s))|| + ||\varepsilon_0 w(t, s)|| \leq e_0, \quad h'(u) \text{ exists}\} < \rho_1.$$

The last inequality with (3.3) gives

$$\beta_1 B_{\mathbf{R}^m} \subset \lambda(s)u(\Sigma_k).$$

and thus

$$\varepsilon_0\beta_1 B_{\mathbf{R}^m} \subset \lambda(s)\varepsilon_0 u(\Sigma_k) = \lambda(s)\varepsilon_0 w(T, \mu) = \phi'(s)\mu, \quad \forall \mu \in \Sigma_k,$$

i.e.,

$$\varepsilon_0\beta_1 B_{\mathbf{R}^m} \subset \phi'(s)\Sigma_k.$$

Finally, for $s \in \Sigma_k$, we put $\psi(s) = h(x(s)(T))$.

Obviously, $\psi(.)$ is continuous and from (3.11), (3.12), (3.13) one has

$$||\psi(s) - \phi(s)|| = ||h(x(s)(T)) - h_0(y(s)(T))|| \leq ||h(x(s)(T)) - h(y(s)(T))|| + ||h(y(s)(T)) - h_0(y(s)(T))||$$
$$\leq l||x(s)(T) - y(s)(T)|| + lb \leq \frac{\varepsilon_0\beta_1}{64} + \frac{\varepsilon_0\beta_1}{64} = \frac{\varepsilon_0\beta_1}{32}.$$

We apply Lemma 2.7 and we find that

$$h(x(0_k)(T)) + \frac{\varepsilon_0\beta_1}{16} B_{\mathbf{R}^m} \subset \psi(\Sigma_k) \subset h(R_F(T)).$$

On the other hand, $||h(z(T)) - h(x(0_k)(T))|| \leq \frac{\varepsilon_0\beta_1}{64}$, so we have $h(z(T)) \in \mathrm{int}(h(R_F(T)))$ and the proof is complete.

Remark 3.2. If $m = n$ and $h(x) \equiv x$, Theorem 3.1 yields Theorem 3.4 in [7].

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