

Coincidence and Fixed Point Results under Generalized Weakly Contractive Condition in Partially Ordered G -metric Spaces

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Abstract. We deduce coincidence and fixed point theorems under generalized weakly contractive conditions in G -metric spaces equipped with partial order. We furnish examples to demonstrate the usage of the results and to distinguish them from the known ones.

1. Introduction

In 2004, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G -metric space [1, 2]. This is a generalization of metric spaces in which to every triplet of elements a non-negative real number is assigned. Analysis of the structure of these spaces was done in details in [2]. Fixed point theory in such spaces was studied in [3]–[5]. Particularly, Banach contraction mapping principle was established in these works. After that several fixed point results were proved in these spaces (see, e.g., [6]–[11]).

Recently, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [12, Theorem 2.1] who presented its applications to matrix equation. Subsequently, Nieto and Rodríguez-López [13] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Further results were obtained, e.g., in [14]–[19].

Fixed point results in partially ordered G -metric spaces were obtained by, e.g., Saadati et al. [20], Shatanawi [21], and Abbas et al. [22].

In this article we deduce coincidence and fixed point theorems under generalized weakly contractive conditions in ordered G -metric spaces. Our results are extensions of the results of, e.g., Harjani and Sadarangani [15, 16], as well as Aydi et al. [9], Shatanawi [11] and other related papers, in the sense that the considered contractive condition is more general, and the problem is treated in the frame of ordered generalized metric spaces. We furnish examples to demonstrate the validity of the results and that these extensions are proper.

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2. Preliminaries

For more details on the following definitions and results, we refer the reader to [2].

Definition 2.1. [2] Let X be a nonempty set and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$;
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables);
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G -metric on X and the pair (X, G) is called a G -metric space.

Definition 2.2. [2] Let (X, G) be a G -metric space and let $\{x_n\}$ be a sequence of points in X .

1. A point $x \in X$ is said to be the limit of sequence $\{x_n\}$ if $\lim_{n,m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G -convergent to x .
2. The sequence $\{x_n\}$ is said to be a G -Cauchy sequence if, for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \rightarrow 0$, as $n, m, l \rightarrow \infty$.
3. (X, G) is said to be G -complete (or a complete G -metric space) if every G -Cauchy sequence in (X, G) is G -convergent in X .

Thus, if $x_n \rightarrow x$ in a G -metric space (X, G) , then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. It was shown in [2] that the G -metric induces a Hausdorff topology and that the convergence, as described in the above definition, is relative to this topology. The topology being Hausdorff, a sequence can converge to at most one point.

Lemma 2.3. [2] Let (X, G) be a G -metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then the following are equivalent:

- (1) $\{x_n\}$ is G -convergent to x .
- (2) $G(x_n, x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.
- (3) $G(x_n, x, x) \rightarrow 0$, as $n \rightarrow \infty$.

Lemma 2.4. [2] If (X, G) is a G -metric space, then the following are equivalent:

- (1) The sequence $\{x_n\}$ is G -Cauchy.
- (2) For every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Lemma 2.5. [2] Let (X, G) , (X', G') be two G -metric spaces. Then a function $f : X \rightarrow X'$ is G -continuous at a point $x \in X$ if and only if it is G -sequentially continuous at x , that is, if $\{fx_n\}$ is G' -convergent to fx whenever $\{x_n\}$ is G -convergent to x .

Definition 2.6. [2] A G -metric space (X, G) is said to be symmetric if

$$G(x, x, y) = G(x, y, y)$$

holds for arbitrary $x, y \in X$. If this is not the case, the space is called asymmetric.

To every G -metric on the set X a standard metric can be associated by

$$d_G(x, y) = G(x, x, y) + G(x, y, y).$$

If G is symmetric, then obviously $d_G(x, y) = 2G(x, y, y)$, but in the case of an asymmetric G -metric, only

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$$

holds for all $x, y, z \in X$.

The following are some easy examples of G -metric spaces.

Example 2.7. (1) Let (X, d) be an ordinary metric space. Define G_s by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in X$. Then it is clear that (X, G_s) is a (symmetric) G -metric space.

(2) Let $X = \{a, b\}$. Define

$$G(a, a, a) = G(b, b, b) = 0, \quad G(a, a, b) = 1, \quad G(a, b, b) = 2,$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear that (X, G) is an asymmetric G -metric space.

Assertions similar to the following lemma were used in the frame of metric spaces in the course of proofs of several fixed point results in various papers (see, e.g., [23, Lemma 2.1]).

Lemma 2.8. Let (X, G) be a G -metric space and let $\{y_n\}$ be a sequence in X such that $\{G(y_n, y_{n+1}, y_{n+1})\}$ is non-increasing and

$$\lim_{n \rightarrow \infty} G(y_n, y_{n+1}, y_{n+1}) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence in (X, G) , then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε when $k \rightarrow \infty$:

$$G(y_{2m_k}, y_{2n_k}, y_{2n_k}), \quad G(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}), \quad G(y_{2m_k+1}, y_{2n_k}, y_{2n_k}), \quad G(y_{2n_k-1}, y_{2m_k+1}, y_{2m_k+1}).$$

Proof. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in (X, G) . Then there exists $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and $G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \geq \varepsilon$, and they can be chosen so that n_k is always the smallest possible, i.e., $G(y_{2m_k}, y_{2n_k-2}, y_{2n_k-2}) < \varepsilon$. Now, applying (G5) we get that

$$\begin{aligned} \varepsilon &\leq G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \leq G(y_{2m_k}, y_{2n_k-2}, y_{2n_k-2}) + G(y_{2n_k-2}, y_{2n_k-1}, y_{2n_k-1}) + G(y_{2n_k-1}, y_{2n_k}, y_{2n_k}) \\ &< \varepsilon + G(y_{2n_k-2}, y_{2n_k-1}, y_{2n_k-1}) + G(y_{2n_k-1}, y_{2n_k}, y_{2n_k}). \end{aligned}$$

Passing to the limit as $k \rightarrow \infty$ we get that $\lim_{k \rightarrow \infty} G(y_{2m_k}, y_{2n_k}, y_{2n_k}) = \varepsilon$.

Now, again by (G5), we have that

$$\begin{aligned} G(y_{2m_k}, y_{2n_k}, y_{2n_k}) &\leq G(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}) + G(y_{2n_k-1}, y_{2n_k}, y_{2n_k}), \\ G(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}) &\leq G(y_{2m_k}, y_{2n_k}, y_{2n_k}) + G(y_{2n_k}, y_{2n_k-1}, y_{2n_k-1}). \end{aligned}$$

Passing to the limit as $k \rightarrow \infty$ we get that $\lim_{k \rightarrow \infty} G(y_{2m_k}, y_{2n_k-1}, y_{2n_k-1}) = \varepsilon$

The proof for the remaining two sequences is similar. \square

Definition 2.9. Let X be a nonempty set. Then (X, G, \leq) is called an ordered G -metric space if:

- (i) (X, G) is a G -metric space, and
- (ii) (X, \leq) is a partially ordered set.

Let (X, \leq) be a partially ordered set. Recall that $x, y \in X$ are called comparable if $x \leq y$ or $y \leq x$ holds. If $S, T : X \rightarrow X$ are such that, for $x, y \in X$, $Sx \leq Sy$ implies $Tx \leq Ty$, then T is said to be S -non-decreasing. Similarly, an S -non-increasing mapping is defined.

Definition 2.10. Let (X, G, \leq) be a partially ordered G -metric space. We say that (X, G, \leq) is regular if the following hypotheses hold:

- (i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- (ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \rightarrow y$ as $n \rightarrow \infty$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

Definition 2.11. [24]. A function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

- (i) ψ is continuous and non-decreasing,
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

Recall also the following notions. Let X be a non-empty set and $S, T : X \rightarrow X$ be given self-maps on X . If $w = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T , and w is called a point of coincidence of S and T . The pair $\{S, T\}$ is said to be weakly compatible if $STx = TSx$, whenever $Tx = Sx$ for some x in X . We will use the following version of compatibility of these maps in a G -metric space.

Definition 2.12. Mappings $S, T : X \rightarrow X$ are said to be compatible in a G -metric space (X, G) if

$$G(TSx_n, STx_n, STx_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n$ in (X, G) .

It is easy to see that S and T are compatible in (X, G) if and only if they are compatible in the associated metric space (X, d_G) .

3. Results

Our first main result is the following theorem.

Theorem 3.1. Let (X, G, \leq) be a complete ordered G -metric space. Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous function with $\varphi(t) = 0$ if and only if $t = 0$ and let ψ be an altering distance function. Suppose that $S, T : X \rightarrow X$ are such that T is S -non-decreasing, $TX \subset SX$ and one of these two subsets of X is closed. Let

$$\psi(G(Tx, Ty, Tz)) \leq \psi(\Theta(x, y, z)) - \varphi(\theta(x, y, z)), \tag{1}$$

where

$$\begin{aligned} \Theta(x, y, z) = \max \left\{ & G(Sx, Sy, Sz), G(Sx, Tx, Tx), G(Sy, Ty, Ty), G(Sz, Tz, Tz), \right. \\ & \frac{1}{2}[G(Sx, Ty, Ty) + G(Sy, Tx, Tx)], \frac{1}{2}[G(Sy, Tz, Tz) + G(Sz, Ty, Ty)], \\ & \frac{1}{2}[G(Sx, Tz, Tz) + G(Sz, Tx, Tx)], \\ & \left. \frac{1}{3}[G(Sx, Ty, Ty) + G(Sy, Tz, Tz) + G(Sz, Tx, Tx)] \right\} \tag{2} \end{aligned}$$

and

$$\theta(x, y, z) = \max\{G(Sx, Sy, Sz), G(Sx, Tx, Tx), G(Sy, Ty, Ty), G(Sz, Tz, Tz)\} \tag{3}$$

for all $x, y, z \in X$ with $Sz \leq Sy \leq Sx$. In addition, we assume that

- (i) S and T are continuous and compatible in the sense of Definition 2.12, or
- (ii) X is regular.

If there exists an $x_0 \in X$ such that $Sx_0 \leq Tx_0$, then T and S have a coincidence point; that is, there exists $z \in X$ such that $Sz = Tz$.

Remark 3.2. The result of this theorem remains valid if the function φ is only lower semi-continuous. For the sake of simplicity, we stay with the given assumptions.

Proof. Let $x_0 \in X$ be such that $Sx_0 \leq Tx_0$. Using that $TX \subset SX$, choose an $x_1 \in X$ such that $Sx_0 < Tx_0 = Sx_1$. Since T is an S -non-decreasing mapping, we have $Sx_0 < Sx_1 = Tx_0 \leq Tx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $Sx_{n+1} = Tx_n$ with

$$Sx_0 < Sx_1 \leq Sx_2 \leq \dots \leq Sx_n \leq Sx_{n+1} \leq \dots .$$

If $Sx_{n_0} = Sx_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \{0, 1, 2, \dots\}$, then x_{n_0} is a coincidence point of S and T and the proof is completed. Thus we shall assume that

$$G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) > 0 \tag{4}$$

for all $n \geq 1$.

We first prove that $\lim_{n \rightarrow \infty} G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) = 0$.

Since $Sx_n \leq Sx_{n+1} \leq Sx_{n+1}$, we can use (1) for the points x_n, x_{n+1}, x_{n+1} . We have, for $n \geq 1$

$$\begin{aligned} &\Theta(x_n, x_{n+1}, x_{n+1}) \\ &= \max\{G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_n, Tx_n, Tx_n), G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1}), G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ &\quad \frac{1}{2}[G(Sx_n, Tx_{n+1}, Tx_{n+1}) + G(Sx_{n+1}, Tx_n, Tx_n)], \\ &\quad \frac{1}{2}[G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1}) + G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1})], \\ &\quad \frac{1}{2}[G(Sx_n, Tx_{n+1}, Tx_{n+1}) + G(Sx_{n+1}, Tx_n, Tx_n)], \\ &\quad \frac{1}{3}[G(Sx_n, Tx_{n+1}, Tx_{n+1}) + G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1}) + G(Sx_{n+1}, Tx_n, Tx_n)]\} \\ &= \max\{G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}), \frac{1}{2}G(Sx_n, Sx_{n+2}, Sx_{n+2}), \\ &\quad \frac{1}{3}[G(Sx_n, Sx_{n+2}, Sx_{n+2}) + G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2})]\}. \end{aligned}$$

By (G5) we have

$$G(Sx_n, Sx_{n+2}, Sx_{n+2}) \leq G(Sx_n, Sx_{n+1}, Sx_{n+1}) + G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}).$$

Thus

$$\begin{aligned} \Theta(x_n, x_{n+1}, x_{n+1}) &= \theta(x_n, x_{n+1}, x_{n+1}) \\ &= \max\{G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2})\}. \end{aligned}$$

From (1) we have

$$\begin{aligned} \psi(G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2})) &= \psi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leq \psi(\Theta(x_n, x_{n+1}, x_{n+1})) - \varphi(\theta(x_n, x_{n+1}, x_{n+1})) \\ &= \psi(\max\{G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2})\}) \\ &\quad - \varphi(\max\{G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2})\}). \end{aligned} \tag{5}$$

We claim that

$$G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) \leq G(Sx_n, Sx_{n+1}, Sx_{n+1}) \tag{6}$$

for all $n \geq 1$. Suppose this is not true, that is, there exists an $n_0 \geq 1$ such that $G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2}) > G(Sx_{n_0}, Sx_{n_0+1}, Sx_{n_0+1})$. Now since $Sx_{n_0} \leq Sx_{n_0+1} \leq Sx_{n_0+1}$, we can use inequality (5) for these elements, and we have

$$\begin{aligned} \psi(G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2})) &\leq \psi(\max\{G(Sx_{n_0}, Sx_{n_0+1}, Sx_{n_0+1}), G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2})\}) \\ &\quad - \varphi(\max\{G(Sx_{n_0}, Sx_{n_0+1}, Sx_{n_0+1}), G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2})\}) \\ &= \psi(G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2})) - \varphi(G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2})). \end{aligned}$$

This implies that $\varphi(G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2})) = 0$ and by the property of φ , we have $G(Sx_{n_0+1}, Sx_{n_0+2}, Sx_{n_0+2}) = 0$, which contradicts to the condition (4). Therefore, (6) is true and so the sequence $\{G(Sx_n, Sx_{n+1}, Sx_{n+1})\}$ is non-increasing and bounded below. Thus there exists $\rho \geq 0$ such that

$$\lim_{n \rightarrow \infty} G(Sx_n, Sx_{n+1}, Sx_{n+1}) = \rho. \tag{7}$$

Now suppose that $\rho > 0$. Taking $n \rightarrow \infty$ in (5), then using (7) and the continuity of ψ and φ , we obtain

$$\psi(\rho) \leq \psi(\rho) - \varphi(\rho).$$

Therefore $\psi(\rho) = 0$ and hence $\rho = 0$. Thus

$$\lim_{n \rightarrow \infty} G(Sx_n, Sx_{n+1}, Sx_{n+1}) = 0.$$

We will prove now that $\{Sx_n\}$ is a G-Cauchy sequence in X . Suppose this is not the case. Then, by Lemma 2.8, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the sequences

$$G(Sx_{2m_k}, Sx_{2n_k}, Sx_{2n_k}), \quad G(Sx_{2m_k}, Sx_{2n_k-1}, Sx_{2n_k-1}), \quad G(Sx_{2m_k+1}, Sx_{2n_k}, Sx_{2n_k}), \quad G(Sx_{2n_k-1}, Sx_{2m_k+1}, Sx_{2m_k+1}).$$

tend to ε when $k \rightarrow \infty$. Now, from the definitions of $\Theta(x, y, z)$ and $\theta(x, y, z)$, and from the obtained limits, we have

$$\lim_{k \rightarrow \infty} \Theta(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1}) = \lim_{k \rightarrow \infty} \theta(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1}) = \varepsilon. \tag{8}$$

Putting $x = x_{2m_k}$, $y = x_{2n_k-1}$, $z = x_{2n_k-1}$ in (1) (which can be done since the sequence $\{Sx_n\}$ is monotone) we have

$$\begin{aligned} \psi(G(Sx_{2m_k+1}, Sx_{2n_k}, Sx_{2n_k})) &= \psi(G(Tx_{2m_k}, Tx_{2n_k-1}, Tx_{2n_k-1})) \\ &\leq \psi(\Theta(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1})) - \varphi(\theta(x_{2m_k}, x_{2n_k-1}, x_{2n_k-1})). \end{aligned}$$

Letting $k \rightarrow \infty$, utilizing (8) and the obtained limits, we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction if $\varepsilon > 0$. We have proved that $\{Sx_n\}$ is a Cauchy sequence in (X, G) . Suppose, e.g., that SX is complete (if TX is complete, the proof is similar). Then there exists an $Sz \in SX$ such that

$$\lim_{n \rightarrow \infty} Sx_n = Sz. \tag{9}$$

Suppose that (i) holds. We have that $Sx_{n+1} = Tx_n$ is a sequence, G-convergent to Sz . Hence, $G(Tx_n, Sz, Sz) \rightarrow 0$ and $G(Sx_n, Sz, Sz) \rightarrow 0$ as $n \rightarrow \infty$. Compatibility of S and T (Definition 2.12) implies that $G(TSx_n, STx_n, STx_n) \rightarrow 0$ as $n \rightarrow \infty$. It follows (using (G5) and continuity of S and T) that

$$\begin{aligned} G(TSz, SSz, SSz) &\leq G(TSz, TSx_n, TSx_n) + G(TSx_n, SSz, SSz) \\ &\leq G(TSz, TSx_n, TSx_n) + G(TSx_n, STx_n, STx_n) + G(STx_n, SSz, SSz) \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, $TSz = SSz$ and Sz is a coincidence point of S and T .

Suppose that (ii) holds. Since $\{Sx_n\}$ is a non-decreasing sequence such that $Sx_n \rightarrow Sz$ and X is regular, it follows that $Sx_n \leq Sz \leq Sz$ for all $n \in \mathbb{N}$. Therefore, we can apply (1) to get

$$\psi(G(Sx_{n+1}, Tz, Tz)) = \psi(G(Tx_n, Tz, Tz)) \leq \psi(\Theta(x_n, z, z)) - \varphi(\theta(x_n, z, z)), \tag{10}$$

where

$$\begin{aligned} \Theta(x_n, z, z) &= \max\{G(Sx_n, Sz, Sz), G(Sx_n, Tx_n, Tx_n), G(Sz, Tz, Tz), G(Sz, Tz, Tz), \\ &\quad \frac{1}{2}[G(Sx_n, Tz, Tz) + G(Sz, Tx_n, Tx_n)], \frac{1}{2}[G(Sz, Tz, Tz) + G(Sz, Tz, Tz)], \\ &\quad \frac{1}{2}[G(Sz, Tx_n, Tx_n) + G(Sx_n, Tz, Tz)], \\ &\quad \frac{1}{3}[G(Sx_n, Tz, Tz) + G(Sz, Tz, Tz) + G(Sz, Tx_n, Tx_n)]\} \\ &= \max\{G(Sx_n, Sz, Sz), G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sz, Tz, Tz), \\ &\quad \frac{1}{2}[G(Sx_n, Tz, Tz) + G(Sz, Sx_{n+1}, Sx_{n+1})], \\ &\quad \frac{1}{3}[G(Sx_n, Tz, Tz) + G(Sz, Tz, Tz) + G(Sz, Sx_{n+1}, Sx_{n+1})]\}, \end{aligned} \tag{11}$$

and

$$\theta(x_n, z, z) = \max\{G(Sx_n, Sz, Sz), G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sz, Tz, Tz)\}. \tag{12}$$

Letting $n \rightarrow \infty$ in inequality (10) and using (9), (11), (12) and the fact that G is continuous in its variables, we obtain

$$\psi(G(Sz, Tz, Tz)) \leq \psi(G(Sz, Tz, Tz)) - \varphi(G(Sz, Tz, Tz)).$$

This implies that $\varphi(G(Sz, Tz, Tz)) = 0$ and hence $Sz = Tz$. Thus z is a coincidence point of S and T . \square

Remark 3.3. Theorem 3.1 is also true if $\Theta(x, y, z)$ and $\theta(x, y, z)$ are replaced, respectively, by

$$\begin{aligned} \Theta_1(x, y, z) &= \max\{G(Sx, Sy, Sz), G(Sx, Sx, Tx), G(Sy, Sy, Ty), G(Sz, Sz, Tz), \\ &\quad \frac{1}{2}[G(x, y, Ty) + G(y, x, Tx)], \frac{1}{2}[G(y, z, Tz) + G(z, y, Ty)], \frac{1}{2}[G(x, z, Tz) + G(z, x, Tx)], \\ &\quad \frac{1}{3}[G(x, y, Ty) + G(y, z, Tz) + G(z, x, Tx)]\} \end{aligned}$$

and

$$\theta_1(x, y, z) = \max\{G(Sx, Sy, Sz), G(Sx, Sx, Tx), G(Sy, Sy, Ty), G(Sz, Sz, Tz)\},$$

and condition (1) by

$$\psi(G(Tx, Ty, Tz)) \leq \psi(\Theta_1(x, y, z)) - \varphi(\theta_1(x, y, z)).$$

We demonstrate the usage of Theorem 3.1 by the following

Example 3.4. Let $X = \{0, 1, 2\}$ and $G : X^3 \rightarrow \mathbb{R}^+$ be given as

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ 1, & \text{if } (x, y, z) \in \{(0, 0, 1), (0, 1, 1), (0, 2, 2), (1, 2, 2)\}, \\ 2, & \text{if } (x, y, z) \in \{(0, 0, 2), (1, 1, 2), (0, 1, 2)\}, \end{cases}$$

and extended by symmetry. Then it is easy to check that X is a G -metric space which is asymmetric since, e.g., $G(0, 0, 2) \neq G(0, 2, 2)$. Define an order relation \leq by $x \leq y \Leftrightarrow x \leq y$ and mappings $S, T : X \rightarrow X$ and functions $\psi, \varphi : [0, +\infty) \rightarrow [0, \infty)$ by

$$S : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \quad T : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \psi(t) = 2t, \quad \varphi(t) = \frac{t}{2}.$$

Then T is S -non-decreasing, $TX \subset SX$, and the space (X, G, \leq) is regular. In order to check the contractive condition (1), consider the following possible cases.

1. If $x = y = z$ or $x, y, z \in \{0, 1\}$, then $G(Tx, Ty, Tz) = 0$ and inequality (1) is trivial.

In all other cases $G(Tx, Ty, Tz) = 1$ and $\psi(G(Tx, Ty, Tz)) = 2$.

2. If $(x, y, z) \in \{(0, 2, 2), (1, 2, 2), \dots\}$ (\dots stays for permutations), then at least one of x, y, z is equal to 2. Let, e.g., $y = 2$. Then $G(Sy, Ty, Ty) = G(2, 1, 1) = 2$ and hence $\Theta(x, y, z) = \theta(x, y, z) = 2$. Thus, the right-hand side of (1) reduces to $\psi(2) - \varphi(2) = 4 - 1 = 3$ and the inequality holds.

3. If $(x, y, z) \in \{(0, 0, 2), (1, 1, 2), (0, 1, 2), \dots\}$, then $G(Sx, Sy, Sz) = 2$, and again $\Theta(x, y, z) = \theta(x, y, z) = 2$. Thus, (1) is again satisfied.

All the assumptions of Theorem 3.1 are fulfilled and S and T have a coincidence point (equal to 0).

Putting $S = i_X$ (the identity map) in Theorem 3.1, we obtain the following corollary.

Corollary 3.5. *Let (X, G, \leq) be a complete ordered G -metric space. Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous function with $\varphi(t) = 0$ if and only if $t = 0$ and let ψ be an altering distance function. Suppose that $T : X \rightarrow X$ is a non-decreasing map, such that*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(\Theta(x, y, z)) - \varphi(\theta(x, y, z)), \tag{13}$$

where

$$\begin{aligned} \Theta(x, y, z) = \max \{ & G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ & \frac{1}{2}[G(x, Ty, Ty) + G(y, Tx, Tx)], \frac{1}{2}[G(y, Tz, Tz) + G(z, Ty, Ty)], \frac{1}{2}[G(x, Tz, Tz) + G(z, Tx, Tx)], \\ & \frac{1}{3}[G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)] \}, \end{aligned}$$

and

$$\theta(x, y, z) = \max \{ G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \}$$

for all $x, y, z \in X$ with $z \leq y \leq x$. In addition, we assume that

(i) T is continuous, or

(ii) X is regular.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point; that is, there exists $z \in X$ such that $Tz = z$.

We furnish the following example to demonstrate the validity of the hypotheses of Corollary 3.5. It also shows that using the order can be crucial.

Example 3.6. Let $X = \{2, 3, 4\}$ and $G : X \times X \times X \rightarrow \mathbb{R}^+$ be defined as follows:

$$G(x, y, z) = \begin{cases} x + y + z, & \text{if } x, y, z \text{ are all distinct} \\ x + z, & \text{if } x = y \neq z \\ 0, & \text{if } x = y = z. \end{cases}$$

Then (X, G) is a complete G -metric space. Let a partial order \leq on X be defined as follows:

$$\leq := \{(2, 2), (3, 3), (4, 4), (4, 2)\}.$$

The topology of (X, G) is discrete, hence all convergent sequences in (X, G) are eventually constant. Thus, the space (X, \leq, G) is regular.

Consider the mapping $T : X \rightarrow X$ defined by

$$T = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 4 & 2 \end{pmatrix},$$

and define functions $\psi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ by

$$\psi(t) = t \text{ and } \varphi(t) = \frac{t}{2}, \text{ for all } t \geq 0.$$

Clearly ψ and φ are altering distance functions. The only nontrivial cases when $z \leq y \leq x$ are $z = 4, x = y = 2$ and $z = y = 4, x = 2$. In both cases the left-hand side of (13) is equal to 0 and the condition is satisfied. Moreover, for $x_0 = 4, x_0 \leq Tx_0$ holds true. Hence, Corollary 3.5 can be applied to conclude that T has a fixed point (which is $z = 2$).

On the other hand, if we consider the same example in the G -metric space (X, G) without order, then the respective conclusion may not be obtained. Indeed, take $x = 2, y = 3, z = 4$. Then $G(Tx, Ty, Tz) = G(2, 4, 2) = 6$ and $\Theta(2, 3, 4) = \theta(2, 3, 4) = 9$, but

$$\psi(6) = 6 > 9 - \frac{9}{2} = \psi(9) - \varphi(9),$$

and the condition (13) is not satisfied.

In a special case when the order in (X, G, \leq) is total (which is equivalent to the case when there is no order and the contractive condition holds for all elements of the space), conclusions of Theorem 3.1 can be improved.

Theorem 3.7. *Under the hypotheses of Theorem 3.1 (case (ii)), and assuming that the order \leq is total, the mappings T and S have a unique point of coincidence, that is, there exists a unique $w \in X$ such that $Sz = Tz = w$ for some $z \in X$. In particular, if T is injective, they have also a unique coincidence point. If, moreover, T and S are weakly compatible, then T and S have a unique common fixed point.*

Proof. The existence of points $z, w \in X$ satisfying $Tz = Sz = w$ was proved in Theorem 3.1, case (ii). Suppose that there exists another point $w_1 \in X, w_1 \neq w$ such that $Tz_1 = Sz_1 = w_1$ for some $z_1 \in X$. Without loss of generality, assume that $G(w_1, w, w) \geq G(w, w_1, w_1)$. Then

$$\begin{aligned} \Theta(z_1, z, z) &= \max \left\{ G(Sz_1, Sz, Sz), G(Sz_1, Tz_1, Tz_1), G(Sz, Tz, Tz), G(Sz, Tz, Tz) \right. \\ &\quad \frac{1}{2} [G(Sz_1, Tz, Tz) + G(Sz, Tz_1, Tz_1)], \frac{1}{2} [G(Sz, Tz, Tz) + G(Sz, Tz, Tz)] \\ &\quad \frac{1}{2} [G(Sz, Tz_1, Tz_1) + G(Sz_1, Tz, Tz)] \\ &\quad \left. \frac{1}{3} [G(Sz_1, Tz, Tz) + G(Sz, Tz, Tz) + G(Sz, Tz_1, Tz_1)] \right\} \\ &= \max \left\{ G(w_1, w, w), \frac{1}{2} [G(w_1, w, w) + G(w, w_1, w_1)], \frac{1}{3} [G(w_1, w, w) + G(w, w_1, w_1)] \right\} \\ &= G(w_1, w, w) = \theta(z_1, z, z). \end{aligned}$$

Applying condition (1) to points z_1, z, z , we get that

$$\psi(G(w_1, w, w)) = \psi(G(Tz_1, Tz, Tz)) \leq \psi(G(w_1, w, w)) - \varphi(G(w_1, w, w)),$$

hence $G(w_1, w, w) = 0$ and $w_1 = w$, a contradiction. Thus, the point of coincidence w of S and T is unique. If T is injective, then also $z = z_1$ and the coincidence point is also unique.

If these two mappings are weakly compatible, it follows that they have a unique common fixed point by a well known result of Jungck. \square

We present a very simple example demonstrating some points of the previous theorem. A bit more complicated example with similar properties can be constructed using [8, Example 2.2].

Example 3.8. Consider the (asymmetric) G -metric space given in Example 2.7(2).

(a) If $S_1, T_1 : X \rightarrow X$ are defined by $S_1 : \begin{pmatrix} a & b \\ a & b \end{pmatrix}, T_1 : \begin{pmatrix} a & b \\ a & a \end{pmatrix}$, then the contractive condition (1) is satisfied (the left-hand side is always equal to 0) and S_1 and T_1 have a unique point of coincidence ($a = S_1a = T_1a$). Moreover, S_1 and T_1 are weakly compatible (since $S_1T_1a = T_1S_1a = a$) and they have a unique common fixed point.

(b) Let now $S_2, T_2 : X \rightarrow X$ be defined by $S_2 : \begin{pmatrix} a & b \\ b & a \end{pmatrix}, T_2 : \begin{pmatrix} a & b \\ a & a \end{pmatrix}$. Then again condition (1) is satisfied and S_2 and T_2 have a unique point of coincidence ($a = S_2b = T_2b$). However, these mappings are not weakly compatible ($T_2S_2b = a \neq b = S_2T_2b$) and they have no common fixed points.

In the special case when $S = i_X$, Theorem 3.7 reduces to

Corollary 3.9. Under the hypotheses of Corollary 3.5 (case (ii)), and assuming that the order \leq is total, the mapping T has a unique fixed point.

As an application, we state a corollary for mappings satisfying conditions of integral type.

Denote by M the set of functions $\mu : [0, +\infty) \rightarrow [0, +\infty)$ satisfying conditions:

- (1) μ is Lebesgue-integrable on each compact subset of $[0, +\infty)$;
- (2) for every $\varepsilon > 0$, $\int_0^\varepsilon \mu(t) dt > 0$.

Corollary 3.10. Let (X, G, \leq) be a complete ordered G -metric space, and let the mappings $S, T : X \rightarrow X$ satisfy all the hypotheses of Theorem 3.1, except that the contractive condition (1) is substituted by

$$\int_0^{\psi(G(Sx, Sy, Sz))} \mu_1(t) dt \leq \int_0^{\psi(\Theta(x, y, z))} \mu_1(t) dt - \int_0^{\varphi(\theta(x, y, z))} \mu_2(t) dt,$$

for some functions $\mu_1, \mu_2 \in M$, and all $x, y, z \in X$ such that $Sz \leq Sy \leq Sx$. Then S and T have a coincidence point. In the case that the order \leq is total, the respective point of coincidence is unique and, if S and T are weakly compatible, they have a unique common fixed point.

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