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A new characterization of markets that don't replicate any option through minimal-lattice subspaces. A computational approach.

Vasilios N. Katsikis[a]

[a]*Department of Mathematics, Technological Education Institute of Piraeus, 12244, Athens, Greece*

Abstract. In this paper the notion of strongly resolving markets with respect to the positive basis of a minimal lattice-subspace Y of $\mathbb{R}^m$ is defined. It is proved that if the number of securities is less than half the dimension of Y, then not a single (non-trivial) option can be replicated. This result extends already known results regarding the notion of a market being strongly resolving. Both theoretical and computational methods are provided in order to establish criteria for the characterization of markets that do not replicate any option.

1. Introduction

In a seminal study, [12], Ross shows that if security markets are resolving then there exist non-redundant options that generate complete security markets. This result poses the following natural question: *Can we ever replicate an option if markets are not complete?* Complementing the work of Ross, the authors in [1] gave a characterization of markets that do not replicate any option. In particular, they show that if security markets are strongly resolving and the number of primitive securities is less than half the number of states, then every option is non-redundant, i.e., not a single (non-trivial) option can be replicated. The replication of options in strongly resolving markets has been studied in [1], [2] and [11]. In [1] the authors defined the notion of strongly resolving markets by considering the payoff matrix with respect to the standard basis of $\mathbb{R}^m$ while in [11], a generalization of the previous definition was presented by taking the payoff matrix with respect to the positive basis of $F_1(X)$, where $F_{\mathbf{1}}(X)$ denotes the completion of X by options i.e., the subspace of $\mathbb{R}^m$ generated by all options written on the elements of $X \cup \{\mathbf{1}\}$. On the other hand, in [2], the result presented in [1] is extended to the case when the condition on the number of primitive securities is not imposed. Since any replicated option can be priced directly, considered as a portfolio of primitive securities it is evident that the replication of options is one of the most important problems in finance.

In this article, we extend the definition of strongly resolving markets by taking the payoff matrix with respect to the positive basis of a minimal lattice-subspace Y, generated by the $x_1, x_2, ..., x_n$ non-redundant securities. In addition, we present a new characterization of markets that do not replicate any option. Our main result states that if the number of securities is less than half the dimension of Y, then not a single (non-trivial) option can be replicated. Besides the theoretical approach, we provide computational methods in order to verify if a market is strongly resolving. To this end, we combine previous knowledge developed

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Email address: vaskats@gmail.com (Vasilios N. Katsikis)

in [3–10] together with some new theoretical and computational ideas. Note that, the vector lattice theory founds great applications in the theory of options since the different kind of options are expressed through lattice operations.

The material in this article is spread out in 6 sections. In section 2, the fundamental properties of lattice-subspaces and vector sublattices are presented. Moreover, we discuss the basic results for vector sublattices, lattice-subspaces and positive bases of $\mathbb{R}^m$ together with the solution to the problem of whether a finite collection of linearly independent, positive vectors of $\mathbb{R}^m$ generates a lattice-subspace or a vector sublattice. In section 3, we discuss the theoretical background for option replication. Also, section 3 emphasis the most important interrelationship between positive bases and the problem of option replication. In section 4, we present three different notions of strongly resolving markets and we prove our main result, theorem 4.4. Section 5 is divided in two subsections; in the first we construct a new Matlab function for verifying if a market is strongly resolving (for each one of the three definitions), whereas in the second we discuss the use of the proposed Matlab function together with important numerical examples. Moreover, subsection 5.2 concludes with three open questions regarding the aforementioned different kinds of strongly resolving markets. Conclusions are provided in section 6.

2. Preliminaries

Let $\mathbb{R}^m = \{x = (x(1), x(2), ..., x(m)) | x(i) \in \mathbb{R}, \textit{ for each } i\}$, where we view $\mathbb{R}^m$ as an ordered space. The *pointwise order* relation in $\mathbb{R}^m$ is defined by

$$x \leq y \textit{ if and only if } x(i) \leq y(i), \textit{ for each } i = 1, ..., m.$$

The positive cone of $\mathbb{R}^m$ is defined by $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x(i) \geq 0, \textit{ for each } i\}$ and if we suppose that X is a vector subspace of $\mathbb{R}^m$ then X ordered by the pointwise ordering is an *ordered subspace* of $\mathbb{R}^m$, with positive cone $X_+ = X \cap \mathbb{R}^m_+$. By $\{e_1, e_2, ..., e_m\}$ we shall denote the standard basis of $\mathbb{R}^m$. A point $x \in \mathbb{R}^m$ is an *upper bound (resp. lower bound)* of a subset $S \subseteq \mathbb{R}^m$ if and only if $y \leq x$ (resp. $x \leq y$), for all $y \in S$. For a two-point set $S = \{x, y\}$, we denote by $x \vee y$ (resp. $x \wedge y$) the *supremum* of S i.e., its least upper bound (resp. the *infimum* of S i.e., its greatest lower bound). Thus, $x \vee y$ (resp. $x \wedge y$) is the componentwise maximum (resp. minimum) of x and y defined by

$$(x \vee y)(i) = \max\{x(i), y(i)\} \; ((x \wedge y)(i) = \min\{x(i), y(i)\}), \; \textit{for all } i = 1, ..., m.$$

For any $x = (x(1), x(2), ..., x(m)) \in \mathbb{R}^m$, the set $supp(x) = \{i | x(i) \neq 0\}$ is the *support* of x. We say that the vectors $x, y \in \mathbb{R}^m$ have *disjoint supports* if $supp(x) \cap supp(y) = \emptyset$.

An ordered subspace X of $\mathbb{R}^k$ is a *lattice-subspace* of $\mathbb{R}^k$ if it is a vector lattice in the induced ordering, i.e., for any two vectors $x, y \in X$ the supremum and the infimum of $\{x, y\}$ both exist in X. Note that the supremum and the infimum of the set $\{x, y\}$ are, in general, different in the subspace from the supremum and the infimum of this set in the initial space. An ordered subspace Z of $\mathbb{R}^m$ is a *vector sublattice* or a *Riesz subspace* of $\mathbb{R}^m$ if for any $x, y \in Z$ the supremum and the infimum of the set $\{x, y\}$ in $\mathbb{R}^m$ belong to Z.

Assume that X is an ordered subspace of $\mathbb{R}^m$ and $B = \{b_1, b_2, ..., b_\mu\}$ is a basis for X. Then B is a *positive basis* of X if for each $x \in X$ it holds that x is positive if and only if its coefficients in the basis B are positive. In other words, B is a positive basis of X if the positive cone X_+ of X has the form,

$$X_+ = \left\{ x = \sum_{i=1}^{\mu} \lambda_i b_i \mid \lambda_i \geq 0, \textit{ for each } i \right\}.$$

Then, for any $x = \sum_{i=1}^{\mu} \lambda_i b_i$ and $y = \sum_{i=1}^{\mu} \varrho_i b_i$ we have $x \leq y$ if and only if $\lambda_i \leq \varrho_i$ for each $i = 1, 2, ..., \mu$. A positive basis $B = \{b_1, b_2, ..., b_\mu\}$ is a *partition of the unit* if the vectors b_i have disjoint supports and $\sum_{i=1}^{\mu} b_i = (1, 1, ..., 1)$.

Recall that a nonzero element x_0 of X_+ is an extremal point of X_+ if, for any $x \in X, 0 \leq x \leq x_0$ implies $x = \lambda x_0$, for a real number λ. Since each element b_i of the positive basis of X is an extremal point of X_+, a positive basis of X is unique in the sense of positive multiples. The existence of positive bases is not always ensured, but in the case where X is a vector sublattice of $\mathbb{R}^m$ then X always has a positive basis. Moreover, it holds that an ordered subspace of $\mathbb{R}^k$ has a positive basis if and only if it is a lattice-subspace of $\mathbb{R}^k$. If $B = \{b_1, b_2, ..., b_n\}$ is a positive basis for a lattice-subspace (or a vector sublattice) X then the lattice operations in X, namely $x \nabla y$ for the supremum and $x \vartriangle y$ for the infimum of the set $\{x, y\}$ in X, are given by

$$x \nabla y = \sum_{i=1}^{n} \max\{\lambda_i, \mu_i\} b_i \quad \textit{and} \quad x \vartriangle y = \sum_{i=1}^{n} \min\{\lambda_i, \mu_i\} b_i,$$

for each $x = \sum\limits_{i=1}^{n} \lambda_i b_i$, $y = \sum\limits_{i=1}^{n} \mu_i b_i \in X$. A vector sublattice is always a lattice-subspace, but the converse is not true. Let $A \subseteq \mathbb{R}^m_+$, $A \neq \emptyset$ and S be the set of lattice-subspaces of $\mathbb{R}^m$ each of which contains A. If $B \in S$ and for any $C \in S$ it holds $C \subseteq B \Rightarrow C = B$, then we say that B is a *minimal lattice-subspace* of $\mathbb{R}^m$ containing A.

The function

$$\beta : \{1, 2, ..., m\} \to \mathbb{R}^n \textit{ such that } \beta(i) = \frac{1}{\sum_{j=1}^{n} |x_j(i)|}(x_1(i), x_2(i), ..., x_n(i))$$

for each $i \in \{1, 2, ..., m\}$ with $\sum_{j=1}^{n} |x_j(i)| \neq 0$ is the *basic function* of the vectors $x_1, x_2, ..., x_n$. The set

$$R(\beta) = \left\{ \beta(i) \mid i = 1, 2, ..., m, \textit{ with } \sum_{j=1}^{n} |x_j(i)| \neq 0 \right\},$$

is the *range* of the basic function and the *cardinal number*, *cardR*(β), of $R(\beta)$ is the number of different elements of $R(\beta)$. Also, $D(\beta)$ denotes the domain of β. If *cardR*$(\beta) = \mu$ then $n \leq \mu \leq m$. We shall denote by K the convex hull of $R(\beta)$ which is, as the convex hull of a finite subset of $\mathbb{R}^m$, a polytope with d vertices and each vertex of K belongs to $R(\beta)$. It is clear that $n \leq d \leq \mu$. We enumerate the range of the basic function as follows, $R(\beta) = \{P_1, P_2, ..., P_\mu\}$ such that the first n vertices $P_1, P_2, ..., P_n$ are linearly independent and $P_1, P_2, ..., P_d$ are the vertices of K. The following theorem is important for our study. We shall present it in a suitable form for our analysis, as in [8].

Theorem 2.1. *[8, Theorem 2]. Suppose that the above assumptions are satisfied. Then,*

(*i*) *X is a vector sublattice of $\mathbb{R}^m$ if and only if $R(\beta)$ has exactly n points (i.e., $\mu = n$). In such a case, a positive basis $b_1, b_2, ..., b_n$ for X is defined by the formula*

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = U^{-1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where U is the $n \times n$ matrix whose ith column is the vector P_i, for each $i = 1, 2, ..., n$.

(*ii*) *Let $\mu > n$. If $I_s = \beta^{-1}(P_s)$, and*

$$x_s = \sum_{i \in I_s} \sum_{j=1}^{n} |x_j(i)| e_i, \ \ s = n+1, n+2, ..., \mu,$$

then

$$Z = [x_1, x_2, ..., x_n, x_{n+1}, ..., x_\mu]$$

is the vector sublattice generated by $x_1, x_2, ..., x_n$ and $\dim Z = \mu$.

Consider the basic function γ of the vectors $x_1, x_2, ..., x_\mu$ with range,

$$R(\gamma) = \{P'_1, P'_2, ..., P'_\mu\}.$$

Then, the relation

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_\mu \end{bmatrix} = V^{-1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\mu \end{bmatrix},$$

where V is the $\mu \times \mu$ matrix with the vectors $P'_1, P'_2, ..., P'_\mu$ specified as columns, defines a positive basis for Z.

(*iv*) *Let $d > n$. If $\xi_i : D(\beta) \to \mathbb{R}_+, i = 1, 2, \ldots, d$ such that $\sum_{i=1}^{d} \xi_i(j) = 1$ and $\beta(j) = \sum_{i=1}^{d} \xi_i(j) P_i$ for each $j \in D(\beta)$, and $x_{n+i}, i = 1, 2, \ldots, d-n$, are the following vectors of $\mathbb{R}^m$:*

$$x_{n+i} = \sum_{j \in D(\beta)} \xi_{n+i}(j) \sum_{j=1}^{n} |x_j(i)| e_j,$$

then

$$Y = [x_1, \ldots, x_n, x_{n+1}, \ldots, x_d]$$

is a minimal lattice-subspace of $\mathbb{R}^m$ containing $x_1, x_2, \ldots, x_n$ and $\dim Y = d$.

In [5, 8] algorithmic procedures as well as computational methods are provided for the calculation of the vector sublattice and the minimal lattice-subspace generated by a finite set of positive vectors of $\mathbb{R}^k$.

In particular, let $X = [x_1, x_2, ..., x_n]$ be the vector subspace generated by the linearly independent, positive vectors $x_1, x_2, ..., x_n$ of $\mathbb{R}^m$. If X is a lattice-subspace or a vector sublattice of $\mathbb{R}^m$ a computational method that determines a positive basis in X is provided in [5]. In the opposite case, the computational method presented in [8], provides a minimal lattice-subspace and a vector sublattice containing X as well as their corresponding positive bases. In addition, in [5, 8] the interconnection between the aforementioned computational methods with problems arising in mathematical economics is further analyzed.

For computational methods in positive bases theory with applications in economics we refer to [3–9].

3. The economic model-Options replication

In our economy there are two time periods, $t = 0, 1$, where $t = 0$ denotes the present and $t = 1$ denotes the future. We consider that at $t = 1$ we have a finite number of states indexed by $s = 1, 2, ..., m$, while at $t = 0$ the state is known to be $s = 0$.

Suppose that, agents trade $x_1, x_2, ..., x_n$ non-redundant (linearly independent) securities in period $t = 0$, future payoffs of $x_1, x_2, ..., x_n$ are collected in a matrix

$$A = \left[x_i(j)\right]_{i=1,2,...,n}^{j=1,2,...,m} \in \mathbb{R}^{m \times n}$$

where $x_i(j)$ is the payoff of one unit of security i in state j. In other words, A is the matrix whose columns are the non-redundant security vectors $x_1, x_2, ..., x_n$. It is clear that the matrix A is of full rank and the *asset span* is denoted by $X = Span(A)$. So, X is the vector subspace of $\mathbb{R}^m$ generated by the vectors x_i. That is, X consists of those income streams that can be generated by trading on the financial market. A *portfolio* is a column vector $\theta = (\theta_1, \theta_2, ..., \theta_n)^T$ of $\mathbb{R}^n$ and the *payoff* of a portfolio θ is the vector $x = A\theta \in \mathbb{R}^m$, which offers payoff $x(i)$ in state i, where $i = 1, ..., m$. A vector in $\mathbb{R}^m$ is said to be *marketed* or *replicated* if x is the payoff of some portfolio θ (called the *replicating portfolio* of x), or equivalently if $x \in X$. If $m = n$, then markets are said to be *complete* and the asset span coincides with the space $\mathbb{R}^m$. On the other hand, if $n < m$, the markets are *incomplete* and some state contingent claim cannot be replicated by a portfolio. Recall that a two-period security market is said to be *resolving* if the collection of securities $x_1, x_2, ..., x_n$ is resolving; in the sense that for any two distinct states j_1 and j_2 there is some security x_i such that $x_i(j_1) \neq x_i(j_2)$. Also, a two-period security market is *strongly resolving* if for any choice of n states and any contingent claim x there is a unique portfolio whose payoff coincides with x on the n selected states, i.e. any $n \times n$ square submatrix of A is non-singular. If a two-period security market is strongly resolving, then it is also resolving. As it is noted in [1] and [2], it is easy to see that the set of security markets that are not strongly resolving is small, therefore the condition that a security market is strongly resolving is not particularly restrictive. Note that the matrix $A = \left[x_i(j)\right]_{i=1,2,...,n}^{j=1,2,...,m}$ is considered regarding the standard basis $\{e_1, ..., e_m\}$ of $\mathbb{R}^m$. Let $\{b_1, ..., b_k\}$ be a positive basis of a vector sublattice or a lattice-subspace of $\mathbb{R}^m$ containing X and $x_i = \sum\limits_{j=1}^{k} x_i^b(j) b_j$ is the expansion of the security $x_i,\ i = 1, ..., n$ in terms of the positive basis $\{b_1, ..., b_k\}$. Then the matrix

$$A^b = \left[x_i^b(j)\right]_{i=1,2,...,n}^{j=1,2,...,k} \in \mathbb{R}^{k \times n}$$

is the *payoff matrix* of vectors x_i with respect to the basis $\{b_i\}$.

Definition 3.1. *A two-period security market is strongly resolving with respect to the (positive) basis $\{b_i\}$ if any $n \times n$ square submatrix of A^b is non-singular.*

In the following, we shall denote by $\mathbf{1}$ the *riskless (or risk-free) bond* i.e., the vector $\mathbf{1} = (1, 1, ..., 1)$. The *call option* written on the vector $x \in \mathbb{R}^m$ with exercise price α is the vector $c(x, a) = (x - \alpha\mathbf{1})^+ = (x - \alpha\mathbf{1}) \vee \mathbf{0}$, where $\mathbf{0} = (0, 0, ..., 0)$. The *put option* written on the vector $x \in \mathbb{R}^m$ with exercise price α is the vector $p(x, a) = (\alpha\mathbf{1} - x)^+ = (\alpha\mathbf{1} - x) \vee \mathbf{0}$. If y is an element of a Riesz space then the following lattice identities hold, $y = y^+ - y^-$ and $y^- = (-y)^+$. It is clear that $x - \alpha\mathbf{1} = (x - \alpha\mathbf{1})^+ - (x - \alpha\mathbf{1})^- = (x - \alpha\mathbf{1})^+ - (\alpha\mathbf{1} - x)^+ = c(x, \alpha) - p(x, \alpha)$. Therefore we have the identity

$$x - a\mathbf{1} = c(x, a) - p(x, a),$$

which is called *put-call parity*. In economic terms, the put-call parity states that a call option on a portfolio x with a given exercise price a is redundant to a put option on x with the same exercise price a, to a riskless bond and a portfolio x.

If both $c(x, \alpha) > 0$ and $p(x, \alpha) > 0$, we say that the call option $c(x, \alpha)$ and the put option $p(x, \alpha)$ are *non trivial* and the exercise price α is a *non trivial exercise price* of x. If $c(x, \alpha)$ and $p(x, \alpha)$ belong to X then we say that $c(x, \alpha)$ and $p(x, \alpha)$ are *replicated*. If we suppose that $\mathbf{1} \in X$ and at least one of $c(x, a)$, $p(x, a)$ is replicated, then both of them are replicated since, $x - \alpha\mathbf{1} = c(x, \alpha) - p(x, \alpha)$. For notation not defined here the interested reader may refer to [7, 11] and the references therein.

Suppose that a security market X is generated by a given collection of linearly independent vectors $x_1, x_2, ..., x_n$ of $\mathbb{R}^m$. In the theory of security markets it is a usual practice to take call and put options with respect to the riskless bond $\mathbf{1} = (1, 1, ..., 1)$. The completion, $F_{\mathbf{1}}(X)$, of X by options is the subspace of $\mathbb{R}^m$ generated by all options written on the elements of $X \cup \{\mathbf{1}\}$. Since the payoff space is $\mathbb{R}^m$, which is a vector lattice, in the case where $\mathbf{1} \in X$ then $F_{\mathbf{1}}(X)$ is exactly the vector sublattice generated by X. In addition, if X is a vector sublattice of $\mathbb{R}^m$ then $F_{\mathbf{1}}(X) = X$ therefore any option is replicated. Note that the vectors $x_1, x_2, ..., x_n$ are not presupposed to be positive. In addition, since $F_{\mathbf{1}}(X)$ is a vector sublattice it has a positive basis B which is a partition of the unit, i.e., $B = \{b_1, b_2, ..., b_\mu\}$ is a positive basis where b_i have disjoint supports and $\sum_{i=1}^{\mu} b_i = \mathbf{1}$.

4. Strongly resolving markets

In [12], Ross shows that if security markets are resolving then there exist non-redundant options that generate complete security markets. Complementing the work of Ross, the authors in [1] gave a characterization of markets that do not replicate any option by showing that if security markets are strongly resolving and the number of primitive securities is less than half the number of states, then every option is non-redundant, i.e., not a single (non-trivial) option can be replicated.

The replication of options in strongly resolving markets has been studied in [1], and [11]. In [1] the authors defined the notion of strongly resolving markets by considering the payoff matrix with respect to the standard basis of $\mathbb{R}^m$ while in [11], a generalization of the previous definition was presented by taking the payoff matrix with respect to the positive basis of $F_1(X)$. In this article, we extend the definition of strongly resolving markets by taking the payoff matrix with respect to the positive basis of a minimal lattice-subspace generated by the $x_1, x_2, ..., x_n$ non-redundant securities. Moreover, we present a new characterization of markets that do not replicate any option.

Now, as before, suppose that agents trade $x_1, x_2, ..., x_n$ non-redundant securities. If $X = [x_1, x_2, ..., x_n]$ then according to (*iv*) from Theorem 2.1 we construct the subspace $Y = [x_1, \ldots, x_n, x_{n+1}, \ldots, x_d]$ which is a minimal lattice-subspace of $\mathbb{R}^m$ containing $x_1, x_2, \ldots, x_n$ and $\dim Y = d$. Suppose that $\{\beta_1, \beta_2, ..., \beta_d\}$ is a positive basis for Y. We expand the vectors $x_1, x_2, ..., x_n$ in terms of the basis $\{\beta_1, \beta_2, ..., \beta_d\}$ so $x_i = \sum_{j=1}^{d} x_i^{\beta}(j)\beta_j$, for each $i = 1, ..., n$. We collect the future payoffs of $x_1, x_2, ..., x_n$ in the following matrix

$$A^{\beta} = \left[x_i^{\beta}(j)\right]_{i=1,2,...,n}^{j=1,2,...,d} \in \mathbb{R}^{d\times n}$$

where $x_i^{\beta}(j)$ is the payoff of one unit of security i in state j. Recall that, a two-period security market is strongly resolving with respect to the basis $\{\beta_i\}$ if any $n \times n$ square submatrix of A^{β} is non-singular. In the next example we shall see that it is possible for a market to be strongly resolving with respect to the positive basis of the minimal lattice-subspace Y but is neither strongly resolving nor strongly resolving with respect to the positive basis of $F_1(X)$. Throughout the paper we shall preserve the notation and terminology presented so far.

Example 4.1. Consider four vectors x_1, x_2, x_3, x_4 in $\mathbb{R}^6$, with

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

and $X = [x_1, x_2, x_3, x_4]$ is the marketed space. Note that $\mathbf{1} = \dfrac{\mathbf{x_3 + x_4}}{\mathbf{3}}$, so the riskless-bond is marketed. It is clear that X is not strongly resolving since the fourth row is equal to the sixth row, i.e., there exists a 4×4 singular submatrix. We use the computational methods presented in [7, 8], so we have that the completion by options, $F_1(X)$, is the subspace of $\mathbb{R}^6$ generated by the vectors

$$y_1 = (1, 1, 1, 1, 2, 1),\ y_2 = (2, 3, 1, 1, 1, 1),\ y_3 = (2, 2, 2, 1, 3, 1), y_4 = (1, 1, 1, 2, 0, 2),\ y_5 = (0, 0, 5, 0, 0, 0).$$

A positive basis (which is also a partition of the unit) of $F_1(X)$ consists of the following five vectors of $\mathbb{R}^6$:

$$b_1 = (1, 0, 0, 0, 0, 0),\ b_2 = (0, 1, 0, 0, 0, 0), b_3 = (0, 0, 0, 1, 0, 1), b_4 = (0, 0, 0, 0, 1, 0), b_5 = (0, 0, 1, 0, 0, 0).$$

Then the matrix

$$A^{b} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

is the payoff matrix of vectors $x_1, ..., x_4$ with respect to the basis $\{b_i\}$. Note that the following 4×4 submatrix of A^b, is singular:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

Therefore X is not strongly resolving with respect to the positive basis of $F_1(X)$.

As before, we use the computational methods presented in [8] in order to construct the minimal lattice-subspace Y generated by x_1, x_2, x_3, x_4. A positive basis for Y consists of the following four vectors:

$$\beta_1 = \left(\frac{7}{2}, 7, 0, 0, 0, 0\right),\ \beta_2 = (0, 0, 0, 5, 0, 5),\ \beta_3 = \left(\frac{5}{2}, 0, 5, 0, 0, 0\right),\ \beta_4 = (0, 0, 0, 0, 6, 0).$$

Note that, the positive basis for Y is not a partition of the unit. Then the matrix

$$A^\beta = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} & \frac{2}{7} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \end{bmatrix}$$

is the payoff matrix of vectors $x_1, ..., x_4$ with respect to the basis $\{\beta_i\}$. Since A^β is non-singular we have that X is strongly resolving with respect to the positive basis of the minimal lattice-subspace Y.

In what follows we shall say that:

- X has the SR-property, if X is strongly resolving.
- X has the SR1-property, if X is strongly resolving with respect to the positive basis of $F_{\mathbf{1}}(X)$.
- X has the SR2-property, if X is strongly resolving with respect to the positive basis of Y.

Remark 4.2. *From [11], theorem 5.2, we have that if $n \geq 2$ and $F_1(X)$ is a proper subspace of $\mathbb{R}^m$, then the market is not strongly resolving. The following example proves that if Y is a proper subspace of $\mathbb{R}^m$, then we cannot say that the market is not strongly resolving. Therefore, theorem 5.2 from [11] does not hold if we replace $F_1(X)$ with Y.*

Example 4.3. Consider the following three vectors x_1, x_2, x_3 in $\mathbb{R}^5$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 5 & 6 & 3 & 2 & 1 \\ 8 & 10 & 11 & 12 & 5 \end{bmatrix}$$

and $X = [x_1, x_2, x_3]$ is the marketed space.

Then, X has the SR-property, the SR1-property and the SR3-property. Moreover, Y is a four dimensional subspace of $\mathbb{R}^5$.

In view of the previous examples and discussion, it is clear that our definition of a market being strongly resolving with respect to the positive basis of the minimal lattice-subspace Y differs from the definition of strongly resolving from [1] and the definition of strongly resolving with respect to the positive basis of $F_1(X)$ from [11]. We note also that, in our definition of strongly resolving markets we do not presuppose that the riskless-bond is marketed, while in the definition of strongly resolving markets presented in [11], the authors made the additional assumption that $\mathbf{1} \in X$ so that the positive basis $\{b_i\}$ is a partition of the unit.

We are now in a position to state and prove our main result:

Theorem 4.4. *If the market X is strongly resolving with respect to the basis $\{\beta_i\}$ of Y, $\mathbf{1} \in X$ and $n \leq \frac{d+1}{2}$ then any non trivial option written on elements of X is non replicated.*

Proof. For $x \in X$, let $y = c(x, a) = (x - \alpha\mathbf{1})^+$ be a non trivial call option, i.e., $y > 0$ and $z = p(x, a) = (\alpha\mathbf{1} - x)^+$ be a non trivial put option. Suppose that y is replicated. Then since $\mathbf{1} \in X$ we have that z is replicated too. Now, let $y = \sum_{i=1}^{d} k_i\beta_i$ and $z = \sum_{i=1}^{d} k'_i\beta_i$. Then, we define the natural numbers

$$p = card(\{i \mid k_i \neq 0\}),\ q = card(\{i \mid k_i = 0\}),\ p' = card(\{i \mid k'_i \neq 0\}),\ q' = card(\{i \mid k'_i = 0\}).$$

It is clear that $p \leq q'$ and $p' \leq q$ and that $p + q = p' + q' = d$. Suppose that $p \leq q$ then $p + q \leq 2q \Rightarrow \frac{d}{2} \leq q$. On the other hand, if $q \leq p$ then $q + p \leq 2p \Rightarrow \frac{d}{2} \leq p \Rightarrow \frac{d}{2} \leq q'$. By our previous analysis it is clear that at least one of the put option or the call option has a number of zero coordinates in the basis β_i greater or equal to $\frac{d}{2}$. Suppose that $\frac{d}{2} \leq q$. We expand y in terms of the non-redundant securities $x_1, x_2, ..., x_n$ so let $y = \sum_{i=1}^{n} \rho_i x_i$ and each one of the x_i can be expressed in terms of the positive basis of Y, therefore we have

$$y = \rho_1 \sum_{i=1}^{d} x_1^{\beta}(i)\beta_i + ... + \rho_n \sum_{i=1}^{d} x_n^{\beta}(i)\beta_i. \tag{1}$$

Since $y = \sum_{i=1}^{d} k_i\beta_i$ we have the following matrix equality

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_d \end{bmatrix} = \begin{bmatrix} x_1^{\beta}(1) & x_2^{\beta}(1) & \cdots & x_n^{\beta}(1) \\ x_1^{\beta}(2) & x_2^{\beta}(2) & \cdots & x_n^{\beta}(2) \\ \vdots & \vdots & & \vdots \\ x_1^{\beta}(d) & x_2^{\beta}(d) & \ldots & x_n^{\beta}(d) \end{bmatrix} \cdot \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} \tag{2}$$

By our hypothesis we have that $n \leq \frac{d+1}{2} \Rightarrow n \leq q + \frac{1}{2} \Rightarrow n \leq q$. Therefore, at least n coordinates $k_{i_1}, ..., k_{i_n}$ of y in the basis β_i are equal to zero. Then, from equation (2), we have the following homogeneous linear system, with ρ_1, ρ_2, ..., ρ_n specified as unknowns:

$$\begin{bmatrix} x_1^{\beta}(i_1) & x_2^{\beta}(i_1) & \cdots & x_n^{\beta}(i_1) \\ x_1^{\beta}(i_2) & x_2^{\beta}(i_2) & \cdots & x_n^{\beta}(i_2) \\ \vdots & \vdots & & \vdots \\ x_1^{\beta}(i_n) & x_2^{\beta}(i_n) & \ldots & x_n^{\beta}(i_n) \end{bmatrix} \cdot \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By our hypothesis, X is strongly resolving with respect to the basis $\{\beta_i\}$ of Y, so the matrix of the system is non-singular and the system has only the trivial solution. In view of equation (1), this is a contradiction since we assumed that $y > 0$. The case $\frac{d}{2} \leq q'$ is similar. □

5. The computational approach

In this section we shall present a computational method that enables us to verify if a market X has the SR-property, the SR1-property and the SR2-property. In order to reach our goal, we shall combine different methods presented in [5, 7] together with some new code for testing if a market is strongly resolving.

Algorithm 1 Strongly resolving properties test

Require: The matrix X, i.e., the payoff matrix with the non-redundant security vectors $x_1, x_2, ..., x_n$ specified as columns.

1: Check if the market has the SR-property.
2: Determine a basic set of marketed securities.
3: Compute the range of the basic curve.
4: Calculate the vector sublattice $F_1(X)$.
5: Calculate a positive basis for $F_1(X)$ which is a partition of the unit.
6: Expand the primitive securities in terms of the positive basis of $F_1(X)$.
7: Check if the market has the SR-1 property.
8: Calculate a positive basis for Y.
9: Expand the primitive securities in terms of the positive basis of Y.
10: Check if the market has the SR-2 property.
11: Compute the output

5.1. Algorithm for verifying if a market has the strongly resolving properties.

We state the algorithm for the Matlab function `srtest` presented in the Appendix. The `srtest` function is our basic tool for verifying each one of the three strongly resolving properties (SR, SR1, and SR2) described in the previous sections.

5.2. Use of the srtest function and numerical examples

In this section, we present carefully selected examples in order to make clear the interconnection between the three presented notions of strongly resolving markets and the theorem 4.4. Moreover, the following numerical examples are presented in such a way as to illustrate how the `srtest` function operates and how to type the initial information. The user should simply retype in the same spaces the input information of his/her own working problem. Note that for the correct performance of the `srtest` function the presence of the `MINlat` function from [8], is needed. Also, recall that, since in the theory of security markets it is usual practice to take call and put options with respect to the riskless bond $\mathbf{1} = (1, 1, ..., 1)$, we consider X such that $\mathbf{1} \in X$.

Example 5.1. Consider the following three vectors x_1, x_2, x_3 in $\mathbb{R}^8$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and $X = [x_1, x_2, x_3]$ is the marketed space.

In order to check the strongly resolving properties we apply the `srtest` function to the given collection by using the code:

```
>> X = [1 1 2 2 0 0 0 0;0 0 0 0 3 3 4 4;1 1 1 1 1 1 1 1];
>> [Pb_Completion,Pb_Minimal_ls] = srtest(X)
```

the results, then, are as follows:

```
Not strongly resolving market.
Strongly resolving with respect to the pb of F1(X).
Strongly resolving with respect to the pb of Y.

Pb_Completion =
                1  0  0  0
```

```
                1  0  0  0
                0  1  0  0
                0  1  0  0
                0  0  1  0
                0  0  1  0
                0  0  0  1
                0  0  0  1


Pb_Minimal_ls =
                0  0  0  2
                0  0  0  2
                0  0  6  0
                0  0  6  0
                4  0  0  0
                4  0  0  0
                0  5  0  0
                0  5  0  0
```

Therefore, X has the SR1-property and the SR2-property.

Example 5.2. Consider the following four vectors x_1, x_2, x_3, x_4 in $\mathbb{R}^6$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 3 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 \end{bmatrix}$$

and $X = [x_1, x_2, x_3, x_4]$ is the marketed space.

Note that $\mathbf{1} = \dfrac{x_3 + x_4}{3}$. We apply the `srtest` function to the given collection by using the code:

```
>> X = [1 2 2 1;1 3 2 1;1 1 2 1;1 1 1 2;2 1 3 0;1 1 1 2];
>> [Pb_Completion,Pb_Minimal_ls] = srtest(X)
```

the results, then, are as follows:

```
Not strongly resolving market.
Not strongly resolving with respect to the pb of F1(X).
Strongly resolving with respect to the pb of Y.

Pb_Completion =
                1  0  0  0  0
                0  1  0  0  0
                0  0  0  0  1
                0  0  1  0  0
                0  0  0  1  0
                0  0  1  0  0


Pb_Minimal_ls =
                 7/2  0  5/2  0
                 7    0  0    0
```

```
0     0  5     0
0     5  0     0
0     0  0     6
0     5  0     0
```

Therefore, X has only the SR2-property. Note that in this example it holds $n > \frac{d+1}{2}$ hence it is possible to have non trivial replicated options written on elements of X.

Our next example presents a market X, without the SR and the SR1 properties. On the other hand X has the SR2-property and it holds $n < \frac{d+1}{2}$, $\mathbf{1} \in X$, hence any non trivial option written on elements of X is non replicated.

Example 5.3. Consider the following four vectors x_1, x_2, x_3, x_4 in $\mathbb{R}^{12}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.1112 & 0.7803 & 0.3897 & 0.5752 & 0.0598 & 0.2348 & 0.6864 & 0.8400 & 0.6245 & 0.1055 & 0.7797 & 0.3874 \\ 0.2417 & 0.4039 & 0.0965 & 0.3532 & 0.8212 & 0.0154 & 0.5948 & 1.2251 & 0.1119 & 0.2382 & 0.3957 & 0.0963 \\ 0.1320 & 0.9421 & 0.9561 & 0.0430 & 0.1690 & 0.6491 & 0.1750 & 1.1110 & 1.6053 & 0.1315 & 0.9404 & 0.9496 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

and $X = [x_1, x_2, x_3, x_4]$ is the marketed space.

The values of the first three vectors were randomly generated and $x_4 = \mathbf{1}$. By following the same procedure, as before, one gets

```
Not strongly resolving market.
Not strongly resolving with respect to the pb of F1(X).
Strongly resolving with respect to the pb of Y.
```

The dimension of the `Pb_Minimal_ls` matrix is 12×8 hence $4 = n < \frac{d+1}{2} = \frac{9}{2}$. Moreover, since the riskless bond belongs to X, by theorem 4.4, we have that any non trivial option written on elements of X is non replicated. The `Pb_Minimal_ls` matrix has been removed from the previous results due to its large size.
We conclude this section with three open questions regarding the properties SR, SR1 and SR2:

1. If Y is a proper subspace of $F_1(X)$, then does X has the SR1-property ?
2. If X has the SR property, then does X has the SR2-property?
3. If X has the SR-1 property, then does X has the SR2-property?

6. Conclusions

In this paper, a characterization of markets that don't replicate any option is presented. Specifically, the notion of strongly resolving markets with respect to the positive basis of a minimal lattice-subspace Y of $\mathbb{R}^m$ is defined. It is proved that if the number of securities is less than half the dimension of Y, then not a single (non-trivial) option can be replicated. This result provides a new characterization of strongly resolving markets. Both theoretical and computational methods are provided and we are hopeful that the results of this work provide an important tool in order to study the interesting problem of option replication of a two-period security market, in which the space of marketed securities is a subspace of $\mathbb{R}^m$.

7. Appendix

The Matlab implementation of Algorithm 5.1 is given below.

```
function [Pb_Completion,Pb_Minimal_ls] = srtest(X)

%***************************%
%   General  Information.   %
%***************************%
% Synopsis:
% SR = srtest(X)
% [SR,Pb_Completion,Pb_Minimal_ls] = srtest(X)
%
% Input:
%    X = the payoff matrix with the non-redundant
%        security vectors x_1, x_2,...,x_n specified
%        as columns.
%
% Output:
%    SR = returns strongly resolving or strongly resolving with respect to
%         the positive basis of F1(X) or stronlgy resolving
%         with respect to the positive basis of a minimal lattice-subspace
%         containing X.
%
%    Pb_Completion = positive basis of F_1(X) which is a partition
%                    of the unit. The i column of the Pb_Completion
%                    matrix is the vector bi of the positive basis.
%
%    Pb_Minimal_ls = positive basis of a minimal lattice-subspace
%                    containing X. The i column of the Pb_Minimal_ls
%                    matrix is the vector bi of the positive basis.
%
% Note that for the correct performance of the srtest function
% the presence of the MINlat function from [1], is needed.
%
% References:
%  [1] V.N. Katsikis, I. Polyrakis, Computation of vector
%  sublattices and minimal lattice-subspaces. Applications in finance.
%  Applied Mathematics and Computation, 218 (2012), 6860-6873.

srtest0;
srtest1;
srtest2;

function srtest0 = srtest0

%***********************************%
%  Strongly resolving market test.  %
%***********************************%

[m,n] = size(X);
combos = combntns(1:m,n);
t = length(combos(:,1));
ranks = zeros(t,1);
for i = 1:t
    Testmatrix = X(combos(i,:),:);
```

```
    ranks(i) = rank(Testmatrix);
end
    if any(ranks < n)
       disp('Not strongly resolving market.')
       return
    else
       disp('Strongly resolving.')
    end
end

%*****************************************************%
%  Strongly resolving market test with respect to the %
%  positive basis of F1(X).                           %
%*****************************************************%

function srtest1 = srtest1
%******************************************************%
% Determination of a basic set of marketed securities. %
%******************************************************%
if any(any(X < 0)) ~= 0
a = max(max(abs(X)));
B= a*ones(size(X)) - X;
 if any(any(B < 0)) ~= 0
B = 2*a*ones(size(X)) - X;
 end
else
B = X;
end
Matrix = zeros(size(B));

%*********************************%
%    Range of the basic curve.    %
%*********************************%

% Determination of the basic curve.
N = length(B(:,1));
for i = 1:N,
  if norm(B(i,:),1) ~= 0,
Matrix(i,:) = 1/norm(B(i,:),1)*B(i,:);
  end
end
% Find the unique elements of the range of the basic curve.
[~,m] = unique(Matrix,'rows','first');
Sort_m = sort(m);
Matrixnew = Matrix(Sort_m,:);
r = length(m);

%**********************************************%
% Calculation of the vector sublattice F_1(X). %
%**********************************************%

% Choose which vectors are linearly independent.
```

```
S = rref(Matrixnew');
[I,J] = find(S);
Linearindep = accumarray(I,J,[rank(Matrixnew),1],@min)';
M = length(B(1,:));
% A) If X=F_1(X).
if r == M
    disp('X is a vector sublattice hence any option is replicated')
end
% B) If X~=F_1(X).
Index1 = 1:r;
Index2 = setdiff(Index1,Linearindep);
Index = 1:N;
YY = sum(B,2)';
TTT = setdiff(Index,Linearindep);
Id = eye(N);
KK = Id(TTT,:);
TT = YY(1,TTT)';
T = diag(TT)*KK;
K = zeros(N);
K(TTT,:) = T;
Vec = zeros(r-M,N);
   DDD = cell(r-M,1);
    for i = 1:length(Index2)
    DD = strmatch(Matrixnew(Index2(i),:),Matrix,'exact');
    R = length(DD);
     if R >= 2,
       Vector = sum(K(DD,:));
     else
       Vector = K(DD,:);
     end
DDD{i,:} = DD;
    Vec(i,:) = Vector;
    end
Sublattice = [B Vec'];

%****************************************************%
% Determination of a positive basis for F_1(X) which %
% is a partition of the unit.                        %
%****************************************************%

% Calculate the new basic curve for F_1(X).
Matrixnew2 = zeros(size(Sublattice));
for i = 1:N,
 if norm(Sublattice(i,:),1) ~= 0,
 Matrixnew2(i,:) = 1/norm(Sublattice(i,:),1)*Sublattice(i,:);
 end
end
u = Matrixnew2([Sort_m(Linearindep)' cell2mat(DDD)'],:);
Test_Pb = u'\Sublattice';
[f,~] = find(Test_Pb);
Pb = Test_Pb(unique(f),:);
% Normalization of the positive basis (Npb).
```

```
Npb1 = diag(1./max(Pb,[ ],2))*Pb;
Npb = Npb1';
Npb(Npb < 10*eps) = 0;
Npb(Npb < 1+10*eps & Npb > 1-10*eps) = 1;
Pb_Completion = Npb;
%********************************************************%
%  Expansion of the primitive securities in terms of the %
%  positive basis (Npb) of F_1(X).                       %
%********************************************************%

X1 = Npb\B;

[mm,nn] = size(X1);
combos = combntns(1:mm,nn);
tt1 = length(combos(:,1));
ranks1 = zeros(tt1,1);
for i = 1:tt1
   Testmatrix1 = X1(combos(i,:),:);
   ranks1(i) = rank(Testmatrix1);
end
   if any(ranks1 < nn)
      disp('Not strongly resolving with respect to the pb of F1(X).')
      return
   else
      disp('Strongly resolving with respect to the pb of F1(X).')
   end
end

%*****************************************************%
%  Strongly resolving market test with respect to the %
%  positive basis of a minimal lattice-subspace       %
%  containing X.                                      %
%*****************************************************%

function srtest2 = srtest2
if any(any(X < 0)) ~= 0
a = max(max(abs(X)));
B= a*ones(size(X)) - X;
 if any(any(B < 0)) ~= 0
B = 2*a*ones(size(X)) - X;
 end
else
B = X;
end
[~,Positivebasis] = MINlat(B);
Pb_Minimal_ls = Positivebasis';
X2 = Positivebasis'\B;
[mmm,nnn] = size(X2);
combos = combntns(1:mmm,nnn);
tt2 = length(combos(:,1));
ranks2 = zeros(tt2,1);
for i = 1:tt2
```

```
   Testmatrix2 = X2(combos(i,:),:);
   ranks2(i) = rank(Testmatrix2);
end
   if any(ranks2 < nnn)
     disp('Not strongly resolving with respect to the pb of Y.')
     return
   else
     disp('Strongly resolving with respect to the pb of Y.')
   end
end
end
```

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