Filomat 27:7 (2013), 1197–1203
DOI 10.2298/FIL1307197B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Random products and product auto-regression

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Abstract. The operation of taking random products of random variables and the notions of infinite divisibility (ID) and stability of distributions under this operation are discussed here. Based on this stationary product auto-regressive time series models are introduced. We investigate some properties of the models, like autocorrelation function, spectral density function, multi-step ahead conditional mean and parameter estimation.

1. Introduction

Klebanov et al. [4] considered the problem of distribution of the product of a random number of random variables (r.vs) and an application of geometric-products in mathematical economics. The discussion therein was based on considering the log-transform of the r.vs so that a product can be treated as a sum (if the range of the r.vs permits this transformation), invoke the result for the sum and then get back to the product.

McKenzie [6] introduced (perhaps for the first time) a product auto-regressive (PAR(1)) model

$$X_n = X_{n-1}^{\alpha} V_n, \tag{1}$$

where $\alpha \in (0, 1)$ This is the product analogue of the AR(1) model

$$Y_n = \alpha Y_{n-1} + \varepsilon_n, \tag{2}$$

where $Y_n = \log X_n$ and $\varepsilon_n = \log V_n$.

McKenzie [6] noticed that the correlation structures of the PAR(1) and AR(1) models with gamma marginals are the same and given by

$$Corr(X_n, X_{n-k}) = \alpha^k,\ k = 0, 1, \ldots$$

Then, he characterized the gamma distribution, among self-decomposable distributions, as the only one having this correlation structure in the PAR (1) model.

2010 *Mathematics Subject Classification*. Primary 62M10
Keywords. Auto-regressive process, conditional least squares, log-gamma, random products, stability
Received: 03 June 2012; Accepted: 08 December 2012
Communicated by Dejan B. Ilić
The second author acknowledges the grant of MNTR 174013 for carrying out this research.
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Motivations of this paper are represented in the following items. Describing the operation of random products of random variables and the notions of infinite divisibility and stability of distributions under this operation. This description enables us to discuss first-order product autoregressive models with random coefficients based on this operation. These models represent nonlinear models with autoregressive correlation structure.

In the next section we discuss certain aspects of random products and then in Section 3 two PAR(1) models related to them. Section 4 investigates some properties of the models, like autocorrelation function, spectral density function, multi-step ahead conditional mean and parameter estimation.

2. Random products and infinite divisibility

Let us note that the divisibility properties of r.vs under the operation of sums or random-sums reflect the corresponding divisibility properties of their log-transforms under the operation of products or random-products. In the ensuing discussion we assume that the range of the r.vs permits this transformation. Hence we may formally have the following.

Definition 2.1. A r.v X is product infinitely divisible (PID) if for each positive integer n, there exists i.i.d r.vs $\{X_{i,n},\ i = 1, \ldots, n\}$ such that $X = \prod_{i=1}^{n} X_{i,n}$.

Definition 2.2. A r.v X is product stable (PS) if for each positive integer n, there exists a $c > 0$ such that $X = \prod_{i=1}^{n} X_{i,n}^{c}$, where $X_{i,n}$, $i = 1, \ldots, n$ are independent copies of X.

More generally we have the following. Since

$$Z_n = \prod_{i=1}^{n} X_i^c \Leftrightarrow \log Z_n = c \sum_{i=1}^{n} \log X_i$$

if $Z_n \xrightarrow{d} Z$ as $n \to \infty$, then $\log Z_n \xrightarrow{d} \log Z$ as $n \to \infty$. Similarly if

$$S_n = b \sum_{i=1}^{n} Y_i \Leftrightarrow e^{S_n} = \prod_{i=1}^{n} e^{bY_i},$$

then conclusions on the weak limit of S_n hold well for $\log S_n$ as above. We may also invoke the transfer theorem for random-sums and obtain results corresponding to random-products. The following conclusions are now clear.

Result 2.3. *Y is ID if and only if $X = e^Y$ is PID.*

Result 2.4. *Y is sum-stable if and only if $X = e^Y$ is product-stable.*

Result 2.5. *Y is random-sum-stable if and only if $X = e^Y$ is random-product-stable.*

The notion of random infinite divisible (N-ID) laws is systematically discussed in Gnedenko and Korolev [2]. To overcome certain limitations in this notion Satheesh [7] introduced φ-ID laws and from its easier definition (Definition 2.3 in Satheesh et al. [9]) it follows that φ-ID laws generalize N-ID laws. Further, notice that the class of geometric-ID (GID) laws forms a subclass of the class of Harris-ID (HID) laws which in turn is a subclass of the class of N-ID laws, see e.g. Satheesh et al. [9]. We are thus in a position to formulate the following results.

Result 2.6. *Y is φ-ID if and only if $X = e^Y$ is product-φ-ID.*

Corollary 2.7. *Y is N-ID if and only if $X = e^Y$ is product-N-ID.*

Corollary 2.8. *Y is HID if and only if $X = e^Y$ is product-HID.*

Corollary 2.9. *Y is GID if and only if $X = e^Y$ is product-GID.*

The following results, which are known in the literature, show the interplay between sums and products of r.vs.

Theorem 2.10. ([1]) *Let $Y_1, Y_2, \ldots$ are i.i.d positive r.vs with non-zero finite mean and N_p a positive geometric(p) r.v independent of Y_1 for every $p \in (0,1)$. Then $Y_1 \stackrel{d}{=} p \sum\limits_{i=1}^{N_p} Y_i$ if and only if Y_1 is exponential.*

Theorem 2.11. ([4]) *Let $X_1, X_2, \ldots$ are i.i.d positive r.vs with $E(\log X_1)$ finite, non-zero and N_p a positive geometric(p) r.v independent of X_1 for every $p \in (0,1)$. Then $X_1 \stackrel{d}{=} \prod_{i=1}^{N_p} X_i^p$ if and only if X_1 is Pareto (log-exponential).*

Theorem 2.12. ([8]) *Let $Y_1, Y_2, \ldots$ are i.i.d positive r.vs with non-zero finite mean and N_p a positive Harris$(1/p, m)$ r.v independent of Y_1 for every $p \in (0,1)$. Then $Y_1 \stackrel{d}{=} p \sum\limits_{i=1}^{N_p} Y_i$ if and only if Y_1 is gamma$(1/m)$.*

Result 2.13. *Let $X_1, X_2, \ldots$ are i.i.d positive r.vs $E(\log X_1)$ finite, non-zero and N_p a positive Harris$(1/p, m)$ r.v independent of X_1 for every $p \in (0,1)$. Then $X_1 \stackrel{d}{=} \prod_{i=1}^{N_p} X_i^p$ if and only if X_1 is log-gamma$(1/m)$.*

3. PAR(1) models

We will consider two PAR(1) models here which are generalizations of (1). We have the following AR(1) model of Lawrance and Lewis [5]

$$Y_n = \begin{cases} V_n, & \text{with probability } p, \\ Y_{n-1} + V_n, & \text{with probability } 1-p. \end{cases} \tag{3}$$

The product analogue of this is

$$X_n = \begin{cases} \varepsilon_n, & \text{with probability } p, \\ X_{n-1}\varepsilon_n, & \text{with probability } 1-p. \end{cases} \tag{4}$$

It is known from [3] that (3) is stationary for each $p \in (0,1)$ if and only if Y_n is GID. Hence we have

Result 3.1. *The PAR(1) model (4) is stationary for each $p \in (0,1)$ if and only if X_n is product-GID.*

The product-stability results, Theorem 2.11 and Result 2.13 above, can be used to model the generalized PAR(1) models which are the multiplicative analogue of those discussed in Satheesh et al. [11] and characterize various distributions that are the log-versions of the distributions therein. For a fixed and known $m > 0$, a generalization of (3) is

$$\sum_{i=1}^{m} Y_{i,n} = \begin{cases} \sum\limits_{i=1}^{m} V_n, & \text{with probability } p, \\ \sum\limits_{i=1}^{m} Y_{n-1} + \sum\limits_{i=1}^{m} V_n, & \text{with probability } 1-p. \end{cases} \tag{5}$$

The product analogue of this (a generalization of (3)) is given by

$$\prod_{i=1}^{m} X_{i,n} = \begin{cases} \prod_{i=1}^{m} \varepsilon_n, & \text{with probability } p, \\ \prod_{i=1}^{m} X_{i,n-1} \prod_{i=1}^{m} \varepsilon_{i,n}, & \text{with probability } 1-p. \end{cases} \tag{6}$$

By Theorem 2.3 in Satheesh et al. [10] the sequence $\{Y_{i,n}\}$ defines the model (5) that is stationary for each $p \in (0,1)$ if and only if $\{Y_{i,n}\}$ is Harris(a,m)-ID, $a = 1/p$. Hence we have

Theorem 3.2. *If for each i fixed, $\{X_{i,n}\}$ describes the PAR(1) model (4) that is stationary for each $p \in (0,1)$ and if for each n, the processes $\{X_{i,n},\ i = 1, 2, \dots, m\}$ ($m > 0$ is known) are independent, then the distribution of $\{X_{i,n}\}$ in (6) is product-H(a,m)-ID, $a = 1/p$ and conversely.*

We now briefly consider log-gamma distributions in this context. If a r.v Y is gamma, then e^Y is log-gamma or if X is log-gamma then $\log X$ is gamma. Thus if Y is gamma(θ,β) r.v with p.d.f

$$f(y) = \frac{\theta^\beta y^{\beta-1} e^{-\theta y}}{\Gamma(\beta)}, \; y > 0, \beta > 0, \theta > 0, \tag{7}$$

then $X = e^Y$ is log-gamma(θ,β) with p.d.f

$$h(x) = \frac{\theta^\beta (\log x)^{\beta-1}}{\Gamma(\beta) x^{\theta+1}}, \; x > 1, \beta > 0, \theta > 0. \tag{8}$$

Since the gamma(θ,β) distribution is GID for $\beta \leq 1$, log-gamma(θ,β) distribution at (8) is product-GID and thus gamma(θ,β) distribution can be used to model the PAR(1) structure (4). Again, since the gamma(θ,β) distribution is H(a,m)-ID for $\beta = 1/m$, $m > 1$ integer, log-gamma$(\theta, 1/m)$ distribution at (8) is multiplicative-H(a,m)-ID and thus gamma$(\theta, 1/m)$ distribution can be used to model the PAR(1) structure (6).

4. Some properties of the PAR(1) models

We now discuss certain distributional and estimation aspects of the PAR(1) model (4). Here we assume $\mu_\varepsilon = E(\varepsilon_n) < \infty$. The stationary PAR(1) model (4) can be rewritten as

$$X_n = X_{n-1}^{A_n} \varepsilon_n, \tag{9}$$

where $\{A_n\}$ is a sequence of i.i.d rvs with $P(A_n = 0) = 1 - P(A_n = 1) = p$ independent of X_{n-l} for $l > 0$ and $\{A_n\}$ and $\{\varepsilon_n\}$ are two mutually independent sequences.

For the process $\{X_n\}$ given by (9), the autocovariance function $\gamma_k = Cov(X_n, X_{n-k})$ is obtained as follows. Using (9) and properties of $\{X_n\}$, we get

$$E\left(X_{n-1}^{A_n}\right) = E\left(E\left(X_{n-1}^{A_n} | X_{n-1}\right)\right) = p + (1-p)\mu_X$$

and

$$E\left(X_{n-1}^{A_n} X_{n-k}\right) = E\left(E\left(X_{n-1}^{A_n} X_{n-k} | X_{n-1}, X_{n-k}\right)\right) = p\mu_X + (1-p)\gamma_{k-1} + (1-p)\mu_X^2,$$

where $\mu_X = E(X_n)$ and $k > 0$. Using the definition of γ_k and results above, we find that

$$\gamma_k = \mu_\varepsilon \left(E\left(X_{n-1}^{A_n} X_{n-k}\right) - \mu_X E\left(X_{n-1}^{A_n}\right)\right) = (1-p)^k \mu_\varepsilon^k \gamma_0.$$

Now, we will show that $|(1-p)\mu_\varepsilon| < 1$. From the stationarity of the process $\{X_n\}$, we obtain that $\mu_X = \frac{p\mu_\varepsilon}{1-(1-p)\mu_\varepsilon}$. This implies that μ_X exists for $1 - (1-p)\mu_\varepsilon \neq 0$. Also, from the definition of the model and stationarity, we have that

$$\mu_X^2 + \sigma_X^2 = p(\mu_\varepsilon^2 + \sigma_\varepsilon^2) + (1-p)(\mu_\varepsilon^2 + \sigma_\varepsilon^2)(\mu_X^2 + \sigma_X^2).$$

Under the condition $1-(1-p)(\mu_\varepsilon^2+\sigma_\varepsilon^2)\neq 0$, the variance of $\{X_n\}$ is given by

$$\sigma_X^2 = \frac{p(1-p)(1-\mu_\varepsilon)^2(\mu_\varepsilon^2+\sigma_\varepsilon^2)+p^2\sigma_\varepsilon^2}{\left(1-(1-p)(\mu_\varepsilon^2+\sigma_\varepsilon^2)\right)(1-(1-p)\mu_\varepsilon)^2}. \tag{10}$$

The variance of $\{X_n\}$ is positive for $1-(1-p)(\mu_\varepsilon^2+\sigma_\varepsilon^2)>0$. We have that

$$\frac{1}{(1-p)^2} > \frac{1}{1-p} > \mu_\varepsilon^2+\sigma_\varepsilon^2 > \mu_\varepsilon^2,$$

which implies that $|(1-p)\mu_\varepsilon|<1$. Hence we have the following theorem.

Theorem 4.1. *The autocorrelation function at lag k of the r.vs X_n and X_{n-k} is given by $Corr(X_n, X_{n-k}) = (1-p)^k\mu_\varepsilon^k$, $k>0$. Further, $|(1-p)\mu_\varepsilon|<1$ and hence the autocorrelation function converges to zero as $k\to\infty$.*

Based on the autocovariance γ_k value, the spectral density function

$$f_{XX}(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k e^{-i\lambda k},\ i=\sqrt{-1}$$

of the PAR(1) process is given by

$$f_{XX}(\lambda) = \frac{\sigma_X^2}{2\pi}\cdot\frac{1-\mu_\varepsilon^2(1-p)^2}{1+\mu_\varepsilon^2(1-p)^2-2\mu_\varepsilon(1-p)\cos\lambda},$$

where σ_X^2 is given in (10).

Further, for this PAR(1) process the one-step ahead conditional mean is

$$E(X_{n+1}|X_n=x) = p\mu_\varepsilon + (1-p)\mu_\varepsilon x,$$

which is linear in x. Also, we can see that

$$E(X_{n+2}|X_n=x) = p\mu_\varepsilon + p(1-p)\mu_\varepsilon^2 + (1-p)^2\mu_\varepsilon^2 x.$$

Hence using induction the k-step ahead conditional mean is given by the following theorem.

Theorem 4.2. *The k-step ahead conditional mean is*

$$E(X_{n+k}|X_n) = p\mu_\varepsilon\sum_{j=0}^{k-1}(1-p)^j\mu_\varepsilon^j + (1-p)^k\mu_\varepsilon^k x.$$

Remark 4.3. It is interesting to note that by virtue of Theorem 4.1

$$\lim_{k\to\infty}E(X_{n+k}|X_n) = \frac{p\mu_\varepsilon}{1-(1-p)\mu_\varepsilon} = \mu_X$$

which is the unconditional mean of $\{X_n\}$.

We now find the conditional least squares (CLS) estimators of PAR(1) parameters. Let $X_1, X_2, \ldots, X_N$ be a realization of PAR(1) process and consider the function

$$Q_N(p,\mu_\varepsilon) = \sum_{n=2}^{N}(X_n - E(X_n|X_{n-1}))^2 = \sum_{n=2}^{N}(X_n - p\mu_\varepsilon - (1-p)\mu_\varepsilon X_{n-1})^2.$$

Then, the CLS estimators of the parameters p and μ_ε are obtained by solving the system of equations $\frac{\partial Q_N(p,\mu_\varepsilon)}{\partial p}=0$ and $\frac{\partial Q_N(p,\mu_\varepsilon)}{\partial \mu_\varepsilon}=0$. The estimators are given in the following theorem.

Theorem 4.4. *The conditional least squares estimators of the PAR(1) parameters are*

$$\hat{\mu}_\varepsilon = \frac{(N-1)\sum_{n=2}^{N} X_n X_{n-1} - \sum_{n=2}^{N} X_n \sum_{n=2}^{N} X_{n-1} + \sum_{n=2}^{N} X_n \sum_{n=2}^{N} X_{n-1}^2 - \sum_{n=2}^{N} X_{n-1} \sum_{n=2}^{N} X_n X_{n-1}}{(N-1)\sum_{n=2}^{N} X_{n-1}^2 - \left(\sum_{n=2}^{N} X_{n-1}\right)^2}$$

and

$$\hat{p} = \frac{\sum_{n=2}^{N} X_n - \hat{\mu}_\varepsilon \sum_{n=2}^{N} X_{n-1}}{\hat{\mu}_\varepsilon \left(N - 1 - \sum_{n=2}^{N} X_{n-1}\right)}.$$

Remark 4.5. *Let $Z_n = \prod_{i=1}^{m} X_{i,n}$, $m > 0$ be fixed and known, we can develop CLS estimators for the model (6) as done above for (4).*

Now we will discuss the asymptotic properties of the obtained CLS estimators. To derive these properties we will need the following lemma.

Lemma 4.6. *The PAR*(1) *process* $\{X_n\}$ *given by* (4) *is a strict stationary and ergodic process.*

Proof. The strict stationarity of the PAR(1) process $\{X_n\}$ follows from the fact that it is a Markov process of the first order and that the random variables $\{X_n\}$ are identically distributed random variables. The ergodicity of the PAR(1) process follows from the Lemma 2 ([12], pp. 408), the fact that the σ-algebra generated by $\{X_n, X_{n-1}, X_{n-2}, \dots\}$ is a subset of the σ-algebra generated by i.i.d. random variables $\{\varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-2}, \dots\}$ and the fact that $\bigcap_{n=0}^{-\infty} \mathcal{F}\{\varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-2}, \dots\}$ is a tail σ-algebra. $\square$

Now, the asymptotical properties of the CLS estimators follow from the following theorem.

Theorem 4.7. *If the PAR*(1) *process given by* (4) *has finite moments $E(X_n^4)$, then the CLS estimator $\hat{\theta} = (\hat{p}, \hat{\mu}_\varepsilon)^T$ of the parameter $\theta = (p, \mu_\varepsilon)^T$ is a strongly consistent estimator and has asymptotical normal distribution, i.e. we have that $\sqrt{N-1}(\hat{\theta} - \theta)$ converges in distribution to $\mathcal{N}(0, U^{-1}RU^{-1})$, as $N \to \infty$, where*

$$U = \begin{bmatrix} \mu_\varepsilon^2 E(1 - X_{n-1})^2 & \mu_\varepsilon E((1 - X_{n-1})(p + (1-p)X_{n-1})) \\ \mu_\varepsilon E((1 - X_{n-1})(p + (1-p)X_{n-1})) & E(p + (1-p)X_{n-1})^2 \end{bmatrix}$$

$$R = \begin{bmatrix} \mu_\varepsilon^2 E\left(\nu_n(1 - X_{n-1})^2\right) & \mu_\varepsilon E(\nu_n(1 - X_{n-1})(p + (1-p)X_{n-1})) \\ \mu_\varepsilon E(\nu_n(1 - X_{n-1})(p + (1-p)X_{n-1})) & E(\nu_n(p + (1-p)X_{n-1})^2) \end{bmatrix}$$

and ν_n is a conditional prediction error of $\{X_n\}$ given by

$$\nu_n = p(\sigma_\varepsilon^2 + (1-p)\mu_\varepsilon^2) - 2p(1-p)\mu_\varepsilon^2 X_{n-1} + (1-p)(p\mu_\varepsilon^2 + \sigma_\varepsilon^2)X_{n-1}^2.$$

Proof. First, we will show that all the conditions of Theorem 3.1 [13] are satisfied. Let $g_n = E(X_n|X_{n-1})$. Then $g_n = \mu_\varepsilon(p + (1-p)X_{n-1})$ and the first derivatives of the function g_n with respect to p and μ_ε are $\partial g_n/\partial p = \mu_\varepsilon(1 - X_{n-1})$ and $\partial g_n/\partial \mu_\varepsilon = p + (1-p)X_{n-1}$, respectively. Then all the conditions from Theorem 3.1 [13] except the condition C2 can be trivially proved. Let us show that the condition C2 is satisfied. Let us suppose that

$$E\left|a_1 \frac{\partial g_n}{\partial p} + a_2 \frac{\partial g_n}{\partial \mu_\varepsilon}\right|^2 = 0.$$

Then it follows that $E\left|a_1\mu_\varepsilon + a_2 p + (a_2 - a_2 p - a_1\mu_\varepsilon)X_{n-1}\right|^2 = 0$. From this condition we obtain that

$$a_1\mu_\varepsilon + a_2 p + (a_2 - a_2 p - a_1\mu_\varepsilon)\mu_X = 0$$

and $a_2 - a_2 p - a_1 \mu_\varepsilon = 0$, which implies that $a_1 = a_2 = 0$. Thus the condition C2 is satisfied and from Theorem 3.1 [13] follows that the CLS estimator $\hat{\theta} = (\hat{p}, \hat{\mu}_\varepsilon)^T$ of the parameter $\theta = (p, \mu_\varepsilon)^T$ is a strongly consistent estimator. Finally, let us prove that the CLS estimator has asymptotical normal distribution. We have that the conditional prediction error of $\{X_n\}$ is given by

$$\begin{aligned} \nu_n &\equiv E\left((X_n - g_n)^2 | X_{n-1}\right) = E(X_n^2 | X_{n-1}) - \mu_\varepsilon^2 (p + (1-p) X_{n-1})^2 \\ &= p(\sigma_\varepsilon^2 + (1-p)\mu_\varepsilon^2) - 2p(1-p)\mu_\varepsilon^2 X_{n-1} + (1-p)(p\mu_\varepsilon^2 + \sigma_\varepsilon^2) X_{n-1}^2. \end{aligned}$$

Since the PAR(1) process given by (4) has finite moments $E(X_n^4)$, it follows that all the elements of matrix R from the condition D1 ([13], Theorem 3.2) are finite. Thus all the conditions of Theorem 3.2 [13] are satisfied. Then the asymptotical normality of the CLS estimator follows from Theorem 3.2 [13]. □

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