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On sequence selection properties

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Abstract. We investigate modifications of the selection principles for various kinds of convergences introduced by L. Bukovský and J. Šupina [9]. We show that if we restrict our selection principles to continuous functions then they split just to two equivalence classes. Considering other families of limit functions we obtain our main result saying that selecting "from every sequence of a sequence of sequences of functions" or just "from infinitely many sequences" can produce properties which can be consistently distinguished. Moreover, we show that our selection principles are characterizations of some properties of a topological space, e.g. hereditarily Hurewicz space or hereditarily $S_1(\Gamma, \Gamma)$-space.

1. Introduction

All topological spaces throughout the paper are assumed to be Hausdorff and infinite. For preliminary definitions see Section 2. We follow mainly [4] and [13]. More about cardinal invariants can be found e.g. in [2].

We investigate sequence selection principles for functions on a topological space. These principles contain principles investigated by A.V. Arkhangel'skiĭ [1], M. Scheepers [37], L. Bukovský and J. Haleš [6] and L. Bukovský and J.Š. [9]. Differences among them are so slight that all investigated selection principles AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) can be consistently with **ZFC** equivalent. However, under some set-theoretical assumptions weaker than **CH**, many of the principles can be distinguished. Hence, statements about some relations are undecidable in set theory **ZFC**.

Let A, B denote one of the following types of convergence: P pointwise, Q quasi–normal, D discrete. Let X be a topological space, $\mathcal{F}, \mathcal{G} \subseteq {}^X\mathbb{R}$ being families of functions containing the zero function on X. We say that X has the **sequence selection property AB($\mathcal{F}$,$\mathcal{G}$)**, if for any functions $f_{n,m} \in C_p(X)$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$, $n, m \in \omega$, such that

a) $f_{n,m} \xrightarrow{A} f_n$ on X for every $n \in \omega$,

b) $f_n \xrightarrow{A} f$ on X,

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there exists an unbounded $\beta \in {}^\omega\omega$ such that

$$f_{n,\beta(n)} \xrightarrow{\mathrm{B}} f \text{ on } X.$$

If there exist an increasing $\alpha \in {}^\omega\omega$ and an unbounded $\beta \in {}^\omega\omega$ such that $f_{\alpha(n),\beta(n)} \xrightarrow{\mathrm{B}} f$ on X then we say that topological space X has the **weak sequence selection property wAB($\mathcal{F}$,$\mathcal{G}$)**. Actually, these weak selection properties turned out to be interesting as well. Moreover, some weak versions of selection properties can be distinguished from their stronger versions under some set-theoretical assumptions. The list of such properties is presented in Theorem 8.3.

Our main focus will be family $\mathcal{B}(X)$ or simply $\mathcal{B}$ of all Borel functions on X, family $\mathrm{C}_p(X)$ of all continuous functions on X, and family containing only zero function $\{0\}$. Interesting results are obtained for family $\mathcal{U}$ or $\mathcal{U}(X)$ of all upper semicontinuous functions on X with values in $[0,1]$. One can easily see that the whole space ${}^X\mathbb{R}$ of all real-valued functions on X as families $\mathcal{F},\mathcal{G}$ in definitions of properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) is in fact abundant. Therefore we shall use the family of all Borel functions on X instead of ${}^X\mathbb{R}$ (i.e., always $\mathcal{F},\mathcal{G} \subseteq \mathcal{B}$). However, in case of family $\mathcal{F}$ we could use the family of all F_σ-measurable functions on X and in case of family $\mathcal{G}$ we could use the family of all second Baire class functions on X. Note that quasi-normal limit of continuous functions in a normal space is $\mathbf{\Delta}^0_2$-measurable, i.e., both F_σ and G_δ-measurable.

Since $f_n \to f$ if and only if $f_n - f \to 0$ for any $f_n, f \in \mathrm{C}_p(X)$, selection property wPP($\{0\}$,$\{0\}$) is equivalent with the property (α_4) for $\mathrm{C}_p(X)$ introduced and investigated by A.V. Arkhangel'skiĭ [1]. Similarly, property PP($\{0\}$,$\{0\}$) is equivalent with the sequence selection property for $\mathrm{C}_p(X)$ introduced by M. Scheepers [37]. These two properties are equivalent for $\mathrm{C}_p(X)$ with similar properties considered by D.H. Fremlin [14], see e.g. [38] and [39]. Property DP($\{0\}$,$\{0\}$) was considered by L. Bukovský and J. Haleš [6]. Note that there is other comprehensive investigation of (α_{i})-like properties introduced by Lj.D.R. Kočinac [22], for more see e.g. [23].

The paper [9] by the author and L. Bukovský is devoted to the study of properties AB($\mathcal{B}$,$\mathcal{B}$) and AB($\mathcal{B}$,$\{0\}$) under notation ASB and ASB*, respectively. An alternative proof of strengthened Recław's Theorem [33] was obtained since it was simply proved that if a perfectly normal topological space X has DP($\mathcal{B}$,$\mathcal{B}$) then X is a σ-set. We consider the application of property QQ($\mathcal{B}$,$\mathcal{B}$) in an alternative proof of Tsaban–Zdomskyy Theorem [40] by the author and L. Bukovský in [9] to be so far the most interesting application of these properties.

D.H. Fremlin [14] investigated properties related to sequential closure of sets and he showed in fact the connection between property PP($\{0\}$,$\{0\}$) and property s_1. Actually, there is a connection between other properties wAB($\mathcal{F}$,$\mathcal{G}$) and properties related to sequential closure operator.

Properties AD($\mathcal{F}$,$\mathcal{G}$) and wAD($\mathcal{F}$,$\mathcal{G}$) for A $\in$ {P, Q} will not be studied in the paper since there is no topological space which has any of them. One can take sequences of constant functions not converging discretely. Therefore these particular properties are never meant when considering selection properties AB($\mathcal{F}$,$\mathcal{G}$) or wAB($\mathcal{F}$,$\mathcal{G}$). Moreover, as it will be shown in Proposition 5.1, all properties DD($\mathcal{F}$,$\mathcal{G}$) and wDD($\mathcal{F}$,$\mathcal{G}$) are equivalent for any perfectly normal space. Hence, these properties will be sometimes omitted from our consideration, e.g from Diagrams 2 - 3.

Directly from the definition we may see that it is meaningless to consider other family than $\{0\}$ in the second argument of properties AB($\{0\}$,$\mathcal{G}$) and wAB($\{0\}$,$\mathcal{G}$), i.e.,

$$\mathrm{AB}(\{0\},\mathcal{G}) \equiv \mathrm{AB}(\{0\},\{0\}), \qquad \mathrm{wAB}(\{0\},\mathcal{G}) \equiv \mathrm{wAB}(\{0\},\{0\}).$$

The introduced selection properties are monotonous, i.e., if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$ then

$$\mathrm{AB}(\mathcal{F}_2,\mathcal{G}_2) \to \mathrm{AB}(\mathcal{F}_1,\mathcal{G}_1),\ \mathrm{wAB}(\mathcal{F}_2,\mathcal{G}_2) \to \mathrm{wAB}(\mathcal{F}_1,\mathcal{G}_1). \tag{1}$$

Since we always assume that $0 \in \mathcal{F},\mathcal{G}$ and $\mathcal{F},\mathcal{G} \subseteq \mathcal{B}$, we immediately obtain that

$$\mathrm{AB}(\mathcal{B},\mathcal{B}) \to \mathrm{AB}(\mathcal{F},\mathcal{G}) \to \mathrm{AB}(\{0\},\{0\}),\ \mathrm{wAB}(\mathcal{B},\mathcal{B}) \to \mathrm{wAB}(\mathcal{F},\mathcal{G}) \to \mathrm{wAB}(\{0\},\{0\}). \tag{2}$$

Diagram 1 describes relations among selection properties for fixed $\mathcal{F},\mathcal{G}\subseteq\mathcal{B}$ that raise from relations among pointwise, quasi-normal and discrete convergences. If we replace A by weaker and B by stronger convergence in AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) then we obtain weaker selection property. Namely (for AB($\mathcal{F}$,$\mathcal{G}$)):

$$\begin{array}{ccccc} & & & & \mathrm{DD}(\mathcal{F},\mathcal{G}) \\ & & & & \downarrow \\ \mathrm{PQ}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{QQ}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{DQ}(\mathcal{F},\mathcal{G}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PP}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{QP}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{DP}(\mathcal{F},\mathcal{G}) \end{array}$$

Diagram 1

If A, B are any of P, Q, D except B = D then by (2) and Diagram 1 we obtain that

$$\mathrm{PQ}(\mathcal{B},\mathcal{B}) \to \mathrm{AB}(\mathcal{F},\mathcal{G}) \to \mathrm{wAB}(\mathcal{F},\mathcal{G}) \to \mathrm{wDP}(\{0\},\{0\}). \tag{3}$$

We formulate some basic equivalences among selection properties in the following lemma.

Lemma 1.1. *Let X be a topological space, $\mathcal{F},\mathcal{G}$ being families of functions. The following hold true.*

a) *If $\mathcal{G}\subseteq\mathcal{F}$, $\mathcal{G}\subseteq \mathrm{C}_p(X)$ and $\mathcal{F}$ is closed under subtraction, then*

$$\mathrm{AB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{AB}(\mathcal{F},\{0\}), \qquad \mathrm{wAB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wAB}(\mathcal{F},\{0\}).^{1)}$$

b) *If $\{\min\{|f|,1\};\ f\in\mathcal{F}\}\subseteq\mathcal{F}$ then X has AB($\mathcal{F}$,{0}) if and only if for any functions $f_{n,m}, f_n \in {}^X[0,1]$, $f_{n,m}\in \mathrm{C}_p(X)$, $f_n\in\mathcal{F}$, $n,m\in\omega$, such that $f_{n,m}\xrightarrow{\mathrm{A}} f_n$ for every $n\in\omega$ and $f_n\xrightarrow{\mathrm{A}} 0$ there exists an unbounded $\beta\in{}^\omega\omega$ such that $f_{n,\beta(n)}\xrightarrow{\mathrm{B}} 0$ on X. Similarly for* wAB($\mathcal{F}$,{0}).

c) wAB({0},{0}) ≡ AB({0},{0}).

d) *If* (A, B) ≠ (P, Q) *and $\mathcal{F}\subseteq \mathrm{C}_p(X)$, then*

$$\mathrm{AB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{AB}(\mathcal{F},\{0\}) \equiv \mathrm{AB}(\{0\},\{0\}) \equiv \mathrm{wAB}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wAB}(\mathcal{F},\{0\}).$$

Proof. a) For $\mathcal{F}\neq\{0\}$ and for functions $f_{n,m}\in \mathrm{C}_p(X)$, $f_n\in\mathcal{F}$, $f\in\mathcal{G}$ consider $f_{n,m}-f\xrightarrow{\mathrm{A}} f_n-f$, $n\in\omega$ and $f_n-f\xrightarrow{\mathrm{A}} 0$ instead of $f_{n,m}\xrightarrow{\mathrm{A}} f_n$, $n\in\omega$ and $f_n\xrightarrow{\mathrm{A}} f$.

b) Let X be a topological space, $f_n, f : X\to\mathbb{R}$, $g_n=\min\{|f_n|,1\}$, $g=\min\{|f|,1\}$ and $\mathrm{A}\in\{\mathrm{P},\mathrm{Q},\mathrm{D}\}$. If $f_n\xrightarrow{\mathrm{A}} f$ then $g_n\xrightarrow{\mathrm{A}} g$. Moreover, if $g_n\xrightarrow{\mathrm{A}} 0$ then $f_n\xrightarrow{\mathrm{A}} 0$.

c) If $f_{n,m}\in {}^X[0,1]$, $n,m\in\omega$ are continuous and $f_{n,m}\to 0$ for any $n\in\omega$ then take functions $g_{n,m}$ defined by $g_{n,m}=\max\{f_{k,m};\ k\le n\}$, $n,m\in\omega$.

d) For $f_{n,m}\in \mathrm{C}_p(X)$, $f_n\in\mathcal{F}$, $f\in\mathcal{G}$ such that $f_{n,m}\xrightarrow{\mathrm{A}} f_n$ on X for every $n\in\omega$ and $f_n\xrightarrow{\mathrm{A}} f$ on X apply AB({0},{0}) to $f_{n,m}-f_n$, $n,m\in\omega$. Similarly for weak properties wAB($\mathcal{F}$,$\mathcal{G}$) and wAB({0},{0}). □

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1) Adopted from [10].

It is noted in [10] that the family of sequence selection properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) for $\mathcal{F},\mathcal{G} \subseteq \mathcal{B}$ and $\{0\} \subseteq \mathcal{F},\mathcal{G}$ can be partially preordered by the relation

$$\mathrm{V} \leq \mathrm{W} \equiv \mathbf{ZFC} \vdash \mathrm{W} \to \mathrm{V}.$$

However, if both V ≤ W and W ≤ V then **ZFC** ⊢ V ≡ W.

By combining all considered types of convergences and families $\mathcal{B}, \mathrm{C}_p(X), \{0\}$ in properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) we obtain 162 principles. 36 of them are meaningless. By aforementioned reductions we obtain 41 principles such that any of properties AB($\mathcal{F}$,$\mathcal{G}$) or wAB($\mathcal{F}$,$\mathcal{G}$) for $\mathcal{F},\mathcal{G} \in \{\mathcal{B}, \mathrm{C}_p(X), \{0\}\}$ has its corresponding equivalent principle among them. Their relations already presented in (1) and Diagram 1 are for stronger properties redrawn to Diagrams 2 - 3.

$$\begin{tikzcd}
\mathrm{PQ}(\mathcal{B},\mathcal{B}) \arrow[r] \arrow[d] & \mathrm{PQ}(\mathrm{C}_p(X),\mathcal{B}) \arrow[d] & \\
\mathrm{PQ}(\mathcal{B},\{0\}) \arrow[r] & \mathrm{PQ}(\mathrm{C}_p(X),\{0\}) \arrow[r] & \mathrm{PQ}(\{0\},\{0\})
\end{tikzcd}$$

Diagram 2

$$\begin{tikzcd}
\mathrm{QQ}(\mathcal{B},\mathcal{B}) \arrow[r] & \mathrm{QP}(\mathcal{B},\mathcal{B}) & & \mathrm{QQ}(\mathcal{B},\{0\}) \arrow[r] & \mathrm{QP}(\mathcal{B},\{0\}) & & \mathrm{DD}(\mathcal{B},\mathcal{B}) \arrow[d] \\
\mathrm{PQ}(\mathcal{B},\mathcal{B}) \arrow[rrr] \arrow[dr] \arrow[ddddd] & & & \mathrm{QQ}(\mathcal{B},\mathcal{B}) \arrow[rrr] \arrow[d] & & & \mathrm{DQ}(\mathcal{B},\mathcal{B}) \arrow[dl] \arrow[ddddd] \\
& \mathrm{PQ}(\mathcal{B},\{0\}) \arrow[rr] \arrow[dr] \arrow[ddd] & & \mathrm{QQ}(\mathcal{B},\{0\}) \arrow[rr] \arrow[d] & & \mathrm{DQ}(\mathcal{B},\{0\}) \arrow[dl] \arrow[ddd] & \\
& & \mathrm{PQ}(\{0\},\{0\}) \arrow[r] \arrow[d] & \mathrm{QQ}(\{0\},\{0\}) \arrow[r] \arrow[d] & \mathrm{DQ}(\{0\},\{0\}) \arrow[d] & & \\
& & \mathrm{PP}(\{0\},\{0\}) \arrow[r] & \mathrm{QP}(\{0\},\{0\}) \arrow[r] & \mathrm{DP}(\{0\},\{0\}) & & \\
& \mathrm{PP}(\mathcal{B},\{0\}) \arrow[rr] \arrow[ur] & & \mathrm{QP}(\mathcal{B},\{0\}) \arrow[rr] \arrow[u] & & \mathrm{DP}(\mathcal{B},\{0\}) \arrow[ul] & \\
\mathrm{PP}(\mathcal{B},\mathcal{B}) \arrow[rrr] \arrow[ur] & & & \mathrm{QP}(\mathcal{B},\mathcal{B}) \arrow[rrr] \arrow[u] & & & \mathrm{DP}(\mathcal{B},\mathcal{B}) \arrow[ul]
\end{tikzcd}$$

Diagram 3

However, in the following we shall show that many properties of Diagrams 2 - 3 are mutually equivalent. E.g. 7 principles related to continuous functions split to two equivalence classes. Finally, relations among weak sequence selection properties can be described by similar diagrams.

By (1), (3) and Diagram 1 the equivalence classes of PQ($\mathcal{B}$,$\mathcal{B}$) and DD($\mathcal{B}$,$\mathcal{B}$) are maximal elements and the equivalence class of wDP($\{0\}$,$\{0\}$) is the smallest element of the partially ordered set of equivalence classes of properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$). It will be shown that for perfectly normal space the greatest element and the smallest element of this partially ordered set are the equivalence classes of QN and wQN, respectively.

The paper is organized as follows. Section 2 contains preliminary definitions and results. We give examples of topological spaces which have investigated sequence selection properties in Section 3 and in Section 4 we show that topological spaces with these selection principles have other interesting properties. Essential part of Theorems 4.1 and 4.6 was already known, however, we prove them in a more general form. Section 5 contains lists of equivalent sequence selection properties. We obtain characterizations of QN-space, wQN-space, hereditarily $S_1(\Gamma,\Gamma)$-space and $S_1(\Gamma,\Gamma)$-space with every open γ-cover shrinkable.

In Section 6 we investigate slightly different principles than properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) which are connected to monotonic convergence of a sequence of functions. We obtain characterizations of hereditarily Hurewicz space and Hurewicz space with every open γ-cover shrinkable. Section 7 is devoted to the relation between properties wPP($\mathcal{F}$,$\mathcal{G}$) and properties of functional space expressed through sequential closure operator or properties investigated by T. Orenshtein [31]. In fact, Corollaries 8.5 and 8.6 answer his Problems 6.0.15 and 6.0.16. Finally, Theorem 8.3 contains a list of properties which can be distinguished under appropriate additional axiom to **ZFC**.

We made an effort to formulate essential results as Theorems and auxiliary results as Lemmas or Propositions.

2. Preliminary definitions

By $C_p(X,A)$ we denote the space of all continuous functions from X to A with topology of pointwise convergence (i.e., subspace topology of Tychonoff product topology on ${}^X\mathbb{R}$). We use symbol 0 to denote so number zero as function with constant zero value.

The notation "$f_n \xrightarrow{\mathrm{P}} f$ on X" or just "$f_n \to f$ on X" means that the sequence of functions $\langle f_n : n \in \omega\rangle$ converges pointwise on X to f. Á. Császár and M. Laczkovich [11] and independently Z. Bukovská [3] introduced and investigated quasi-normal convergence. A sequence of real-valued functions $\langle f_n : n \in \omega\rangle$ converges quasi-normally on X to f, written $f_n \xrightarrow{\mathrm{Q}} f$ on X, if there exists a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ (a control) converging to 0 such that for every $x \in X$ we have $|f_n(x) - f(x)| < \varepsilon_n$ for all but finitely many $n \in \omega$. A sequence of functions $\langle f_n : n \in \omega\rangle$ converges discretely on X to f, written $f_n \xrightarrow{\mathrm{D}} f$ on X, if for every $x \in X$ we have $f_n(x) = f(x)$ for all but finitely many $n \in \omega$. A sequence $\langle f_n : n \in \omega\rangle$ of real-valued functions on X converges monotonically to a function $f : X \to \mathbb{R}$, shortly $f_n \xrightarrow{\mathrm{M}} f$, if $f_n \xrightarrow{\mathrm{P}} f$ and $f_{n+1} \le f_n$ for any $n \in \omega$.

L. Bukovský, I. Recław and M. Repický [7] say that a topological space X is a QN-space if each sequence of continuous functions converging to 0 on X converges to 0 quasi-normally as well. X is a wQN-space if each sequence of continuous functions converging to 0 on X contains a subsequence that converges to 0 quasi-normally.

A topological space X is a σ-set if every F_σ subset of X is G_δ set in X. A topological space X is a λ-set if every countable subset of X is G_δ set in X. A subset A of a topological space X is called perfectly meager[2)] if for any perfect set $P \subseteq X$ the intersection $A \cap P$ is meager in the subspace P. For more about these notions see e.g. [28] or [4].

A function f is said to be upper semicontinuous if the set $f^{-1}((-\infty,r)) = \{x \in X : f(x) < r\}$ is open in a topological space X for every real number r. By $\min\{f_1,\dots,f_n\}$ and $\max\{f_1,\dots,f_n\}$ for real-valued functions $f_1,\dots,f_n$ on a topological space X we mean functions defined by $\min\{f_1,\dots,f_n\}(x) = \min\{f_1(x),\dots,f_n(x)\}$, $\max\{f_1,\dots,f_n\}(x) = \max\{f_1(x),\dots,f_n(x)\}$, respectively, for any $x \in X$.

[2)] or always of the first category

H. Ohta and M. Sakai [30] say that a topological space X has property USC (USC$_s$), if for any sequence $\langle f_n : n \in \omega\rangle$ of upper semicontinuous functions with values in $[0,1]$ converging to 0, there is a sequence $\langle g_m : m \in \omega\rangle$ of continuous functions converging to 0 (and an increasing sequence $\{n_m\}_{m=0}^{\infty}$ of natural numbers) such that $f_m \leq g_m,\ m \in \omega$ ($f_{n_m} \leq g_m,\ m \in \omega$). A topological space X has a property USC$_m$, if for any sequence $\langle f_n : n \in \omega\rangle$ of upper semicontinuous functions with values in $[0,1]$ converging to 0 monotonically, there is a sequence $\langle g_m : m \in \omega\rangle$ of continuous functions converging to 0 such that $f_m \leq g_m,\ m \in \omega$.

Let Y be a topological space and $A \subseteq Y$. By $\mathrm{scl}(A,Y)$ we denote a sequential closure of A in Y, i.e., a set $\{y \in Y;\ (\exists \{y_n\}_{n=0}^{\infty} \in {}^{\omega}A)\ y_n \to y\}$. D.H. Fremlin [14] investigates the following families:

$$\mathrm{scl}_0(A,Y) = A,$$

$$\mathrm{scl}_\alpha(A,Y) = \mathrm{scl}\left(\bigcup_{\beta<\alpha} \mathrm{scl}_\beta(A,Y), Y\right),$$

He says that a topological space X is an s_1-space if $\mathrm{scl}_{\omega_1}(\mathcal{F}, \mathrm{C}_p(X)) = \mathrm{scl}_1(\mathcal{F}, \mathrm{C}_p(X))$ for every $\mathcal{F} \subseteq \mathrm{C}_p(X)$.

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a cover of a topological space X if $X = \bigcup \mathcal{A}$ and $X \notin \mathcal{A}$. A cover $\mathcal{A}$ of X is an ω-cover if for any finite subset F of X there is $A \in \mathcal{A}$ such that $F \subseteq A$. An infinite cover $\mathcal{A}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$. A cover $\mathcal{V}$ is said to be a refinement of $\mathcal{A}$ if for any $V \in \mathcal{V}$ there is $U \in \mathcal{A}$ such that $V \subseteq U$. A γ-cover $\mathcal{A}$ is shrinkable if there exists a closed γ-cover $\mathcal{V}$ which is a refinement of $\mathcal{A}$.

J. Gerlits and Zs. Nagy [18] introduced the notion of a γ-set. A topological space X is a γ-set if any open ω-cover of X contains γ-subcover. By M. Scheepers [36] a topological space X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space if for every sequence $\langle \mathcal{A}_n : n \in \omega\rangle$ of open γ-covers there exist sets $U_n \in \mathcal{A}_n, n \in \omega$ such that $\{U_n;\ n \in \omega\}$ is a γ-cover.

W. Hurewicz [20] introduced and investigated properties E^* and E^{**} which are nowadays called Hurewicz properties.[3)] We say that a topological space X possesses Hurewicz property (or property $\mathcal{U}_{fin}(\mathcal{O},\Gamma)$ or is a Hurewicz space) if for any sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of countable open covers not containing a finite subcover there are finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n, n \in \omega$ such that $\{\bigcup \mathcal{V}_n;\ n \in \omega\}$ is a γ-cover.

For a property $\mathcal{A}$ of a topological space X we say that X is hereditarily $\mathcal{A}$-space, shortly h$\mathcal{A}$-space, or X possesses $\mathcal{A}$ hereditarily if any subset of X is an $\mathcal{A}$-space.

L. Bukovský, I. Recław and M. Repický [7] proved that any perfectly normal wQN-space has Hurewicz property and M. Scheepers [37], [39] proved that any $\mathrm{S}_1(\Gamma,\Gamma)$-space is a wQN-space. By J. Haleš [19] any perfectly normal QN-space is hereditarily $\mathrm{S}_1(\Gamma,\Gamma)$-space (any normal QN-space is an $\mathrm{S}_1(\Gamma,\Gamma)$-space by L. Bukovský and J. Haleš [6] or M. Sakai [35]). By J. Haleš [19] and M. Sakai [35] it follows that every open γ-cover of perfectly normal hereditarily $\mathrm{S}_1(\Gamma,\Gamma)$-space is shrinkable. We say that X is sh$\mathrm{S}_1(\Gamma,\Gamma)$-space if X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space and every open γ-cover of X is shrinkable.Thus for perfectly normal space we have

$$\mathrm{QN} \to \mathrm{hS}_1(\Gamma,\Gamma) \to \mathrm{shS}_1(\Gamma,\Gamma) \to \mathrm{S}_1(\Gamma,\Gamma) \to \mathrm{wQN} \to \mathcal{U}_{fin}(\mathcal{O},\Gamma). \tag{4}$$

3. Spaces satisfying selection principles

All investigated sequence seletion principles are meaningful. Actually, the author and L. Bukovský using Tsaban–Zdomskyy Theorem [40] showed in [9] that statements "X is a QN-space", "X has PQ($\mathcal{B}$,$\mathcal{B}$)" and "X has DD($\mathcal{B}$,$\mathcal{B}$)" are equivalent for any perfectly normal space X. Thus

Proposition 3.1 (L. Bukovský – J. Šupina). *Any perfectly normal* QN*-space has all properties* AB($\mathcal{F}$,$\mathcal{G}$) *and* wAB($\mathcal{F}$,$\mathcal{G}$).

[3)] Note that our definition of Hurewicz property corresponds to property E^{**}_{ω} of L. Bukovský and J. Haleš [5] rather than to original properties E^* and E^{**}.

Consequently, all introduced selection principles hold in any countable set of reals (in fact, in any set of reals of cardinality less than $\mathfrak{b}$, see L. Bukovský, I. Recław and M. Repický [7]).

However, the assumption about topological space to be perfectly normal in Proposition 3.1 is in many cases abundant. It was shown in [9] that any QN-space has property QQ($\mathcal{B}$,$\mathcal{B}$), therefore we obtain Proposition 3.2. Moreover, see Theorem 3.5 as well.

Proposition 3.2 (L. Bukovský – J. Šupina). *Any* QN*-space has selection property* AB($\mathcal{F}$,$\mathcal{G}$) *for each* A $\neq$ P *and each* B $\neq$ D*.*

Proposition 3.2 is one of two crucial assertions of [9].

L. Bukovský, I. Recław and M. Repický [7] proved that any $\mathfrak{b}$-Sierpiński set[4)] is a QN-set. Hence, we obtain the following.

Corollary 3.3. *Any* $\mathfrak{b}$*-Sierpiński set has all selection properties* AB($\mathcal{F}$,$\mathcal{G}$) *and* wAB($\mathcal{F}$,$\mathcal{G}$)*.*

Let $\mathcal{N}$ be a σ-ideal of all Lebesgue measure zero subsets of reals. If $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ there is a $\mathfrak{b}$-Sierpiński set of cardinality $\mathfrak{b}$. Moreover, if there is a $\mathfrak{b}$-Sierpiński set then $\mathrm{non}(\mathcal{N}) \leq \mathfrak{b} \leq \mathrm{cov}(\mathcal{N})$ (see e.g. [4]).

We prove that some weak sequence selection properties are satisfied in γ-sets.

Theorem 3.4. *Any γ-set has property* wAB($\mathcal{F}$,{0}) *for each* A *and each* B $\neq$ D*.*

Proof. We shall show that any γ-set has property wPQ($\mathcal{B}$,{0}). Let $f_{n,m}, f_n : X \to [0,1]$ be such that $f_{n,m}$ are continuous, $f_{n,m} \to f_n, n \in \omega$ and $f_n \to 0$. Moreover, let us pick a sequence $\{x_k\}_{k=0}^{\infty}$ of distinct points of X. We define

$$\mathcal{A} = \{f_{n,m}^{-1}((-\frac{1}{2^k}, \frac{1}{2^k})) \setminus \{x_k\};\ k \leq n, n, m, k \in \omega\}.$$

$\mathcal{A}$ is an ω-cover of X. Actually, for any $A \in [X]^{<\omega}$ and any $k \in \omega$ there is n_0 such that

$$x_n \notin A \wedge (x \in A \to |f_n(x)| < \frac{1}{2^k})$$

for any $n \geq n_0$. For $\varepsilon_n = \frac{1}{2^k} - \max\{|f_n(x)|;\ x \in A\}$ there is a sequence $\{m_n\}_{n=n_0}^{\infty}$ such that $|f_{n,m}(x) - f_n(x)| < \varepsilon_n$ for any $x \in A, m \geq m_n, n \geq n_0$. Hence, $|f_{n,m}(x)| < \frac{1}{2^k}$ for any $x \in A, m \geq m_n, n \geq n_0$.

Since X is γ-set there is a γ-cover $\mathcal{V} \subseteq \mathcal{A}$. Due to definition of $\mathcal{A}$ we obtain

$$|\mathcal{V} \cap \{f_{n,m}^{-1}((-\frac{1}{2^k}, \frac{1}{2^k})) \setminus \{x_k\};\ k \leq n, k < l, n, m, k \in \omega\}| < \aleph_0, l \in \omega.$$

Therefore there are increasing functions $\alpha, \gamma \in {}^{\omega}\omega$ and a function $\beta \in {}^{\omega}\omega$ such that

$$\{f_{\alpha(l),\beta(l)}^{-1}((-\frac{1}{2^{\gamma(l)}}, \frac{1}{2^{\gamma(l)}})) \setminus \{x_{\gamma(l)}\};\ l \in \omega\}$$

is a γ-cover. One can easily see that $f_{\alpha(l),\beta(l)} \xrightarrow{\mathrm{Q}} 0$ with the control $\{\frac{1}{2^{\gamma(l)}}\}_{l=0}^{\infty}$. $\square$

As we shall see later, Theorem 3.4 will help us to distinguish between a number of selection properties. Well-known set-theoretic assumption for the existence of a γ-set is presented in Section 8. Finally, by Theorem 3.5 some selection properties hold in arbitrary wQN-space.

Theorem 3.5. *Let X be a topological space.*

a) *X is a* QN*-space if and only if X has* PQ($\mathrm{C}_p(X)$,{0})*.*

4) A set $X \subseteq \mathbb{R}$ is $\mathfrak{b}$-Sierpiński set if $|X| \geq \mathfrak{b}$ and $|A \cap X| < \mathfrak{b}$ for any Lebesgue measure zero set $A \subseteq \mathbb{R}$. For more see e.g. [4].

b) *X is a* wQN*-space if and only if X has* PQ({0},{0}).

c) *X is a* wQN*-space if and only if X has* wPQ($C_p(X)$,{0}).

Proof. a) Let us assume that $f_{n,m}, f_n \in C_p(X)$ are such that $f_{n,m} \xrightarrow{P} f_n, n \in \omega$ and $f_n \xrightarrow{P} 0$. Then $f_{n,m} - f_n \xrightarrow{P} 0, n \in \omega$ and by PP({0},{0}) there is an unbounded $\beta \in {}^{\omega}\omega$ such that $f^n_{\beta(n)} - f_n \xrightarrow{P} 0$. Moreover, since X is a QN-space both convergences are quasi-normal, i.e., $f^n_{\beta(n)} - f_n \xrightarrow{Q} 0$ and $f_n \xrightarrow{Q} 0$. Then $f^n_{\beta(n)} = f^n_{\beta(n)} - f_n + f_n \xrightarrow{Q} 0$.

Let us assume that X has property PQ($C_p(X)$,{0}) and $f_n : X \to \mathbb{R}, n \in \omega$ are continuous functions such that $f_n \to 0$. Let $f_{n,m} = f_n$ for any $n, m \in \omega$. By PQ($C_p(X)$,{0}) there is $\beta \in {}^{\omega}\omega$ such that $f_n = f_{n,\beta(n)} \xrightarrow{Q} 0$. Thus X is a QN-space.

b) and c) If X is a wQN-space then X has PP({0},{0}) by D.H. Fremlin [15] and thus X has wPP($C_p(X)$,{0}) by Lemma 1.1. We prove that X has wPQ($C_p(X)$,{0}). Let $f_{n,m}, f_n : X \longrightarrow \mathbb{R}, n, m \in \omega$ be continuous functions such that $f_{n,m} \xrightarrow{P} f_n, n \in \omega$ and $f_n \xrightarrow{P} 0$ on X. By wPP($C_p(X)$,{0}) there is an unbounded $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \xrightarrow{P} 0$. Since X is a wQN-space there is an increasing $\alpha \in {}^{\omega}\omega$ such that $f_{\alpha(l),\beta(\alpha(l))} \xrightarrow{Q} 0$.

X has PP({0},{0}) if and only if X has wPP({0},{0}) by Lemma 1.1. Thus if X has PQ({0},{0}) or wPQ($C_p(X)$,{0}) then X has PP({0},{0}). However, M. Scheepers [39] showed that if X has PP({0},{0}) then X is a wQN-space.

Finally, we prove that if X has PP({0},{0}) then X has PQ({0},{0}). Let us assume that $f_{n,m} : X \longrightarrow \mathbb{R}$, $n, m \in \omega$ are continuous functions such that $f_{n,m} \xrightarrow{P} 0$ on X for every $n \in \omega$ and

$$g_{n,m} = 2^n \cdot f_{n,m}, n, m \in \omega.$$

By PP({0},{0}) there is $\beta \in {}^{\omega}\omega$ such that $g_{n,\beta(n)} \xrightarrow{P} 0$. Then $f_{n,\beta(n)} \xrightarrow{Q} 0$ with the control $\{\frac{1}{2^n}\}_{n=0}^{\infty}$. □

Hence, any QN-space has property AB($\mathcal{F}$,$\mathcal{G}$) for B ≠ D, (A, B) ≠ (P, Q), $\mathcal{F} \subseteq C_p(X)$ and any wQN-space has properties AB({0},{0}) for B ≠ D and wAB($\mathcal{F}$,{0}) for B ≠ D, $\mathcal{F} \subseteq C_p(X)$.

4. Properties of spaces with selection principles

Topological spaces satisfying investigated selection principles possess other interesting properties. The following was proved by L. Bukovský and J. Haleš [6] for normal space and property DP({0},{0}). We show that the assumption of normality is redundant.

Theorem 4.1. *A topological space X has property* wDP({0},{0}) *if and only if X is a* wQN*-space.*

Proof. If X is a wQN-space then X has wDP({0},{0}) by Theorem 3.5, b).

Let X have wDP({0},{0}). By Lemma 1.1, d) X has DP({0},{0}). Let $f_{n,m} : X \longrightarrow [0,1]$, $n, m \in \omega$ be continuous functions such that $f_{n,m} \xrightarrow{P} 0$ on X for every $n \in \omega$ and

$$g_{n,m} = \max\{\frac{1}{2^n}, f_{n,m}\} - \frac{1}{2^n}, n, m \in \omega.$$

Then $g_{n,m} \xrightarrow{D} 0$ for any $n \in \omega$ and by DP({0},{0}) there is an unbounded $\beta \in {}^{\omega}\omega$ such that $g_{n,\beta(n)} \xrightarrow{P} 0$.

Finally, $f_{n,\beta(n)} \xrightarrow{P} 0$. Actually, let $x \in X$ and $\varepsilon > 0$. There is n_0 such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$ and $g_{n,\beta(n)}(x) < \frac{\varepsilon}{2}$ for any $n \geq n_0$. Then

$$f_{n,\beta(n)}(x) \leq \max\{\frac{1}{2^n}, f_{n,\beta(n)}(x)\} < \frac{1}{2^n} + \frac{\varepsilon}{2} < \varepsilon$$

for any $n \geq n_0$ and the result follows. □

Hence, by (3) all investigated selection properties imply wQN-property.

Corollary 4.2. *A topological space with any of the selection properties* AB($\mathcal{F}$,$\mathcal{G}$) *or* wAB($\mathcal{F}$,$\mathcal{G}$) *is a* wQN*-space.*

Consequently we obtain the following assertion about dimension. J. Haleš [19] says that a topological space X is an nCM-space (or non-continuously mappable space) if X cannot be continuously mapped onto [0,1]. It is well-known that any normal nCM-space X has $\mathrm{Ind}(X) = 0$ (see e.g. [4], Theorem 8.22).

Corollary 4.3. *Any topological space which has any of the properties* AB($\mathcal{F}$,$\mathcal{G}$) *or* wAB($\mathcal{F}$,$\mathcal{G}$) *is an* nCM*-space. Hence,* $\mathrm{Ind}(X) = 0$ *for any normal space X which has any of the selection properties* AB($\mathcal{F}$,$\mathcal{G}$) *or* wAB($\mathcal{F}$,$\mathcal{G}$)*.*

L. Bukovský, I. Recław and M. Repický [7] proved that a wQN-subspace of metric separable space is perfectly meager.

Corollary 4.4. *A subset of metric separable space which has any of the selection properties* AB($\mathcal{F}$,$\mathcal{G}$) *or* wAB($\mathcal{F}$,$\mathcal{G}$) *is perfectly meager.*

L. Bukovský, I. Recław and M. Repický [7] showed that every perfectly normal space of less cardinality than $\mathfrak{b}$ is a QN-space and there is a set of reals of cardinality $\mathfrak{b}$ which is not a wQN-space. non(AB($\mathcal{F}$,$\mathcal{G}$)), non(wAB($\mathcal{F}$,$\mathcal{G}$)) denote the minimal cardinality of a perfectly normal space which does not have AB($\mathcal{F}$,$\mathcal{G}$) or wAB($\mathcal{F}$,$\mathcal{G}$), respectively. Hence, by Proposition 3.1 and Theorem 4.1 we obtain

Corollary 4.5. non(AB($\mathcal{F}$,$\mathcal{G}$)) = non(wAB($\mathcal{F}$,$\mathcal{G}$)) = $\mathfrak{b}$

By Proposition 3.1 all selection properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) hold in perfectly normal QN-space. Vice versa, Theorem 4.6 says that arbitrary topological space which has some of them is a QN-space. The author and L. Bukovský proved in [9] that a perfectly normal space X is a QN-space if and only if X has DD({0},{0}). We prove the following.

Theorem 4.6. *Let X be a topological space, $\mathcal{F}$, $\mathcal{G}$ being families of functions.*

a) *If* $\mathrm{C}_p(X) \subseteq \mathcal{F}$ *and X has property* PQ($\mathcal{F}$,$\mathcal{G}$) *then X is a* QN*-space.*

b) *If X has property* wDD($\mathcal{F}$,$\mathcal{G}$) *then X is a* QN*-space.*

Proof. a) follows by Theorem 3.5.

b) Let us assume that X has property wDD({0},{0}), $f_m : X \longrightarrow [0,1]$, $m \in \omega$ are continuous functions such that $f_m \xrightarrow{\mathrm{P}} 0$ on X and

$$g_{n,m} = \max\{\frac{1}{2^n}, f_m\} - \frac{1}{2^n}, n, m \in \omega.$$

Then $g_{n,m} \xrightarrow{\mathrm{D}} 0$ for any $n \in \omega$. Let

$$f_{n,m} = \sum_{k=m}^{\infty} \min\{\frac{1}{2^{k+1}}, g_{n,k}\}, n, m \in \omega.$$

Then $f_{n,m} \xrightarrow{\mathrm{D}} 0$ for any $n \in \omega$. Since X has DD({0},{0}) by part d) of Lemma 1.1, there is an increasing $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{D}} 0$.

Let $\varepsilon_m = 1$ for $m < \beta(0)$ and $\varepsilon_m = \frac{1}{2^n}$ for $\beta(n) \leq m < \beta(n+1)$. Then $f_m \xrightarrow{\mathrm{Q}} 0$ with the control $\{\varepsilon_m\}_{m=0}^{\infty}$. Actually, let $x \in X$. There is n_0 such that $f_{n,\beta(n)}(x) = 0$ for any $n \geq n_0$. Moreover, $g_{n,k}(x) = 0$ for any $k \geq \beta(n), n \geq n_0$. Then

$$f_k(x) \leq \max\{\frac{1}{2^n}, f_k(x)\} = \frac{1}{2^n}$$

for any $k \geq \beta(n), n \geq n_0$ and the result follows. □

Consequently, by Proposition 3.2 and Theorem 4.6 we obtain the position of QN-space in preordered set of all properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) for arbitrary topological space. Moreover, we obtain the following relations among investigated sequence selection properties for arbitrary topological space.

Corollary 4.7. *Let X be a topological space. If X has* $\mathrm{PQ}(\mathrm{C}_p(X),\{0\})$ *or* $\mathrm{wDD}(\{0\},\{0\})$ *then X has* $\mathrm{QQ}(\mathcal{B},\mathcal{B})$.

The following assertion was proved in [9] for properties $\mathrm{DP}(\mathcal{B},\mathcal{B})$ and $\mathrm{DP}(\mathcal{B},\{0\})$. One may easily see that property $\mathrm{wDP}(\mathcal{U},\mathcal{B})$ could be used in the proof.

Proposition 4.8 (L. Bukovský – J. Šupina). *If a perfectly normal topological space X has* $\mathrm{wDP}(\mathcal{U},\mathcal{B})$ *or* $\mathrm{DP}(\mathcal{B},\{0\})$ *then X is a σ-set.*

Proof of the second part of Proposition 4.8 employees a characterization of σ-set for perfectly normal space. It is called (γ,γ)-shrinkable space and is due to M. Sakai [35]. A space X is (γ,γ)-shrinkable if for every open γ-cover $\langle U_n : n \in \omega\rangle$ of X there exists a closed γ-cover $\langle F_n : n \in \omega\rangle$ of X such that $F_n \subseteq U_n$ for each $n \in \omega$. H. Ohta and M. Sakai [30] proved that a normal space X is (γ,γ)-shrinkable if and only if X possesses property USC.

Almost the same proof as the one of the second part of Proposition 4.8 using property $\mathrm{DP}(\mathcal{U},\{0\})$ gives part a) of Proposition 4.9 and using property $\mathrm{wDP}(\mathcal{U},\{0\})$ gives part b) of the same theorem. The second part of b) follows by Corollary 2.11 in [30], since H. Ohta and M. Sakai showed that every open γ-cover of a normal space X is shrinkable if and only if X possesses property USC_s.

Proposition 4.9. *Let X be a perfectly normal space, $\mathcal{G}$ being any family of functions, $\mathcal{U} \subseteq \mathcal{F}$.*

a) *If X has property* $\mathrm{AB}(\mathcal{F},\mathcal{G})$ *then X is (γ,γ)-shrinkable. Hence, X is a σ-set and possesses property* USC.
b) *If X has property* $\mathrm{wAB}(\mathcal{F},\mathcal{G})$ *then every open γ-cover of X is shrinkable. Hence, X possesses property* USC_s.

Proof. a) Let X have property $\mathrm{DP}(\mathcal{U},\{0\})$, $\langle U_n : n \in \omega\rangle$ being an open γ-cover of X. Then $\chi_{X\setminus U_n} \xrightarrow{\mathrm{D}} 0$. Since X is a perfectly normal space there is a non-decreasing sequence $\langle F_{n,m} : m \in \omega\rangle$ of closed sets such that $\bigcup_{m=0}^{\infty} F_{n,m} = U_n$ and continuous functions $f_{n,m} : X \longrightarrow [0,1]$ such that $f_{n,m}(x) = 0$ for $x \in F_{n,m}$ and $f_{n,m}(x) = 1$ for $x \in X \setminus U_n$. Then $f_{n,m} \xrightarrow{\mathrm{D}} \chi_{U_n}$. By $\mathrm{DP}(\mathcal{U},\{0\})$ there is an unbounded $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{P}} 0$. If we denote $F_n = f^{-1}_{n,\beta(n)}([0,\frac{1}{2}])$, then $F_n \subseteq U_n, n \in \omega$ are closed sets and $\langle F_n : n \in \omega\rangle$ is a γ-cover of X. Thus X is (γ,γ)-shrinkable space and by M. Sakai [35] a σ-set. Finally, X has USC by H. Ohta and M. Sakai [30].

b) Similarly to a). □

A perfectly normal space with properties $\mathrm{wAB}(\mathcal{U},\{0\})$ is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.

Proposition 4.10. *If $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{G}$ is any family of functions then a perfectly normal space X with the property* $\mathrm{wAB}(\mathcal{F},\mathcal{G})$ *is an* $\mathrm{S}_1(\Gamma,\Gamma)$*-space.*

Proof. We shall show that a perfectly normal space X with the property $\mathrm{wDP}(\mathcal{U},\{0\})$ is an $\mathrm{S}_1(\Gamma,\Gamma)$-space. Let $\langle U_{n,m} : m \in \omega\rangle$ be a γ-cover for any $n \in \omega$. By Proposition 4.9 there is a γ-cover $\langle F_{n,m} : m \in \omega\rangle$ of closed sets and increasing functions $\alpha_n \in {}^{\omega}\omega$ for any $n \in \omega$ such that $F_{n,m} \subseteq U_{n,\alpha_n(m)}$. Therefore there are continuous functions $f_{n,m} : X \to [0,1]$ such that $F_{n,m} = f^{-1}_{n,m}(0)$ and $X \setminus U_{n,\alpha_n(m)} = f^{-1}_{n,m}(1)$. Then $f_{n,m} \xrightarrow{\mathrm{D}} 0$ for any $n \in \omega$. Since X has $\mathrm{DP}(\{0\},\{0\})$ by (1) and Lemma 1.1, c), there is $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \to 0$. A family $\{U_{n,\alpha_n(\beta(n))};\ n \in \omega\}$ is a γ-cover of X. □

J. Haleš [19] proved that a perfectly normal space X is hereditarily $\mathrm{S}_1(\Gamma,\Gamma)$-space if and only if X is both, a σ-set and an $\mathrm{S}_1(\Gamma,\Gamma)$-space. This holds for Tychonoff space as well, see T. Orenshtein and B. Tsaban [32]. Consequently, by Propositions 4.9 - 4.10 we obtain Corollary 4.11. It was observed already in [9] for property $\mathrm{DP}(\mathcal{B},\{0\})$.

Corollary 4.11. *If $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{G}$ is any family of functions then a perfectly normal space with* $\mathrm{AB}(\mathcal{F},\mathcal{G})$ *or* $\mathrm{wAB}(\mathcal{F},\mathcal{B})$ *is hereditarily* $\mathrm{S}_1(\Gamma,\Gamma)$*-space.*

A proof of following lemma was motivated by L. Bukovský, I. Recław and M. Repický [7] and their Theorem 4.1. For the purpose of following two assertions the notation $\{0\}(X)$ means a set containing only zero function on X.

Lemma 4.12. *Let C be F_σ-subset of a perfectly normal space X and $\mathcal{F} \in \{\{0\}, \mathcal{U}, \mathcal{B}\}$, $\mathcal{G} \in \{\{0\}, \mathcal{B}\}$. If X has* AB($\mathcal{F}(X)$,$\mathcal{G}(X)$) *then C has* AB($\mathcal{F}(C)$,$\mathcal{G}(C)$) *as well. If X has* wAB($\mathcal{F}(X)$,$\mathcal{G}(X)$) *then A has* wAB($\mathcal{F}(C)$,$\mathcal{G}(C)$) *as well.*

Proof. Let us assume that $F_n, n \in \omega$ are closed subsets of X such that $F_n \subseteq F_{n+1}, n \in \omega$ and $C = \bigcup_{n\in\omega} F_n$, $G_{n,m}, n, m \in \omega$ are open subsets of X such that $G_{n,m+1} \subseteq G_{n,m}, n, m \in \omega$ and $F_n = \bigcap_{m\in\omega} G_{n,m}$. Let $f_{n,m} \in \mathrm{C}_p(C)$, $f_n \in \mathcal{F}(C)$ and $f \in \mathcal{G}(C)$ be such that $f_{n,m} \xrightarrow{\mathrm{A}} f_n, n \in \omega$ and $f_n \xrightarrow{\mathrm{A}} f$.

We define $g_{n,m} \in \mathrm{C}_p(X)$, $g_n \in \mathcal{F}(X)$ and $g \in \mathcal{G}(X)$ such that $g_{n,m} \xrightarrow{\mathrm{A}} g_n, n \in \omega$ and $g_n \xrightarrow{\mathrm{A}} g$. Let $g(x) = 0$ for $x \in X \setminus C$ and $g(x) = f(x)$ for $x \in C$. Similarly, let $g_n(x) = 0$ for $x \in X \setminus F_n$ and $g_n(x) = f_n(x)$ for $x \in F_n$. Finally, by Tietze–Urysohn Theorem (see e.g. [13], Theorem 2.1.8) there are continuous functions $g_{n,m}, n, m \in \omega$ on X such that $g_{n,m}(x) = 0$ for $x \in X \setminus G_{n,m}$ and $g_{n,m}(x) = f_{n,m}(x)$ for $x \in F_n$. One can easily see that functions $g_{n,m}, g_n, g$ possess aforementioned properties.

By AB($\mathcal{F}$,$\mathcal{G}$) there is $\beta \in {}^\omega\omega$ such that $g_{n,\beta(n)} \xrightarrow{\mathrm{B}} g$. The sequence $\langle f_{n,\beta(n)} : n \in \omega\rangle$ is the desired one since for any $x \in C$ there is n_x such that $g_{n,\beta(n)}(x) = f_{n,\beta(n)}(x)$ for any $n \geq n_x$.

Similarly for wAB($\mathcal{F}$,$\mathcal{G}$). □

The following is motivated by a proof of Theorem 4 in a paper [19] by J. Haleš. We show that some stronger selection properties are hereditary which is not surprising after one has seen Proposition 4.8.

Theorem 4.13. *Let $\mathcal{G} \in \{\{0\}, \mathcal{B}\}$. Any metric space X has* AB($\mathcal{B}$,$\mathcal{G}$) *if and only if X has* AB($\mathcal{B}$,$\mathcal{G}$) *hereditarily. Any metric space X has* wAB($\mathcal{B}$,$\mathcal{B}$) *if and only if X has* wAB($\mathcal{B}$,$\mathcal{B}$) *hereditarily.*

Proof. Let C be a subset of X, $f_{n,m} \in \mathrm{C}_p(C)$, $f_n : C \to \mathbb{R}$ and $f \in \mathcal{G}(C)$ being such that $f_{n,m} \xrightarrow{\mathrm{A}} f_n, n \in \omega$ and $f_n \xrightarrow{\mathrm{A}} f$. By Kuratowski Theorems (see e.g. [24], §31, I and VI, Théorème) there are G_δ-set $G \supseteq C$, continuous extensions $g_{n,m} : G \to \mathbb{R}$ of $f_{n,m}$ for all $n, m \in \omega$ and Borel extensions $g_n, g : X \to \mathbb{R}$ of f_n, f, respectively (take the zero function on X if f is zero function on C).

We shall firstly consider A = P. Let

$$E_{\mathrm{P}} = \{x \in G;\ g_n(x) \to g(x) \wedge (\forall n \in \omega)\ g_{n,m}(x) \to g_n(x)\}.$$

E_{P} is Borel subset of X and $C \subseteq E_{\mathrm{P}} \subseteq G$. By Proposition 4.8 any Borel subset of X is F_σ-set, so E_{P} is F_σ-set. By Lemma 4.12 there is $\beta \in {}^\omega\omega$ such that $g_{n,\beta(n)} \xrightarrow{\mathrm{B}} g$ on E_{P}. Hence, $f_{n,\beta(n)} \xrightarrow{\mathrm{B}} f$ on C.

To prove the assertion for A $\in \{\mathrm{D}, \mathrm{Q}\}$ one can use sets E_{D} and E_{Q} instead of E_{P}, where

$$\begin{aligned}
E_{\mathrm{D}} =& \{x \in G;\ (\exists n_x)(\forall n \geq n_x)\ g_n(x) = g(x) \wedge (\forall n \in \omega)(\exists m_{n,x})(\forall m \geq m_{n,x})\ g_{n,m}(x) = g_n(x)\}\\
E_{\mathrm{Q}} =& \{x \in G;\ (\exists n_x)(\forall n \geq n_x)\ |g_n(x) - g(x)| < \varepsilon_n\\
&\wedge (\forall n \in \omega)(\exists m_{n,x})(\forall m \geq m_{n,x})\ |g_{n,m}(x) - g_n(x)| < \varepsilon_{n,m}\}
\end{aligned}$$

and sequences $\{\varepsilon_{n,m}\}_{m=0}^{\infty}, n \in \omega$, $\{\varepsilon_n\}_{n=0}^{\infty}$ of reals are controls of quasi-normal convergences of sequences $\langle f_{n,m} : m \in \omega\rangle, n \in \omega$, $\langle f_n : n \in \omega\rangle$, respectively.

Similarly for wAB($\mathcal{B}$,$\mathcal{B}$). □

Note that by results of Section 5 some other selection properties are hereditary in the sense of Lemma 4.12 or Theorem 4.13. It follows by properties of wQN-space, QN-space and hereditarily $\mathrm{S}_1(\Gamma, \Gamma)$-space.

5. Equivalences

We shall see that sequence selection properties related to continuous functions can be completely described by quasi-normal convergence properties of a sequence of continuous functions. If $\mathcal{F}, \mathcal{G} \in \{\mathrm{C}_p(X), \{0\}\}$ then properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) are equivalent either to QN-property or to wQN-property. By (1) and Diagram 1 this can be strengthen for other families of continuous functions. However, if we consider families of functions containing sufficient amount of other Borel functions, then we very often do not know how many nonequivalent classes of properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) there are. By proposition 8.1 it is possible consistently with **ZFC**, that there is only one class. Theorem 8.3 contains a list of properties which can be distinguished if additional hypotheses is added to **ZFC**.

By (2), Diagram 1, Proposition 3.1 and Theorem 4.6 we obtain

Proposition 5.1. *Let X be a perfectly normal space, $\mathcal{F}$, $\mathcal{G}$ being families of functions.*

a) *X has* DD($\mathcal{F}$,$\mathcal{G}$) *if and only if X is a* QN-*space.*
b) *X has* wDD($\mathcal{F}$,$\mathcal{G}$) *if and only if X is a* QN-*space.*

Large group of properties related only to continuous functions is in equivalence class of wQN. By Theorems 4.1, 3.5 and Diagram 3 we obtain the following.

Proposition 5.2. *Let X be a topological space,* A, B *being any of* P, Q, D *except* B = D. *Then*

$$X \text{ satisfies } \mathrm{AB}(\{0\},\{0\}) \text{ if and only if } X \text{ is a wQN-space.}$$

Note that by Proposition 5.2 wQN can be added to equivalences of part d) of Lemma 1.1 except B = D.

Naturally, properties PQ($\mathcal{F}$,$\mathcal{G}$) resemble stronger versions of QN-property. L. Bukovský, I. Recław and M. Repický [8] say that a topological space X is an $\overline{\mathrm{m}}$QN-space if each sequence of continuous functions converging monotonically to a function $f : X \to [0,1]$ (not necessarily continuous) converges to f quasi-normally as well. By Proposition 3.1 and Theorem 4.6 we obtain

Proposition 5.3. *Let X be a perfectly normal space, $\mathcal{F}$, $\mathcal{G}$ being families of functions such that* $\mathrm{C}_p(X) \subseteq \mathcal{F}$.

a) *X has* PQ($\mathcal{F}$,$\mathcal{G}$) *if and only if X is a* QN-*space.*
b) *If $\mathcal{U} \subseteq \mathcal{G}$ then X has* wPQ($\mathcal{F}$,$\mathcal{G}$) *if and only if X is a* QN-*space.*

Proof. By Proposition 3.1 if X is a QN-space then X has any of PQ($\mathcal{F}$,$\mathcal{G}$) or wPQ($\mathcal{F}$,$\mathcal{G}$).

a) If X has PQ($\mathcal{F}$,$\mathcal{G}$) then X has PQ($\mathrm{C}_p(X)$,{0}) and thus X is a QN-space by Theorem 4.6.

b) If X has wPQ($\mathcal{F}$,$\mathcal{G}$) then X has wPQ($\mathrm{C}_p(X)$,$\mathcal{U}$). We shall show that X is an $\overline{\mathrm{m}}$QN-space. Then X is a QN-space by Theorem 5.10 in L. Bukovský, I. Recław and M. Repický [8].[5)]

Let us assume that X has property wPQ($\mathrm{C}_p(X)$,$\mathcal{U}$) and $f_n : X \to \mathbb{R}, n \in \omega$ are continuous functions such that $f_n \to f$ for $f : X \to [0,1]$ and $f_{n+1} \leq f_n, n \in \omega$. Then f is upper semicontinuous (see e.g. [13], 1.7.14 (a)). Let $f_{n,m} = f_n$ for any $n, m \in \omega$. By wPQ($\mathrm{C}_p(X)$,$\mathcal{U}$) there is $\beta \in {}^{\omega}\omega$ such that $f_{\beta(n)} \xrightarrow{\mathrm{Q}} f$. Finally, $f_n \xrightarrow{\mathrm{Q}} f$ as well. $\square$

In fact the following was proved in [9].

Proposition 5.4 (L. Bukovský – J. Šupina). *For a perfectly normal space X the following are equivalent.*

a) *X is a* QN-*space.*
b) *X has* QQ($\mathcal{B}$,$\mathcal{B}$).
c) *X has* wQQ($\mathcal{B}$,$\mathcal{B}$).

Proof. a) → b) by Theorem 16 in [9].

b) → c) is trivial.

c) → a) by a proof similar to that of (3) → (1) in [9], Theorem 16. One can go through that proof and use property wQQ($\mathcal{B}$,$\mathcal{B}$) instead of property QQ($\mathcal{B}$,$\mathcal{B}$). Our Corollary 4.2 and Proposition 4.8 are necessary. $\square$

Similarly to Lemma 14 in [10] we can prove

5) Note that their definition of an $\overline{\mathrm{m}}$QN-space uses arbitrary real-valued functions. Hovewer, they proved in fact that an $\overline{\mathrm{m}}$QN-space in our sense is a characterization of a QN-space.

Proposition 5.5. *Let X be a topological space, $\{0\} \subseteq \mathcal{F} \subseteq \mathcal{B}$ being closed under multiplication by a positive real. X has* DP($\mathcal{F}$,{0}) *if and only if X has* DQ($\mathcal{F}$,{0}) *and X has* wDP($\mathcal{F}$,{0}) *if and only if X has* wDQ($\mathcal{F}$,{0}).

Proof. Let $f_{n,m}, f_n : X \to \mathbb{R}$ be such that $f_{n,m}$ are continuous, $f_{n,m} \xrightarrow{\mathrm{D}} f_n, n \in \omega$ and $f_n \xrightarrow{\mathrm{D}} 0$. If we consider functions $g_{n,m} = 2^{n+1} f_{n,m}, n, m \in \omega$ and $g_n = 2^{n+1} f_n, n \in \omega$ then $g_{n,m} \xrightarrow{\mathrm{D}} g_n, n \in \omega$ and $g_n \xrightarrow{\mathrm{D}} 0$. By DP($\mathcal{B}$,{0}) there is $\beta \in {}^\omega\omega$ such that $g_{n,\beta(n)} \to 0$. Hence, $f_{n,\beta(n)} \xrightarrow{\mathrm{Q}} 0$ with the control $\{2^{-(n+1)}\}_{n=0}^\infty$. Similarly the latter case. □

By Lemmas 14 and 15 of [10] we obtain

Proposition 5.6 (L. Bukovský – J. Šupina). *For a normal space X the following statements are equivalent:*

X has QQ($\mathcal{B}$,{0}), *X has* QP($\mathcal{B}$,{0}), *X has* DQ($\mathcal{B}$,{0}), *X has* DP($\mathcal{B}$,{0}).

By latter lemmas of [10] we have the same also for weak properties.

Proposition 5.7 (L. Bukovský – J. Šupina). *For a normal space X the following statements are equivalent:*

X has wQQ($\mathcal{B}$,{0}), *X has* wQP($\mathcal{B}$,{0}), *X has* wDQ($\mathcal{B}$,{0}), *X has* wDP($\mathcal{B}$,{0}).

Moreover, family $\mathcal{G}$ in properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) can be substituted by simple {0} in case of $\mathcal{G}$ being a family of suitable limits of continuous functions. More precisely

Proposition 5.8. *Let us assume that X is a topological space, $\mathcal{F}, \mathcal{G}$ are families of functions on X and*

i) *$\mathcal{F}$ is closed under subtraction of continuous functions,*

ii) *for any $g \in \mathcal{G}$ there is a sequence $\langle g_n : n \in \omega \rangle$ of continuous functions on X such that $g_n \xrightarrow{\mathrm{C}} g$,*

iii) *for any sequence $\langle g_n : n \in \omega \rangle$ of continuous functions on X if $g_n \xrightarrow{\mathrm{C}} g$ then $g_n \xrightarrow{\mathrm{A}} g$ and $g_n \xrightarrow{\mathrm{B}} g$.*

Then

a) *X has* AB($\mathcal{F}$,{0}) *if and only if X has* AB($\mathcal{F}$,$\mathcal{G}$).

b) *X has* wAB($\mathcal{F}$,{0}) *if and only if X has* wAB($\mathcal{F}$,$\mathcal{G}$).

Proof. a) Let us assume that X has AB($\mathcal{F}$,{0}) and functions $f_{n,m} \in \mathrm{C}_p(X)$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$ are such that $f_{n,m} \xrightarrow{\mathrm{A}} f_n, n \in \omega$ and $f_n \xrightarrow{\mathrm{A}} f$. By ii) there is a sequence $\langle g_n : n \in \omega \rangle$ of continuous functions on X such that $g_n \xrightarrow{\mathrm{C}} f$. Then $f_{n,m} - g_n \xrightarrow{\mathrm{A}} f_n - g_n, n \in \omega$ and by iii) also $f_n - g_n \xrightarrow{\mathrm{A}} 0$. Moreover, $f_n - g_n \in \mathcal{F}$ by i). Thus by AB($\mathcal{F}$,{0}) there is $\beta \in {}^\omega\omega$ such that $f_{n,\beta(n)} - g_n \xrightarrow{\mathrm{B}} 0$. Finally, by iii) we obtain $f_{n,\beta(n)} = f_{n,\beta(n)} - g_n + g_n \xrightarrow{\mathrm{B}} f$.
b) Similarly to a). □

Let us recall that the family of all F_σ-measurable functions on a perfectly normal space corresponds to the family of all pointwise limits of continuous functions. Consequently we have

Corollary 5.9. *Let X be a perfectly normal space. Then X has* PP($\mathcal{B}$,{0}) *if and only if X has* PP($\mathcal{B}$,$\mathcal{B}$).

Proof. Perfectly normal space X with PP($\mathcal{B}$,{0}) is a σ-set by Proposition 4.9. Thus any Borel function on X is F_σ-measurable and so a pointwise limit of continuous functions. □

By results of this section, Diagrams 2 and 3 can be simplified to Diagram 4. Similar diagrams to Diagrams 2 - 3 for weak properties can be simplified to Diagram 5. Every selection property AB($\mathcal{F}$,$\mathcal{G}$) or wAB($\mathcal{F}$,$\mathcal{G}$) for $\mathcal{F}, \mathcal{G} \in \{\mathcal{B}, \mathrm{C}_p(X), \{0\}\}$ belongs to the equivalence class of some sequence selection property in Diagrams 4 - 5. We do not know if there are any properties in Diagrams 4 - 5 which are equivalent. For those, which cannot be equivalent, see Theorem 8.3.

$$\begin{tikzcd}
\mathrm{QN} \arrow[rrrr] \arrow[ddd] & & & & \mathrm{DQ}(\mathcal{B},\mathcal{B}) \arrow[ddd] \\
& & \mathrm{wQN} & & \\
& & & \mathrm{DP}(\mathcal{B},\{0\}) \arrow[ul] & \\
\mathrm{PP}(\mathcal{B},\mathcal{B}) \arrow[r] & \mathrm{QP}(\mathcal{B},\mathcal{B}) \arrow[rrr] & & & \mathrm{DP}(\mathcal{B},\mathcal{B}) \arrow[ul]
\end{tikzcd}$$

Diagram 4

$$\begin{tikzcd}
\mathrm{QN} \arrow[rrrrr] \arrow[ddd] \arrow[dr] & & & & & \mathrm{wDQ}(\mathcal{B},\mathcal{B}) \arrow[ddd] \\
& \mathrm{wPQ}(\mathcal{B},\{0\}) \arrow[d] & & \mathrm{wQN} & & \\
& \mathrm{wPP}(\mathcal{B},\{0\}) \arrow[rrrr] & & & \mathrm{wDP}(\mathcal{B},\{0\}) \arrow[ul] & \\
\mathrm{wPP}(\mathcal{B},\mathcal{B}) \arrow[ur] \arrow[rr] & & \mathrm{wQP}(\mathcal{B},\mathcal{B}) \arrow[rrr] & & & \mathrm{wDP}(\mathcal{B},\mathcal{B}) \arrow[ul]
\end{tikzcd}$$

Diagram 5

In the following we proceed with presenting the equivalences among some of the sequence selection properties related to upper semicontinuous functions. We found helpful the properties introduced by H. Ohta and M. Sakai [30].

Lemma 5.10. *Let X be a topological space.*

a) *If X has properties* USC *and* $\mathrm{PP}(\mathrm{C}_p(X),\{0\})$ *then X has* $\mathrm{PP}(\mathcal{U},\{0\})$.

b) *If X has properties* USC_s *and* $\mathrm{wPQ}(\mathrm{C}_p(X),\{0\})$ *then X has* $\mathrm{wPQ}(\mathcal{U},\{0\})$.

Proof. a) Let $f_{n,m} : X \to [0,1]$ be continuous and $f_n : X \to [0,1]$ be upper semicontinuous functions such that $f_{n,m} \to f_n, n \in \omega$ and $f_n \to 0$. By USC there are continuous functions $g_n : X \to [0,1]$ such that $g_n \to 0$ and $f_n \le g_n, n \in \omega$. We define continuous functions $f_{n,m} : X \to [0,1]$ by

$$g_{n,m} = \max\{f_{n,m}, g_n\}.$$

We have $g_{n,m} \to g_n$ for any $n \in \omega$. By $\mathrm{PP}(\mathrm{C}_p(X),\{0\})$ there is an unbounded $\beta \in {}^\omega\omega$ such that $g_{n,\beta(n)} \to 0$. Hence, $f_{n,\beta(n)} \to 0$ as well.

b) Similarly to a). □

Already mentioned results by J. Haleš [19] and H. Ohta and M. Sakai [30] have the following consequences.

Theorem 5.11. *Let X be a perfectly normal space,* A, B *being any of* P, Q, D *except* B = D.

a) *If* (A, B) ≠ (P, Q) *then X has* AB($\mathcal{U}$,{0}) *if and only if X is hereditarily* $S_1(\Gamma,\Gamma)$*-space.*

b) *X has* wAB($\mathcal{U}$,{0}) *if and only if X is an* $S_1(\Gamma,\Gamma)$*-space and every open γ-cover of X is shrinkable.*

Proof. a) If X has DP($\mathcal{U}$,{0}) then X is hereditarily $S_1(\Gamma,\Gamma)$-space by Corollary 4.11.

If X is hereditarily $S_1(\Gamma,\Gamma)$-space then X has USC by J. Haleš [19] and H. Ohta and M. Sakai [30]. Moreover, X has PP($C_p(X)$,{0}) by (4), Lemma 1.1, d) and Proposition 5.2. Hence, X has PP($\mathcal{U}$,{0}) by Lemma 5.10.

If X has QP($\mathcal{U}$,{0}) then X has QQ($\mathcal{U}$,{0}) by a slight modification of a proof of Lemma 15 in [10] (assumption of Lemma 15 in [10] is not satisfied by our family $\mathcal{U}$ but can be appropriately weakened).

b) If every open γ-cover of X is shrinkable then X has USC_s by H. Ohta and M. Sakai [30]. If X is an $S_1(\Gamma,\Gamma)$-space then by (4) and Theorem 3.5 X has wPQ($C_p(X)$,{0}). Hence, if X is an $S_1(\Gamma,\Gamma)$-space and every open γ-cover of X is shrinkable then X has wPQ($\mathcal{U}$,{0}) by Lemma 5.10.

If X has wAB($\mathcal{U}$,{0}) then X is an $S_1(\Gamma,\Gamma)$-space by Proposition 4.10 and every open γ-cover of X is shrinkable by Proposition 4.9, b). □

6. Monotonic convergence

As it is presented in the preceding section, sequence selection properties AB($\mathcal{F}$,$\mathcal{G}$) and wAB($\mathcal{F}$,$\mathcal{G}$) can describe other properties of a topological space, namely to be a QN-space, to be a wQN-space, to be hereditarily $S_1(\Gamma,\Gamma)$-space and to be an $S_1(\Gamma,\Gamma)$-space with every open γ-cover shrinkable. M. Scheepers [37] found characterization of Hurewicz property by a sequence selection principle called monotonic sequence selection property. Therefore it is not so surprising that if we add monotonic convergence to the list of convergences in definitions of introduced sequence selection properties we can describe some additional topological properties by such selection principles. Theorem 6.5 motivates us to define slightly more general sequence selection properties with added monotonic convergence but for restricted ranges of functions. However, we proceed in investigation of these properties only in cases of monotonic and pointwise convergences.

Let A, B, C denote one of the following types of convergence: P pointwise, Q quasi–normal, D discrete, M monotonic. Let $\mathcal{F},\mathcal{G} \subseteq {}^X[0,1]$ be families of functions containing the zero function on X. We say that X has the **sequence selection property ABC($\mathcal{F}$,$\mathcal{G}$)**, if for any functions $f_{n,m} \in C_p(X,[0,1])$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$, $n, m \in \omega$, such that

a) $f_{n,m} \xrightarrow{\text{A}} f_n$ on X for every $n \in \omega$,

b) $f_n \xrightarrow{\text{B}} f$ on X,

there exists an unbounded $\beta \in {}^\omega\omega$ such that

$$f_{n,\beta(n)} \xrightarrow{\text{C}} f \text{ on X.}$$

If there exist an increasing $\alpha \in {}^\omega\omega$ and an unbounded $\beta \in {}^\omega\omega$ such that $f_{\alpha(n),\beta(n)} \xrightarrow{\text{C}} f$ on X then we say that topological space X has the **weak sequence selection property wABC($\mathcal{F}$,$\mathcal{G}$)**.

Note that properties ABM($\mathcal{F}$,$\mathcal{G}$) and wABM($\mathcal{F}$,$\mathcal{G}$) are meaningless, i.e., there is no topological space which has any of them. Thus there are three versions of properties ABP($\mathcal{F}$,$\mathcal{G}$) with monotonic and pointwise convergences for fixed families $\mathcal{F},\mathcal{G}$. Relations among them which raise from relations of monotonic and pointwise convergences are described by Diagram 6. Similar diagram would describe properties wABP($\mathcal{F}$,$\mathcal{G}$).

$$\begin{array}{ccc} \mathrm{PPP}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{MPP}(\mathcal{F},\mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{PMP}(\mathcal{F},\mathcal{G}) & \longrightarrow & \mathrm{MMP}(\mathcal{F},\mathcal{G}) \end{array}$$

Diagram 6

If $f_n \in {}^X[0,1], n \in \omega$ are upper semicontinuous and $f \in {}^X[0,1]$ such that $f_n \xrightarrow{\mathrm{M}} f$ on X then f is upper semicontinuous (see e.g. [13], 1.7.14 (a)). If $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$ then

$$\mathrm{MPP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{MPP}(\mathcal{U},\mathcal{G}), \qquad \mathrm{wMPP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wMPP}(\mathcal{U},\mathcal{G})$$
$$\mathrm{MMP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{MMP}(\mathcal{U},\mathcal{G}), \qquad \mathrm{wMMP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wMMP}(\mathcal{U},\mathcal{G})$$

and if moreover $\mathcal{U} \subseteq \mathcal{G} \subseteq \mathcal{B}$ then

$$\mathrm{MMP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{MMP}(\mathcal{U},\mathcal{U}), \qquad \mathrm{wMMP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wMMP}(\mathcal{U},\mathcal{U}).$$

If $\{0\} \subseteq \mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{G} \subseteq \mathcal{B}$ then

$$\mathrm{PMP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{PMP}(\mathcal{F},\mathcal{U}), \qquad \mathrm{wPMP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{wPMP}(\mathcal{F},\mathcal{U}).$$

Property MMP({0},{0}) is monotonic sequence selection property introduced by M. Scheepers [37].[6)] A topological space with any monotonic property ABP($\mathcal{F}$,$\mathcal{G}$) for $\mathrm{A}, \mathrm{B} \in \{\mathrm{P}, \mathrm{M}\}$ has property MMP({0},{0}). M. Scheepers [37] proved that a topological space X has MMP({0},{0}) if and only if X has the Hurewicz property.

By definitions we have that

$$\mathrm{PPP}(\{0\},\{0\}) \equiv \mathrm{PMP}(\{0\},\{0\}), \qquad \mathrm{MPP}(\{0\},\{0\}) \equiv \mathrm{MMP}(\{0\},\{0\})$$
$$\mathrm{wPPP}(\{0\},\{0\}) \equiv \mathrm{wPMP}(\{0\},\{0\}), \qquad \mathrm{wMPP}(\{0\},\{0\}) \equiv \mathrm{wMMP}(\{0\},\{0\})$$

Similarly as in Lemma 1.1, c) - d), for $\{0\} \subseteq \mathcal{F} \subseteq \mathrm{C}_p(X)$, $\{0\} \subseteq \mathcal{G} \subseteq \mathcal{B}$ and $\mathrm{A}, \mathrm{B} \in \{\mathrm{P}, \mathrm{M}\}$ we have

$$\mathrm{ABP}(\mathcal{F},\mathcal{G}) \equiv \mathrm{ABP}(\{0\},\{0\}) \equiv \mathrm{wABP}(\{0\},\{0\}) \equiv \mathrm{wABP}(\mathcal{F},\mathcal{G}).$$

Theorem 6.1. *Let X be a perfectly normal space.*

a) *X has* MPP($\mathcal{B}$,{0}) *if and only if X has* MPP($\mathcal{B}$,$\mathcal{U}$).
b) *X has* wMPP($\mathcal{B}$,{0}) *if and only if X has* wMPP($\mathcal{B}$,$\mathcal{U}$).

Proof. a) Let us assume that X has MPP($\mathcal{B}$,{0}) and $f_{n,m} \in \mathrm{C}_p(X,[0,1])$, $f_n \in \mathcal{B}$, $f \in \mathcal{U}$ $n, m \in \omega$ are such that $f_{n,m} \xrightarrow{\mathrm{M}} f_n$ on X for every $n \in \omega$ and $f_n \xrightarrow{\mathrm{P}} f$ on X. There are $g_n \in \mathrm{C}_p(X,[0,1])$ such that $g_n \xrightarrow{\mathrm{M}} f$ on X (see e.g. [13], 1.7.14 (c)). We define

$$h_{n,m} = \max\{f_{n,m} - g_n, 0\}, \quad h_n = \max\{f_n - g_n, 0\}, \quad n, m \in \omega.$$

[6)]Although his monotonic sequence selection property is defined for arbitrary real-valued functions, we may consider $g_{n,m} = \min\{f_{n,m}, 1\}$ instead of $f_{n,m}$ for any $n, m \in \omega$.

We have that $h_{n,m} \xrightarrow{\mathrm{M}} h_n$ and $h_n \xrightarrow{\mathrm{P}} 0$. By MPP($\mathcal{B}$,{0}) there is an unbounded $\beta \in {}^\omega\omega$ such that $h_{n,\beta(n)} \xrightarrow{\mathrm{P}} 0$ on X. We shall show that $f_{n,\beta(n)} \xrightarrow{\mathrm{P}} 0$.

Actually, let $x \in X$ and $\varepsilon > 0$. There is $n_0 \in \omega$ such that $h_{n,\beta(n)}(x) < \varepsilon$, $g_n(x) - f(x) < \varepsilon$ and $|f_n(x) - f(x)| < \varepsilon$ for any $n \geq n_0$. Let $n \geq n_0$. If $f_{n,\beta(n)}(x) \geq g_n(x)$ then $f_{n,\beta(n)}(x) - g_n(x) < \varepsilon$ and $0 \leq f_{n,\beta(n)}(x) - f(x) < 2\varepsilon$. If $f(x) \leq f_{n,\beta(n)}(x) < g_n(x)$ then $0 \leq f_{n,\beta(n)}(x) - f(x) < g_n(x) - f(x) < \varepsilon$. Finally, if $f_n(x) \leq f_{n,\beta(n)}(x) < f(x)$ then $0 < f(x) - f_{n,\beta(n)}(x) \leq f(x) - f_n(x) < \varepsilon$.

b) Similarly to a). □

Corollary 6.2. *Let X be a perfectly normal space.*

a) *If X has* MPP($\mathcal{B}$,{0}) *then X has* MMP($\mathcal{B}$,$\mathcal{B}$).
b) *If X has* wMPP($\mathcal{B}$,{0}) *then X has* wMMP($\mathcal{B}$,$\mathcal{B}$).

So as in Theorem 5.11, properties introduced by H. Ohta and M. Sakai [30] are useful for description of some properties ABP($\mathcal{F}$,$\mathcal{G}$) or wABP($\mathcal{F}$,$\mathcal{G}$).

Proposition 6.3. *Let X be a perfectly normal space.*

a) *If X has property* MPP($\mathcal{B}$,{0}) *then X possesses property* USC. *Hence, X is a σ-set and (γ,γ)-shrinkable.*
b) *If X has property* wMPP($\mathcal{B}$,{0}) *then X possesses property* USC$_s$. *Hence, every open γ-cover of X is shrinkable.*

Proof. a) Let $f_n : X \to [0,1]$ be upper semicontinuous functions such that $f_n \xrightarrow{\mathrm{P}} 0$ on X. There are $f_{n,m} \in \mathrm{C}_p(X,[0,1])$ such that $f_{n,m} \xrightarrow{\mathrm{M}} f_n$ on X for every $n \in \omega$ (see e.g. [13], 1.7.14 (c)). By MPP($\mathcal{B}$,{0}) there is an unbounded $\beta \in {}^\omega\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{P}} 0$ on X. Thus X has USC and X is (γ,γ)-shrinkable space by H. Ohta and M. Sakai [30]. Finally, X is a σ-set by M. Sakai [35].

b) Similarly to a). □

Lemma 6.4. *Let X be a topological space.*

a) *If X has* MMP({0},{0}) *and* USC *then X has* MPP($\mathcal{B}$,{0}).
b) *If X has* MMP({0},{0}) *and* USC$_s$ *then X has* wMPP($\mathcal{B}$,{0}).
c) *If X has* MMP({0},{0}) *and* USC$_m$ *then X has* MMP($\mathcal{B}$,{0}).

Proof. a) Let us assume that X has MMP({0},{0}) and USC. Then X has MPP(C$_p(X)$,{0}) by the equivalences above Theorem 6.1. Let $f_{n,m} \in \mathrm{C}_p(X,[0,1])$, $f_n \in \mathcal{B}$ $n, m \in \omega$ be such that $f_{n,m} \xrightarrow{\mathrm{M}} f_n$ on X for every $n \in \omega$ and $f_n \xrightarrow{\mathrm{P}} 0$ on X. By property USC there are functions $g_n \in \mathrm{C}_p(X,[0,1])$ such that $g_n \to 0$ and $f_n \leq g_n$, $n \in \omega$. We define $g_{n,m} = \max\{f_{n,m}, g_n\}$ and we have $g_{n,m} \xrightarrow{\mathrm{M}} g_n$ for any $n \in \omega$. By MPP(C$_p(X)$,{0}) there is an unbounded $\beta \in {}^\omega\omega$ such that $g_{n,\beta(n)} \xrightarrow{\mathrm{P}} 0$ on X. We can conclude that $f_{n,\beta(n)} \xrightarrow{\mathrm{P}} 0$ as well.

b) and c) Similarly to a). □

A perfectly normal space X has Hurewicz property hereditarily if and only if X is a σ-set and has the Hurewicz property, see B. Tsaban and L. Zdomskyy [40] or T. Orenshtein and B. Tsaban [32]. H. Ohta and M. Sakai [30] showed that every normal countably paracompact space has property USC$_m$ and every space with the property USC$_m$ is countably paracompact. Let us recall that every perfectly normal space is countably paracompact (see e.g. [13], Corollary 5.2.5). Hence, we obtain

Theorem 6.5. *Let X be a perfectly normal space.*

a) *X has* MPP($\mathcal{B}$,{0}) *if and only if X possesses Hurewicz property hereditarily.*
b) *X has* wMPP($\mathcal{B}$,{0}) *if and only if X possesses Hurewicz property and every open γ-cover of X is shrinkable.*
c) *X has* MMP($\mathcal{B}$,{0}) *if and only if X has Hurewicz property.*

Proof. a) Let X have MPP($\mathcal{B}$,{0}). Then X has MMP({0},{0}) and by M. Scheepers [37] X has Hurewicz property. X is a σ-set by Proposition 6.3. Finally, X possesses Hurewicz property hereditarily by B. Tsaban and L. Zdomskyy [40].

If X possesses Hurewicz property hereditarily then X has MMP({0},{0}) by M. Scheepers [37] and X is a σ-set by D.H. Fremlin and A.W. Miller [16] (see the paper by T. Orenshtein and B. Tsaban [32] for appropriate strengthening). Thus X has USC by H. Ohta and M. Sakai [30]. Finally, by Lemma 6.4 we obtain that X has MPP($\mathcal{B}$,{0}).

b) and c) Similarly to a). □

Corollary 6.6. *A perfectly normal space X has* MMP($\mathcal{B}$,{0}) *if and only if X has* wMMP($\mathcal{B}$,{0}).

H. Ohta and M. Sakai [30] proved that every separable metrizable space which has USC_s is perfectly meager. The unit interval $[0,1]$ or Cantor set have MMP($\mathcal{B}$,{0}) but neither of them has USC_s. Hence, we obtain that

$$\mathrm{wMPP}(\mathcal{B},\{0\}) \not\equiv \mathrm{MMP}(\mathcal{B},\{0\}).$$

7. Sequential closure

D.H. Fremlin [14] showed that any topological space X is an s_1-space if and only if X has PP({0},{0}). By a referee of [9] we were warned about the thesis [31] by T. Orenshtein and about the connection between sequence selection principles investigated in [9] and properties described by sequential closure operator.

Let us recall that a topological space X is an s_1-space if and only if $\mathrm{scl}_{\omega_1}(\mathcal{F}, \mathrm{C}_p(X)) = \mathrm{scl}_1(\mathcal{F}, \mathrm{C}_p(X))$ for every $\mathcal{F} \subseteq \mathrm{C}_p(X)$. Moreover, we know even more. By Theorem 3.5 and Proposition 5.2 a topological space X has wPQ($\mathrm{C}_p(X)$,{0}) if and only if X has PP({0},{0}). Thus any topological space X is an s_1-space if and only if any function from $\mathrm{scl}_{\omega_1}(\mathcal{F}, \mathrm{C}_p(X))$ is a quasi-normal limit of a sequence of functions from $\mathcal{F}$ for every $\mathcal{F} \subseteq \mathrm{C}_p(X)$.

T. Orenshtein [31] calls a topological space X an $\mathrm{s}_1({}^X\mathbb{R})$-space if $\mathrm{scl}_{\omega_1}(\mathcal{F}, {}^X\mathbb{R}) = \mathrm{scl}_1(\mathcal{F}, {}^X\mathbb{R})$ for every $\mathcal{F} \subseteq \mathrm{C}_p(X)$. Note that by Lebesque's Theorem there are functions of all Baire classes on the unit interval $[0,1]$, therefore $[0,1]$ is not an $\mathrm{s}_1({}^X\mathbb{R})$-space. One can easily see that if a topological space X has property wPP($\mathcal{B}$,$\mathcal{B}$) then X is an $\mathrm{s}_1({}^X\mathbb{R})$-space.

Proposition 7.1. *Any topological space X has property* wPP($\mathcal{B}$,$\mathcal{B}$) *if and only if X is an $\mathrm{s}_1({}^X\mathbb{R})$-space. In particular:*

a) *If X is a perfectly normal* QN*-space then X is an $\mathrm{s}_1({}^X\mathbb{R})$-space.*
b) *If X is a perfectly normal* QN*-space then any function $f \in \mathrm{scl}_{\omega_1}(\mathcal{F}, {}^X\mathbb{R})$ is quasi-normal limit of functions from $\mathcal{F}$ for any $\mathcal{F} \subseteq \mathrm{C}_p(X)$.*
c) *If X is a perfectly normal* QN*-space then any function $f \in \mathrm{scl}_{\omega_1}(\mathrm{C}_p(X), {}^X\mathbb{R})$ is discrete limit of functions from $\mathrm{C}_p(X)$.*

Proof. The fisrt statement follows by definitions. By Proposition 3.1 any perfectly normal QN-space X has property wPQ($\mathcal{B}$,$\mathcal{B}$) which gives (1) and (2). Assertion (3) follows by (2) and by the fact that quasi-normal limit of a sequence of continuous functions on normal space X is a discrete limit of a sequence of continuous functions on X (see e.g. first section of [9]). □

By Propositions 4.8 and 7.1 we obtain the following.

Corollary 7.2. *Any perfectly normal $\mathrm{s}_1({}^X\mathbb{R})$-space is a σ-set.*

T. Orenshtein [31] studies property $\binom{\mathcal{F}}{\mathcal{G}}$ which was introduced by M. Scheepers [36] under notation Sub($\mathcal{F}$, $\mathcal{G}$). A topological space X possesses property $\binom{\mathrm{S}'_0}{\Gamma_0}$ if for any set $\mathcal{F} \subseteq \mathrm{C}_p(X) \setminus \{0\}$ with $0 \in \mathrm{scl}_{\omega_1}(\mathcal{F}, {}^X\mathbb{R})$ there is a sequence $\langle f_n : n \in \omega \rangle$ of functions from $\mathcal{F}$ such that $f_n \to 0$. Thus a topological space X possesses property $\binom{\mathrm{S}'_0}{\Gamma_0}$ if and only if $\mathrm{scl}_{\omega_1}(A, {}^X\mathbb{R}) \cap \mathrm{C}_p(X) = \mathrm{scl}_1(A, {}^X\mathbb{R}) \cap \mathrm{C}_p(X)$ for any $A \subseteq \mathrm{C}_p(X)$. Consequently by definition of $\mathrm{s}_1({}^X\mathbb{R})$-space and by Proposition 7.1 we obtain the following. One can easily see that any $\mathrm{s}_1({}^X\mathbb{R})$-space possesses property $\binom{S_0'}{\Gamma_0}$.

Corollary 7.3. *If a topological space X has property* wPP($\mathcal{B}$,$\mathcal{B}$) *then X possesses* $\binom{S_0'}{\Gamma_0}$. *In particular, any perfectly normal* QN*-space possesses* $\binom{S_0'}{\Gamma_0}$.

T. Orenshtein [31] proves that if a perfectly normal space X possesses $\binom{S_0'}{\Gamma_0}$ then X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space. In Problem 6.0.17 he asks if any perfectly normal $\mathrm{S}_1(\Gamma,\Gamma)$-space possesses $\binom{S_0'}{\Gamma_0}$.

Similarly, we may express other properties in terms of sequential closure. Propositions 7.4 and 7.6 follow easily by definitions.

Proposition 7.4. *Any topological space X has property* wPP($\mathcal{B}$,{0}) *if and only if*

$$\mathrm{scl}_2(\mathcal{F},{}^X\mathbb{R})\cap \mathrm{C}_p(X)=\mathrm{scl}_1(\mathcal{F},{}^X\mathbb{R})\cap \mathrm{C}_p(X)$$

for any $\mathcal{F}\subseteq \mathrm{C}_p(X)$.

Consequently by the above natural characterization of $\binom{S_0'}{\Gamma_0}$ we obtain the following. The former follows by Proposition 4.9.

Corollary 7.5. *If a topological space X possesses* $\binom{S_0'}{\Gamma_0}$ *then X has property* wPP($\mathcal{B}$,{0}). *Hence, every open* γ*-cover of X is shrinkable.*

Proposition 7.6. *Any topological space X has property* wPP($\mathrm{C}_p(X)$,$\mathcal{B}$) *if and only if*

$$\mathrm{scl}(\mathrm{scl}_{\omega_1}(\mathcal{F},\mathrm{C}_p(X)),{}^X\mathbb{R})=\mathrm{scl}_1\mathcal{F},{}^X\mathbb{R})$$

for any $\mathcal{F}\subseteq \mathrm{C}_p(X)$.

8. Consistency results

B. Tsaban and L. Zdomskyy [40] have realized the consequences of the result by A. Dow [12] on relation of QN-space and wQN-space, i.e., wQN is equivalent to QN in the Laver model. The Laver model is that constructed by A. Laver [25]. Later, A.W. Miller and B. Tsaban [29] slightly improved the result and obtained that the statement "X is a wQN-space" is in the Laver model equivalent to "$|X|<\mathfrak{b}$" for any set of reals X.

Proposition 8.1. *Let* $\mathcal{F},\mathcal{G},\mathcal{Q},\mathcal{H}$ *be families of functions,* A, B, M, N *being any of* P, Q, D. *Then the theory*

$$\mathbf{ZFC}+\mathrm{AB}(\mathcal{F},\mathcal{G})\equiv \mathrm{MN}(\mathcal{Q},\mathcal{H})\equiv \mathrm{wAB}(\mathcal{F},\mathcal{G})\equiv \mathrm{wMN}(\mathcal{Q},\mathcal{H})$$

is consistent. Added equivalences are understood to hold for any perfectly normal space.

Hence, the answer to Problem 6.0.17 by T. Orenshtein [31] is "consistently Yes".

J. Gerlits and Zs. Nagy [18] showed that the minimal cardinality of a set which is not a γ-set is $\mathfrak{p}$. By Corollary 4.5 we obtain

Proposition 8.2. *If* $\mathfrak{p}<\mathfrak{b}$ *then* $\mathrm{AB}(\mathcal{F},\mathcal{G})\not\to\gamma$ *nor* $\mathrm{wAB}(\mathcal{F},\mathcal{G})\not\to\gamma$.

F. Galvin and A.W. Miller [17] showed that assuming **MA** ($\mathfrak{p} = \mathfrak{c}$ is sufficient) there is a γ-set of reals of cardinality $\mathfrak{c}$ which is $\mathfrak{c}$-concentrated on a countable subset. If $\mathrm{cf}(\kappa) > \aleph_0$ then set of cardinality κ which is κ-concentrated on a countable subset cannot be a λ-set (see e.g. M. Sakai [34]). M. Scheepers [38] adapted the construction of an $S_1(\Gamma,\Gamma)$-set by W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki [21] and obtained the following: If $\mathfrak{t} = \mathfrak{b}$ then there is an $S_1(\Gamma,\Gamma)$-set X of real numbers of cardinality $\mathfrak{b}$ such that X is not a λ-set (see e.g. [4] as well). Finally, T. Orenshtein and B. Tsaban [32] showed that if $\mathfrak{p} = \mathfrak{b}$ then there is a γ-set of real numbers of cardinality $\mathfrak{b}$ which is $\mathfrak{p}$-concentrated on a countable subset. Note that $\mathfrak{p} = \mathfrak{t}$ by M. Malliaris and S. Shelah [26].

Theorem 8.3 contains several sequence selection properties which can be distinguished. By (1) and Diagram 1 it follows that there is a number of others.

Theorem 8.3. *Let* (A, B) *be any couple of* P, Q, D *with the exception of* (P,D), (Q,D), (D,D), (P, Q). *If* $\mathfrak{p} = \mathfrak{b}$ *then*

a) $\mathrm{AB}(\mathcal{F},\{0\}) \not\equiv \mathrm{AB}(C_p(X),\{0\})$ *for* $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$.
b) $\mathrm{wAB}(\mathcal{F},\mathcal{B}) \not\equiv \mathrm{wAB}(\mathcal{F},\{0\})$ *for* $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$.
c) $\mathrm{AB}(\mathcal{F},\{0\}) \not\equiv \mathrm{wAB}(\mathcal{F},\{0\})$ *for* $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$.
d) $\mathrm{PQ}(C_p(X),\{0\}) \not\equiv \mathrm{PQ}(\{0\},\{0\})$.
e) $\mathrm{PQ}(C_p(X),\{0\}) \not\equiv \mathrm{wPQ}(C_p(X),\{0\})$.
f) $\mathrm{MPP}(\mathcal{B},\{0\}) \not\equiv \mathrm{wMPP}(\mathcal{B},\{0\})$.

The most interesting diagram of relations is Diagram 7 which describes what we know about relations among properties $\mathrm{AB}(\mathcal{F},\{0\})$ and $\mathrm{wAB}(\mathcal{F},\{0\})$ for $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$. Note that for $\mathcal{F} \in \{\mathcal{U},\mathcal{B}\}$ and normal space we know that any property $\mathrm{AB}(\mathcal{F},\{0\})$ or $\mathrm{wAB}(\mathcal{F},\{0\})$ has its corresponding equivalent principle in Diagram 7. Negative arrow represents non-provability of implication.

$$
\begin{tikzcd}
 & \mathrm{QN} \arrow[r] \arrow[d, shift right] & \mathrm{PP}(\mathcal{F},\{0\}) \arrow[r] \arrow[d, shift right] & \mathrm{DP}(\mathcal{F},\{0\}) \arrow[r] \arrow[d, shift right] & hS_1(\Gamma,\Gamma) \arrow[r] \arrow[d, shift right] & \sigma \\
\gamma \arrow[r] & \mathrm{wPQ}(\mathcal{F},\{0\}) \arrow[r] \arrow[u, shift right, "/" marking] & \mathrm{wPP}(\mathcal{F},\{0\}) \arrow[r] \arrow[u, shift right, "/" marking] & \mathrm{wDP}(\mathcal{F},\{0\}) \arrow[r] \arrow[u, shift right, "/" marking] & S_1(\Gamma,\Gamma) \arrow[r] \arrow[u, shift right, "/" marking] & \mathrm{wQN}
\end{tikzcd}
$$

$\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$

Diagram 7

A.W. Miller [27] showed that the theory **ZFC**+"any σ-set of reals is countable" is consistent. Any second-countable space X with $\mathrm{ind}(X) = 0$ is homeomorphic to a subset of reals (see e.g. [13], Theorem 6.2.16). Thus any separable metrizable space with properties $\mathrm{AB}(\mathcal{F},\mathcal{G})$ or $\mathrm{wAB}(\mathcal{F},\mathcal{B})$ for $\mathcal{U} \subseteq \mathcal{F}$ is by Corollaries 4.3 and 4.11 homeomorphic to a hereditarily $S_1(\Gamma,\Gamma)$-set of reals. By J. Haleš [19] such set is a σ-set. Moreover, by Corollary 4.5 any countable set of reals has investigated sequence selection properties. However, there is an uncountable γ-set in Miller's model by T. Orenshtein and B. Tsaban [32]. Let us recall that any second-countable topological space X is metrizable if and only if X is regular (see e.g. [13], Theorem 4.2.9).

Proposition 8.4. *Let* (A, B) *be any couple of* P, Q, D *with the exception of* (P,D), (Q,D), (D,D), (P, Q). *Then the theory*

$$\mathbf{ZFC} + \text{“}\mathrm{AB}(\mathcal{B},\mathcal{B}) \equiv \mathrm{AB}(\mathcal{U},\{0\}) \equiv \mathrm{wAB}(\mathcal{U},\mathcal{B})\text{”} + \text{“}\mathrm{AB}(\mathcal{U},\{0\}) \not\equiv \mathrm{wAB}(\mathcal{U},\{0\})\text{”}$$

is consistent. Added equivalences are understood to hold for any separable metrizable space.

Diagram 8 describes in more detail the relations in Miller's model.

$$
\begin{array}{c}
\mathrm{QN} \equiv \mathrm{PQ}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\mathcal{F},\mathcal{B}) \equiv \mathrm{wAB}(\mathcal{F},\mathcal{B}) \equiv \mathrm{AB}(\mathcal{F},\{0\}) \equiv \mathrm{hS}_1(\Gamma,\Gamma) \equiv \sigma\text{-set} \\
\downarrow \quad \not\uparrow \\
\mathrm{wAB}(\mathcal{F},\{0\}) \\
\downarrow \\
\mathrm{wQN} \equiv \mathrm{wPQ}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\mathrm{C}_p(X),\{0\}) \equiv \mathrm{AB}(\{0\},\{0\}) \equiv \mathrm{PQ}(\{0\},\{0\})
\end{array}
$$

$$(\mathrm{A},\mathrm{B}) \neq (\mathrm{P},\mathrm{Q}),\ \mathrm{B} \neq \mathrm{D},\ \mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{B}$$

Diagram 8

T. Orenshtein [31] proved that any perfectly normal $\mathrm{s}_1({}^X\mathbb{R})$-space is an $\mathrm{S}_1(\Gamma,\Gamma)$-space and he asked in Problem 6.0.15 if the reversed implication could hold.

Corollary 8.5. *The statement*

"any perfectly normal $\mathrm{S}_1(\Gamma,\Gamma)$*-space is an* $\mathrm{s}_1({}^X\mathbb{R})$*-space"*

is undecidable in **ZFC***.*

Proof. By (4), Propositions 5.2, 7.1, 8.1 and by the above result by T. Orenshtein [31] we obtain that the theory **ZFC** + $\mathrm{s}_1({}^X\mathbb{R}) \equiv \mathrm{S}_1(\Gamma,\Gamma)$ is consistent. Added equivalence is understood to hold for any perfectly normal space.

By Theorem 7.2 an $\mathrm{s}_1({}^X\mathbb{R})$-space is a σ-set. Hence, the aforementioned set by M. Scheepers [38] is an $\mathrm{S}_1(\Gamma,\Gamma)$-set which is not an $\mathrm{s}_1({}^X\mathbb{R})$-space. □

By definitions an $\mathrm{s}_1({}^X\mathbb{R})$-space possesses $\binom{\mathrm{S}'_0}{\Gamma_0}$. T. Orenshtein [31] asked again in Problem 6.0.16 if the reversed implication could hold. Note that J. Gerlits and Zs. Nagy [18] showed that a Tychonoff space X is γ-set if and only if $\mathrm{C}_p(X)$ is Fréchet-Urysohn. Hence any γ-set has $\binom{\mathrm{S}'_0}{\Gamma_0}$.

Corollary 8.6. *The statement*

"any perfectly normal space possessing $\binom{\mathrm{S}'_0}{\Gamma_0}$ *is an* $\mathrm{s}_1({}^X\mathbb{R})$*-space"*

is undecidable in **ZFC***.*

Proof. By Propositions 7.1, 8.1 and Corollaries 7.3, 7.5 we obtain that the theory **ZFC** + $\binom{\mathrm{S}'_0}{\Gamma_0} \equiv \mathrm{s}_1({}^X\mathbb{R})$ is consistent. Added equivalence is understood to hold for any perfectly normal space.

By Theorem 7.2 an $\mathrm{s}_1({}^X\mathbb{R})$-space is a σ-set. Hence, the γ-set by T. Orenshtein and B. Tsaban [32] cannot be an $\mathrm{s}_1({}^X\mathbb{R})$-space but possesses $\binom{\mathrm{S}'_0}{\Gamma_0}$ by J. Gerlits and Zs. Nagy [18]. □

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