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Semisymmetric cubic graphs of orders $36p$, $36p^2$

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Abstract. A cubic graph is said to be semisymmetric if its full automorphism group acts transitively on its edge set but not on its vertex set . The semisymmetric cubic graphs of orders $6p$ and $6p^2$ were classified in (Com. in Algebra, 28 (6) (2000) 2685-2715) and (Science in China Ser. A Mathematics, 47 (2004) No.1 1-17), respectively. In this paper we first classify all connected cubic semisymmetric graphs of order $36p$ for each prime p and also classify all connected cubic semisymmetric graphs of order $36p^2$, where $p \neq 5$ and 7 is a prime.

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notations not defined here we refer the reader to [22, 25].

A regular graph is said to be semisymmetric if its full automorphism group acts transitively on its edge set but not on its vertex set.

Covering techniques have long been known as a powerful tool in topology and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. The class of semisymmetric graphs was introduced by Folkman [10]. He constructed several infinite families of such graphs and posed eight open problems. Afterwards, Bouwer [4], Titov [23], Klin [15], A.A. Ivanov and M.E. Iofinova [11], A.V. Ivanov [12], Du and Xu [9] and others did much work on semisymmetric graphs. They gave new constructions of such graphs by combinatorial and group-theoretical methods. A census of all semisymmetric cubic graphs up to 768 vertices has been obtained by Conder et al. [5]. By now, the answers to most of Folkman's open problems are known. By using the covering technique, semisymmetric cubic graphs of order $8p$, $6p^2$, $8p^2$, $16p^2$ and $2p^3$ were classified in [1, 2, 3, 16, 20]. Some general methods of elementary abelian coverings were developed in [8, 17, 18, 19, 21]. In this paper we first classify all connected cubic semisymmetric graphs of order $36p$ for each prime p. Also classify all connected cubic semisymmetric graphs of order $36p^2$, where $p \neq 5$ and 7 is a prime.

2. Preliminaries

Given a graph X, we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the *vertex set*, the *edge set*, the *arc set* and the *full automorphism group* of X, respectively. For u, $v \in V(X)$, we denote by uv the edge incident to u and

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v in X If a subgroup G of Aut(X) acts transitively on $V(X)$ or $E(X)$, we say that X is *G-vertex-transitive* or *G-edge-transitive*, respectively. In the special case when G =Aut(X) we say that X is vertex-transitive or edge-transitive, respectively. If G acts transitively on both $V(X)$ and $E(X)$ we say that X is *Symmetric*. It can be shown that a G-edge-transitive but not G-vertex-transitive graph X is necessarily bipartite, where the two parts of the bipartition are orbits of $G \leq$Aut(X). Moreover, if X is regular then these two parts have the same cardinality. A regular G-edge-transitive but not G-vertex-transitive graph will be referred to as a *G-semisymmetric graph*. In particular, if G =Aut(X) the graph is said to be semisymmetric.

A permutation group G on a set Ω is said to be semiregular if for each $\alpha \in \Omega$, the stabilizer G_α of α in G is the identity group, and regular if it is semiregular and transitive.

An epimorphism $\wp : X \to Y$ of connected graphs is a *regular covering projection* if it arises essentially as a factorization $X \to X/N \cong Y$, where the action of $N \leq$Aut(X) is *semiregular* (that is, fixed point free) on both vertices and edges of X . Note that the graph Y may not be simple even if X is. The graph X is called the *covering graph* and Y is the *base graph*. The preimage $\wp^{-1}(v), v \in V(Y)$, corresponds to an orbit of N on $V(X)$ and is called the *(vertex)-fibre* over v. Similarly, *edge-fibres* correspond to orbits of N on $E(X)$.

Let X be a graph and let K be a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or, *K-voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called *voltages*, and K is the *voltage group*. The graph $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times_\phi K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$. Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $\wp : X \times_\phi K \to X$, which is called the *natural projection*. By defining $(u, g')^g = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, K becomes a subgroup of Aut($X \times_\phi K$) which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a *K-covering*. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment.

The next proposition is a special case of [24, proposition 2.5].

Proposition 2.1. *Let X be a G-semisymmetric cubic graph with bipartition sets $U(X)$ and $W(X)$. Moreover, suppose that N is a normal subgroup of G. Then,*
(1) If N is intransitive on the bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is a regular N-covering of the G/N-semisymmetric graph X_N.
(2) If 3 does not divide $|Aut(X)/N|$, then N is semisymmetric on X.

Proposition 2.2 [20, Proposition 2.4]. *The vertex stabilizers of a connected G-semisymmetric cubic graph X have order $2^r \cdot 3$, where $0 \leq r \leq 7$. Moreover, if u and v are two adjacent vertices, then the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

Note that the structure of the pair (G_u, G_v), where u and v are two adjacent vertices of the G-semisymmetric cubic graph X was completely determined in [13].

The following result can be obtained from [14, pp. 12-14] and [7].

Proposition 2.3. *Let p be a prime and G be a non-abelian simple group whose order divides $2^{r+1} \cdot 3^3 \cdot p^2$ for some non-negative integer $r \leq 7$. Then, G is isomorphic to A_5, A_6, PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), or PSU(3, 3) of orders $2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3^2 \cdot 5$, $2^3 \cdot 3 \cdot 7$, $2^3 \cdot 3^2 \cdot 7$, $2^4 \cdot 3^2 \cdot 17$, $2^4 \cdot 3^3 \cdot 13$, $2^5 \cdot 3^3 \cdot 7$, respectively.*

3. Main Results

The following theorem, which is one of the main results of this paper, shows that every edge-transitive cubic graph of order $36p$ is a symmetric graph, where p is a prime.

Theorem 3.1. *There is no semisymmetric cubic graph of order* $36p$*, where* p *is a prime.*

Proof. If $p < 23$, by [5] there is no semisymmetric cubic graph of order $36p$. We can assume that $p \geq 23$. By way of contradiction, let X be a semisymmetric cubic graph of order $36p$. Set $A := \mathrm{Aut}(X)$. By Proposition 2.2, $|A_v| = 2^r \cdot 3$, where $0 \leq r \leq 7$, and hence $|A| = 2^{r+1} \cdot 3^3 \cdot p$. Let $Q := O_p(A)$ be the maximal normal p-subgroup of A. We show that $|Q| = p$ as follows.

Let N be a minimal normal subgroup of A. Thus, $N \cong T \times T \times \cdots \times T = T^k$, where T is a simple group. Let N be unsolvable. Therefore, T is isomorphic to one of the simple groups mentioned in Proposition 2.3, contrary to the fact that $p \geq 23$. Therefore, N is solvable and so elementary abelian. It follows that N acts intransitively on bipartition sets of X and hence it is semiregular on each partition. Thus, $|N| \mid 18p$.

Suppose first that $Q = 1$. It implies three cases: $N \cong Z_2$, Z_3 or $Z_3 \times Z_3$. We get a contradiction in each case as follows.

case (I): $N \cong Z_2$**.**

By Proposition 2.1, X_N is a A/N-semisymmetric cubic graph of order $18p$. Let M/N be a minimal normal subgroup of A/N. By a similar argument as above M/N is solvable and so elementary abelian. Therefore, M/N acts intransitively on bipartition sets of X_N and by Proposition 2.1, it is semiregular on each partition, which force $|M/N| \mid 9p$.

If $|M/N| = p$, then $|M| = 2p$ and so the Sylow p-subgroup of M is characteristic and consequently normal in A. It contradicts our assumption that $Q = 1$. If $|M/N| = 3$, $|M| = 6$. Thus, X_M is a A/M-semisymmetric cubic graph of order $6p$. Let K/M be a minimal normal subgroup of A/M. Again, K/M is solvable and so elementary abelian. Hence, K/M is semiregular on each partition of X_M and so $|K/M| \mid 3p$. If $|K/M| = p$, $|K| = 6p$. It follows that the Sylow p-subgroup of K is normal in A, a contradiction. So, $|K/M| = 3$ and hence $|K| = 18$. So, X_K is a A/K-semisymmetric cubic graph of order $2p$. Let H/K be a minimal normal subgroup of A/K. Again, H/K is solvable and so elementary abelian. If H/K acts transitively on one partition set of X_K, then by [25, Proposition 4.4], H/K is regular. It implies that $|H| = 18p$. Since $p \geq 23$, the Sylow p-subgroup of H is characteristic and consequently normal in A, a contradiction. Thus, H/K acts intransitively on the two partition sets of X_K and so it is semiregular, forcing $|H| = 18p$, a contradiction as above. Therefore, $|M/N| = 9$ and so $|M| = 18$. We get a contradiction as $|K| = 18$.

case (II): $N \cong Z_3$**.**

Therefore, X_N is a A/N-semisymmetric cubic graph of order $12p$. Let L/N be a minimal normal subgroup of A/N. It is easy to see that L/N is solvable and so elementary abelian. Therefore, it is semiregular on each partition of X_N, which force $|L/N| \mid 6p$. Since $Q = 1$, $|L/N| = 2$ or 3. One can see that the case $|L/N| = 2$ is impossible as in case (I). Hence, $|L/N| = 3$ and so $|L| = 9$. Then, X_L is a A/L-semisymmetric cubic graph of order $4p$. Again, we consider a minimal normal subgroup R/L of A/L and by a similar argument as case (I), one can show that $|R| = 18$ or $18p$. This leads to a contradiction as in case (I).

case (III): $N \cong Z_3 \times Z_3$**.**

So, X_N is a A/N-semisymmetric cubic graph of order $4p$. This case was rejected in case (II).

Therefore, $|Q| = p$. By Proposition 2.1, X_Q is A/Q-semisymmetric cubic graph of order 36. But, by [5, 6] there is no edge-transitive cubic graph of order 36, a contradiction. The result now follows. ■

The graph $S144$ is the only cubic graph of order 144 whose diameter and girth are equal to 8, and it is Z_3^2-covering of Möbius-Kantor (symmetric) cubic graph $F16$. The second main result of this paper as follows:

Theorem 3.2. *The graph* $S144$ *is only semisymmetric cubic graph of order* $36p^2$*, where* $p \neq 5, 7$ *is a prime.*

Proof. If $p < 5$, by [5] the graph $S144$ is the only semisymmetric cubic graph of order $36p^2$. Hence we can assume that $p \geq 11$. By way of contradiction, let X be a semisymmetric cubic graph of order $36p^2$. Set $A := \mathrm{Aut}(X)$. By Proposition 2.2, $|A| = 2^{r+1} \cdot 3^3 \cdot p^2$, where $0 \leq r \leq 7$. Let $Q := O_p(A)$ be the maximal normal p-subgroup of A. We show that $|Q| = p^2$ as follows.

Let N be a minimal normal subgroup of A. Thus, $N \cong T \times T \times \cdots \times T = T^k$, where T is a simple group. Let N be unsolvable. By Proposition 2.3, T is isomorphic either to $\mathrm{PSL}(2, 17)$ or to $\mathrm{PSL}(3, 3)$ of orders $2^4 \cdot 3^2 \cdot 17$

and $2^3 \cdot 3^3 \cdot 13$. Hence $k = 1$. If $N \cong PSL(3,3)$ then 3 dose not divide $|A/N|$. By Proposition 2.1, N is semisymmetric on X and so $18p^2 \mid |N|$, a contradiction. Also, if $N \cong PSL(2,17)$ then N acts intransitively on the bipartition sets of X and by Proposition 2.1, it is semiregular on each partition, which forces $|N| \mid 18p^2$, a contradiction. Therefore, N is solvable and so elementary abelian. It follows that N is semiregular on each partition of X_N. Thus, $|N| \mid 18p^2$.

Suppose first that $Q = 1$. It implies three cases: $N \cong Z_2$, Z_3 or $Z_3 \times Z_3$. We get a contradiction in each case as in the previous lemma.

We suppose now that $|Q| = p$. Let P be a Sylow p-subgroup of A and $C := C_A(Q)$ the centralizer of Q in A. Clearly, $Q < P$ and also $P \leq C$ because P is abelian. Thus, $p^2 \mid |C|$. By [22, pp.236], $p \nmid |C' \bigcap Z(C)|$, which implies that $C' \bigcap Q = 1$. This forces $p^2 \nmid |C'|$, and so C' acts intransitively on each partition set of X. According to Proposition 2.1, C' is semiregular on each partition set of X and hence $|C'|18p^2$. Note that PC'/C' is a Sylow p-subgroup of C/C', and C/C' is abelian. Then PC'/C' is characteristic in C/C' and since $C/C' \trianglelefteq A/C'$, we have $PC'/C' \trianglelefteq A/C'$. Hence $PC' \lhd A$. Clearly $|PC'| = tp^2$, where $t \mid 18$. If 2 does not divide $|C'|$, then by Sylow theorem P is normal in PC' because $p > 7$. Thus P is characteristic in PC', implying $P \lhd A$, contradicting $|Q| = p$. If 2 divides $|C'|$, then PC' has a characteristic subgroup of index 2 which is normal in A, and by the same argument as in the previous case a similar contradiction is obtained (replacing PC' by this normal subgroup).

Therefore, $|Q| = p^2$. By Proposition 2.1, X_Q is A/Q-semisymmetric cubic graph of order 36, a contradiction. The result now follows. ■

Remark. First author et al. have classified the cubic semisymmetric graphs of orders $12p$, $12p^2$, $18p$, $18p^2$. Also, cubic semisymmetric graphs of orders $6p, 6p^2$ were classified in [9, 16]. So, by the previous theorems, the classification of cubic semisymmetric graphs of order $2^i \cdot 3^j \cdot p^k$ $(1 \leq i, j, k \leq 2)$ will be completed if one can verify the existence of semisymmetric cubic graphs of orders $2^2 \cdot 3^2 \cdot 5^2$ and $2^2 \cdot 3^2 \cdot 7^2$.

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