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Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind

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Abstract. In the paper, by establishing a new and explicit formula for computing the n-th derivative of the reciprocal of the logarithmic function, the author presents new and explicit formulas for calculating Bernoulli numbers of the second kind and Stirling numbers of the first kind. As consequences of these formulas, a recursion for Stirling numbers of the first kind and a new representation of the reciprocal of the factorial $n!$ are derived. Finally, the author finds several identities and integral representations relating to Stirling numbers of the first kind.

1. Introduction

It is general knowledge that the n-th derivative of the logarithmic function $\ln x$ for $x > 0$ is

$$(\ln x)^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n} \tag{1.1}$$

for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. One may ask a question: What is the formula for the n-th derivative of the reciprocal of the logarithmic function $\ln x$? There have been some literature to deal with this question. For example, Lemma 2 in [7] reads that for any $m \geq 0$ we have

$$\left[\frac{1}{\ln(1+t)}\right]^{(m)} = \frac{1}{(1+t)^m}\sum_{i=0}^{m}(-1)^i i!\frac{s(m,i)}{[\ln(1+t)]^{i+1}}, \tag{1.2}$$

where $s(n,k)$ are Stirling numbers of the first kind, which may be generated by

$$\frac{[\ln(1+x)]^m}{m!} = \sum_{k=m}^{\infty}\frac{s(k,m)}{k!}x^k, \quad |x| < 1. \tag{1.3}$$

The first aim of this paper is to establish a new and explicit formula for computing the n-th derivative of the reciprocal of the logarithmic function. As consequences of this formula, a recursion for Stirling numbers of the first kind and a new representation of the reciprocal of the factorial $n!$ are derived.

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The Bernoulli numbers $b_0, b_1, b_2, \dots, b_n, \dots$ of the second kind may be defined by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n. \tag{1.4}$$

The first few Bernoulli numbers b_n of the second kind are

$$b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24}, \quad b_4 = -\frac{19}{720}, \quad b_5 = \frac{3}{160}.$$

For more information, please refer to [5, 6] and closely related references therein. By the way, we note that the so-called Cauchy number of the first kind may be defined by $n!b_n$. See [2, 7] and plenty of references cited therein. One may also ask a natural question: Can one discover an explicit formula for computing b_n for $n \in \mathbb{N}$? There have been several formulas and recurrence relations for computing b_n. For example, it is derived in [8] that

$$b_n = \frac{1}{n!} \sum_{k=0}^{n} \frac{s(n,k)}{k+1}, \tag{1.5}$$

where $s(n,k)$ may also be generated by

$$\prod_{k=0}^{n-1} (x-k) = \sum_{k=0}^{n} s(n,k) x^k. \tag{1.6}$$

We remark that two definitions of $s(n,k)$ by (1.3) and (1.6) are coincident.

The second aim is to derive a new and explicit formula for calculating Bernoulli numbers b_n of the second kind.

Finally, we will find several identities and integral representations relating to Stirling numbers of the first kind $s(n,k)$.

2. Explicit formula for derivatives of the logarithmic function

In this section, we establish a new and explicit formula for computing the n-th derivative of the reciprocal of the logarithmic function, which will be applied in next section to derive an explicit formula for calculating Bernoulli numbers of the second kind.

Theorem 2.1. *For $n \in \mathbb{N}$, we have*

$$\left(\frac{1}{\ln x}\right)^{(n)} = \frac{(-1)^n}{x^n} \sum_{i=2}^{n+1} \frac{a_{n,i}}{(\ln x)^i}, \tag{2.1}$$

where

$$a_{n,2} = (n-1)! \tag{2.2}$$

and, for $n+1 \ge i \ge 3$,

$$a_{n,i} = (i-1)!(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{i-3}=1}^{\ell_{i-4}-1} \frac{1}{\ell_{i-3}} \sum_{\ell_{i-2}=1}^{\ell_{i-3}-1} \frac{1}{\ell_{i-2}}. \tag{2.3}$$

Proof. An easy differentiation gives

$$\begin{aligned}
\left(\frac{1}{\ln x}\right)^{(n+1)} &= \left[\left(\frac{1}{\ln x}\right)^{(n)}\right]' \\
&= \left[\frac{(-1)^n}{x^n}\sum_{i=2}^{n+1}\frac{a_{n,i}}{(\ln x)^i}\right]' \\
&= (-1)^n\sum_{i=2}^{n+1}a_{n,i}\left[\frac{1}{x^n(\ln x)^i}\right]' \\
&= \frac{(-1)^{n+1}}{x^{n+1}}\sum_{i=2}^{n+1}a_{n,i}\frac{i+n\ln x}{(\ln x)^{i+1}} \\
&= \frac{(-1)^{n+1}}{x^{n+1}}\left[\sum_{i=2}^{n+1}\frac{ia_{n,i}}{(\ln x)^{i+1}}+\sum_{i=2}^{n+1}\frac{na_{n,i}}{(\ln x)^i}\right] \\
&= \frac{(-1)^{n+1}}{x^{n+1}}\left[\sum_{i=3}^{n+2}\frac{(i-1)a_{n,i-1}}{(\ln x)^i}+\sum_{i=2}^{n+1}\frac{na_{n,i}}{(\ln x)^i}\right] \\
&= \frac{(-1)^{n+1}}{x^{n+1}}\left[\frac{na_{n,2}}{(\ln x)^2}+\sum_{i=3}^{n+1}\frac{(i-1)a_{n,i-1}+na_{n,i}}{(\ln x)^i}+\frac{(n+1)a_{n,n+1}}{(\ln x)^{n+2}}\right].
\end{aligned}$$

Equating coefficients of $(\ln x)^i$ for $2 \le i \le n+2$ on both sides of

$$\frac{(-1)^{n+1}}{x^{n+1}}\sum_{i=2}^{n+2}\frac{a_{n+1,i}}{(\ln x)^i} = \frac{(-1)^{n+1}}{x^{n+1}}\left[\frac{na_{n,2}}{(\ln x)^2}+\sum_{i=3}^{n+1}\frac{(i-1)a_{n,i-1}+na_{n,i}}{(\ln x)^i}+\frac{(n+1)a_{n,n+1}}{(\ln x)^{n+2}}\right]$$

yields the recursion formulas of the coefficients $a_{n,i}$ satisfying

$$a_{n+1,2} = na_{n,2}, \tag{2.4}$$
$$a_{n+1,n+2} = (n+1)a_{n,n+1}, \tag{2.5}$$

and

$$a_{n+1,i} = (i-1)a_{n,i-1} + na_{n,i} \tag{2.6}$$

for $3 \le i \le n+1$.

From

$$\left(\frac{1}{\ln x}\right)' = -\frac{1}{x(\ln x)^2},$$

it follows that

$$a_{1,2} = 1. \tag{2.7}$$

Combining (2.7) with (2.4) and (2.5) respectively results in (2.2) and

$$a_{n,n+1} = n!. \tag{2.8}$$

Letting $i = 3$ in (2.6) and using (2.2) produce

$$a_{n+1,3} = 2a_{n,2} + na_{n,3} = 2(n-1)! + na_{n,3} \tag{2.9}$$

for $n \ge 2$. Utilizing (2.8) for $n = 2$ as an initial value and recurring (2.9) figure out

$$a_{n,3} = 2!(n-1)!\sum_{k=1}^{n-1}\frac{1}{k} \tag{2.10}$$

for $n \ge 2$.

Taking $i = 4$ in (2.6) and employing (2.10) give

$$a_{n+1,4} = 3a_{n,3} + na_{n,4} = 3\times 2(n-1)!\sum_{k=1}^{n-1}\frac{1}{k} + na_{n,4} \tag{2.11}$$

for $n \ge 3$. Making use of (2.8) for $n = 3$ as an initial value and recurring (2.11) reveal

$$a_{n,4} = 3!(n-1)!\sum_{i=1}^{n-1}\frac{1}{i}\sum_{k=1}^{i-1}\frac{1}{k} \tag{2.12}$$

for $n \ge 3$.

By similar arguments to the deduction of (2.10) and (2.12), we have

$$a_{n,5} = 4!(n-1)!\sum_{j=1}^{n-1}\frac{1}{j}\sum_{i=1}^{j-1}\frac{1}{i}\sum_{k=1}^{i-1}\frac{1}{k} \tag{2.13}$$

for $n \ge 4$ and

$$a_{n,6} = 5!(n-1)!\sum_{\ell=1}^{n-1}\frac{1}{\ell}\sum_{j=1}^{\ell-1}\frac{1}{j}\sum_{i=1}^{j-1}\frac{1}{i}\sum_{k=1}^{i-1}\frac{1}{k} \tag{2.14}$$

for $n \ge 5$.

From (2.10), (2.12), (2.13), and (2.14), we inductively conclude the formula (2.3). The proof of Theorem 2.1 is thus completed. □

Corollary 2.1. *The coefficients $a_{n,i}$ in* (2.1) *satisfies the recursion* (2.6) *for* $3 \le i \le n+1$.

Proof. This follows from the proof of Theorem 2.1. □

Corollary 2.2. *For $n \in \mathbb{N}$, the factorial $n!$ meets*

$$\frac{1}{n!} = \sum_{\ell_1=1}^{n}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{n-1}=1}^{\ell_{n-2}-1}\frac{1}{\ell_{n-1}}\sum_{\ell_n=1}^{\ell_{n-1}-1}\frac{1}{\ell_n}. \tag{2.15}$$

Proof. This follows from combining (2.3) and (2.8) and simplifying. □

Corollary 2.3. *Stirling numbers of the first kind $s(n,i)$ for $1 \le i \le n$ may be computed by*

$$s(n,i) = (-1)^{n+i}(n-1)!\sum_{\ell_1=1}^{n-1}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{i-2}=1}^{\ell_{i-3}-1}\frac{1}{\ell_{i-2}}\sum_{\ell_{i-1}=1}^{\ell_{i-2}-1}\frac{1}{\ell_{i-1}}. \tag{2.16}$$

Proof. This is a direct consequence of comparing the formulas (1.2) and (2.1) and rearranging. □

Corollary 2.4. *For $1 \le i \le n$, Stirling numbers of the first kind $s(n,i)$ satisfies the recursion*

$$s(n+1,i) = s(n,i-1) - ns(n,i). \tag{2.17}$$

Proof. Comparing formulas (1.2) and (2.1) reveals that

$$a_{n,i} = (-1)^{n+i-1}(i-1)!s(n, i-1) \tag{2.18}$$

for $2 \le i \le n+1$. Substituting this into (2.6) and simplifying lead to (2.17). □

Remark 2.1. The recursion (2.17) is called in [1, p. 101] the "triangular" relation which is the most basic recurrence. Corollary 2.4 recovers this triangular relation.

Remark 2.2. It is helps to include a table of concrete values of the coefficients $a_{n,i}$ for small n. See Table 1. Basing on the data listed in Table 1, we conjecture that the sequence $a_{n,i}$ for $n \in \mathbb{N}$ and $2 \le i \le n+1$ is

Table 1: The coefficients $a_{n,i}$

$a_{n,i}$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$n = 1$	1				
$n = 2$	1!	2!			
$n = 3$	2!	6	3!		
$n = 4$	3!	22	36	4!	
$n = 5$	4!	100	210	240	5!
$n = 6$	5!	548	1350	2040	1800
$n = 7$	6!	3528	9744	17640	21000
$n = 8$	7!	26136	78792	162456	235200
$n = 9$	8!	219168	708744	1614816	2693880
$n = 10$	9!	2053152	7036200	17368320	32319000
$n = 11$	10!	21257280	76521456	201828000	410031600

$a_{n,i}$	$i = 7$	$i = 8$	$i = 9$	$i = 10$	$i = 11$
$n = 6$	6!				
$n = 7$	15120	7!			
$n = 8$	231840	141120	8!		
$n = 9$	3265920	2751840	1451520	9!	
$n = 10$	45556560	47628000	35078400	16329600	10!
$n = 11$	649479600	795175920	731808000	479001600	199584000

increasing with respect to n while it is unimodal with respect to i.

Remark 2.3. The elementary method and idea in the proof of Theorem 2.1 has been employed in [10] to establish an explicit formula for computing the n-th derivatives of the tangent and cotangent functions. This explicit formula for the n-th derivative of the cotangent function has been applied in [12] to build the limit formulas for ratios of two polygamma functions at their singularities.

3. Explicit formula for Bernoulli numbers of the second kind

In this section, basing on Theorem 2.1, we establish a new and explicit formula for calculating Bernoulli numbers b_i of the second kind for $i \in \mathbb{N}$.

Theorem 3.1. *For $n \ge 2$, Bernoulli numbers b_n of the second kind can be computed by*

$$b_n = (-1)^n \frac{1}{n!}\left(\frac{1}{n+1} + \sum_{k=2}^{n} \frac{a_{n,k} - n a_{n-1,k}}{k!}\right), \tag{3.1}$$

where $a_{n,k}$ are defined by (2.2) *and* (2.3).

Proof. Differentiating the left-hand side of (1.4) and making use of Theorem 2.1 give

$$
\begin{aligned}
\left[\frac{x}{\ln(1+x)}\right]^{(i)} &= x\left[\frac{1}{\ln(1+x)}\right]^{(i)} + i\left[\frac{1}{\ln(1+x)}\right]^{(i-1)}\\
&= \frac{(-1)^i x}{(1+x)^i}\sum_{k=2}^{i+1}\frac{a_{i,k}}{[\ln(1+x)]^k} + \frac{(-1)^{i-1} i}{(1+x)^{i-1}}\sum_{k=2}^{i}\frac{a_{i-1,k}}{[\ln(1+x)]^k}\\
&= \frac{(-1)^i}{(1+x)^i}\left\{x\sum_{k=2}^{i+1}\frac{a_{i,k}}{[\ln(1+x)]^k} - i(1+x)\sum_{k=2}^{i}\frac{a_{i-1,k}}{[\ln(1+x)]^k}\right\}\\
&= \frac{(-1)^i}{(1+x)^i}\frac{1}{[\ln(1+x)]^{i+1}}\left\{x\sum_{k=2}^{i+1}a_{i,k}[\ln(1+x)]^{i-k+1} - i(1+x)\sum_{k=2}^{i}a_{i-1,k}[\ln(1+x)]^{i-k+1}\right\}.
\end{aligned}
$$

Applying L'Hôspital rule consecutively and by induction, we have

$$
\begin{aligned}
&\lim_{x\to0}\frac{x\sum_{k=2}^{i+1}a_{i,k}[\ln(1+x)]^{i-k+1} - i(1+x)\sum_{k=2}^{i}a_{i-1,k}[\ln(1+x)]^{i-k+1}}{[\ln(1+x)]^{i+1}}\\
&= \lim_{u\to0}\frac{(e^u-1)\sum_{k=2}^{i+1}a_{i,k}u^{i-k+1} - ie^u\sum_{k=2}^{i}a_{i-1,k}u^{i-k+1}}{u^{i+1}}\\
&= \lim_{u\to0}\frac{a_{i,i+1}(e^u-1) - \sum_{k=2}^{i}a_{i,k}u^{i-k+1} + \sum_{k=2}^{i}(a_{i,k}-ia_{i-1,k})\left(e^u u^{i-k+1}\right)}{u^{i+1}}\\
&= \frac{1}{(i+1)!}\lim_{u\to0}\left[a_{i,i+1}(e^u-1)^{(i+1)} - \sum_{k=2}^{i}a_{i,k}\left(u^{i-k+1}\right)^{(i+1)} + \sum_{k=2}^{i}(a_{i,k}-ia_{i-1,k})\left(e^u u^{i-k+1}\right)^{(i+1)}\right]\\
&= \frac{1}{(i+1)!}\lim_{u\to0}\left[a_{i,i+1}e^u + \sum_{k=2}^{i}(a_{i,k}-ia_{i-1,k})\left(e^u u^{i-k+1}\right)^{(i+1)}\right]\\
&= \frac{1}{(i+1)!}\left[a_{i,i+1} + \lim_{u\to0}\sum_{k=2}^{i}(a_{i,k}-ia_{i-1,k})\sum_{m=0}^{i+1}\binom{i+1}{m}e^u\left(u^{i-k+1}\right)^{(m)}\right]\\
&= \frac{1}{(i+1)!}\left[i! + \sum_{k=2}^{i}(a_{i,k}-ia_{i-1,k})\binom{i+1}{i-k+1}(i-k+1)!\right]\\
&= \frac{1}{(i+1)!}\left[i! + \sum_{k=2}^{i}(a_{i,k}-ia_{i-1,k})\frac{(i+1)!}{k!}\right]\\
&= \frac{1}{i+1} + \sum_{k=2}^{i}\frac{a_{i,k}-ia_{i-1,k}}{k!}.
\end{aligned}
$$

This means that

$$
\lim_{t\to0}\left[\frac{x}{\ln(1+x)}\right]^{(i)} = (-1)^i\left(\frac{1}{i+1} + \sum_{k=2}^{i}\frac{a_{i,k}-ia_{i-1,k}}{k!}\right). \tag{3.2}
$$

Differentiating the right-hand side of (1.4) and taking limit generate

$$
\lim_{x\to0}\left[\left(\sum_{n=0}^{\infty}b_n x^n\right)^{(i)}\right] = \lim_{x\to0}\sum_{n=i}^{\infty}b_n\frac{n!}{(n-i)!}x^{n-i} = i!b_i. \tag{3.3}
$$

Equating (3.2) and (3.3) leads to (3.1). The proof of Theorem 3.1 is complete. □

Corollary 3.1. *For $i \in \mathbb{N}$, we have*

$$\left[\frac{x}{\ln(1+x)}\right]^{(i)} = \frac{(-1)^i}{(1+x)^i}\sum_{k=2}^{i+1}\frac{xa_{i,k}-i(1+x)a_{i-1,k}}{[\ln(1+x)]^k} \tag{3.4}$$

and

$$\left[\frac{x}{\ln(1+x)}\right]^{(i)} = \frac{(-1)^i}{(1+x)^i}\sum_{k=1}^{i}\frac{(-1)^{i+k}k![xs(i,k)+i(1+x)s(i-1,k)]}{[\ln(1+x)]^{k+1}}, \tag{3.5}$$

where $a_{i-1,i+1}=0$ and $s(i-1,i)=0$.

Proof. The formula (3.4) can be deduced from the proof of Theorem 3.1. Substituting (2.18) into (3.4) and simplifying result in (3.5). The proof is complete. □

Remark 3.1. The formula (3.5) is a recovery and reformulation of [7, (10), Lemma 2].

4. Integral representations of Stirling numbers of the first kind

In this section, we will find several identities and integral representations relating to Stirling numbers of the first kind $s(n,k)$.

Theorem 4.1. *For $1 \le k \le n+1$, we have*

$$\sum_{i=k-1}^{n}(-1)^{n+i}\frac{i!(i+1)!s(n,i)}{(i-k+1)!} = \int_0^\infty \frac{\Gamma(u+n)}{\Gamma(u)}\left[\sum_{\ell=0}^{k-1}(-1)^\ell c_{k,\ell}u^{k-\ell}\right]e^{-u}\,\mathrm{d}\,u, \tag{4.1}$$

where $\Gamma(u)$ is the classical Euler gamma function which may be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t \tag{4.2}$$

for $\Re z > 0$ and

$$c_{k,\ell} = \binom{k}{\ell}\binom{k-1}{\ell}\ell! \tag{4.3}$$

for all $0 \le \ell \le k-1$.

Proof. In [15], it was obtained that

$$\frac{1}{\ln(1+x)} = \int_0^\infty \frac{1}{(1+x)^u}\,\mathrm{d}\,u, \quad x>0. \tag{4.4}$$

Utilizing this integral representation in (1.2) gives

$$\int_0^\infty \frac{(-1)^m\Gamma(u+m)}{\Gamma(u)}\frac{1}{(1+t)^{u+m}}\,\mathrm{d}\,u = \frac{1}{(1+t)^m}\sum_{i=0}^{m}(-1)^i i!\frac{s(m,i)}{[\ln(1+t)]^{i+1}}.$$

Simplifying this yields

$$\int_0^\infty \frac{\Gamma(u+m)}{\Gamma(u)}\frac{1}{(1+t)^u}\,\mathrm{d}\,u = \sum_{i=0}^{m}(-1)^{m+i}i!\frac{s(m,i)}{[\ln(1+t)]^{i+1}}. \tag{4.5}$$

Substituting t for $\frac{1}{\ln(1+t)}$ in (4.5) brings out

$$\int_0^\infty \frac{\Gamma(u+m)}{\Gamma(u)} e^{-u/t} \,\mathrm{d}\,u = \sum_{i=0}^m (-1)^{m+i} i! s(m,i) t^{i+1}. \tag{4.6}$$

Differentiating $1 \le k \le m+1$ times with respect to t on both sides of (4.6) generates

$$\int_0^\infty \frac{\Gamma(u+m)}{\Gamma(u)} \left(e^{-u/t}\right)^{(k)} \,\mathrm{d}\,u = \sum_{i=k-1}^m (-1)^{m+i} i! s(m,i) \frac{(i+1)!}{(i-k+1)!} t^{i-k+1}.$$

Further letting $t \to 1$ in the above equality produces

$$\sum_{i=k-1}^m (-1)^{m+i} i! s(m,i) \frac{(i+1)!}{(i-k+1)!} = \int_0^\infty \frac{\Gamma(u+m)}{\Gamma(u)} \lim_{t\to 1} \left[\left(e^{-u/t}\right)^{(k)}\right] \mathrm{d}\,u. \tag{4.7}$$

In [13, 14, 16] and [17, Theorem 2.2], it was obtained that

$$\left(e^{-1/t}\right)^{(i)} = \frac{1}{e^{1/t} t^{2i}} \sum_{k=0}^{i-1} (-1)^k c_{i,k} t^k \tag{4.8}$$

for $i \in \mathbb{N}$ and $t \ne 0$, where $c_{i,k}$ is defined by (4.3). Combining this with

$$\frac{\mathrm{d}^i f(ut)}{\mathrm{d}\,t^i} = u^i f^{(i)}(ut)$$

turns out

$$\left(e^{-u/t}\right)^{(k)} = \frac{u^k}{e^{u/t} t^{2k}} \sum_{\ell=0}^{k-1} (-1)^\ell \frac{c_{k,\ell}}{u^\ell} t^\ell$$

which tends to

$$\frac{u^k}{e^u} \sum_{\ell=0}^{k-1} (-1)^\ell \frac{c_{k,\ell}}{u^\ell} = e^{-u} \sum_{\ell=0}^{k-1} (-1)^\ell c_{k,\ell} u^{k-\ell}$$

as $t \to 1$. Substituting this into (4.7) builds (4.1). Theorem 4.1 is proved. □

Theorem 4.2. *For $1 \le k \le m+1$, we have*

$$\sum_{i=k-1}^m \frac{(-1)^{m+i} i! (i+1)! s(m,i)}{(i-k+1)!} = m! \left\{ \lim_{t\to 1} \frac{\mathrm{d}^k}{\mathrm{d}\,t^k} \left[\frac{e^{m/t}}{(e^{1/t}-1)^{m+1}}\right] + \int_1^\infty \frac{1}{[\ln(u-1)]^2 + \pi^2} \lim_{t\to 1} \frac{\mathrm{d}^k}{\mathrm{d}\,t^k} \left[\frac{e^{m/t}}{(e^{1/t}-1+u)^{m+1}}\right] \mathrm{d}\,u \right\}. \tag{4.9}$$

Proof. In [15], it was recited that

$$\frac{1}{\ln(1+z)} = \frac{1}{z} + \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{\mathrm{d}\,t}{z+t}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{4.10}$$

Here we remark that this formula corrects an error appeared in the proof of [3, Theorem 1.3, p. 2130]. Therefore, by (1.2), it is easy to see that

$$(-1)^m m! \left[\frac{1}{t^{m+1}} + \int_1^\infty \frac{1}{[\ln(u-1)]^2 + \pi^2} \frac{\mathrm{d}\,u}{(t+u)^{m+1}}\right] = \frac{1}{(1+t)^m} \sum_{i=0}^m (-1)^i i! \frac{s(m,i)}{[\ln(1+t)]^{i+1}}.$$

Further replacing $\frac{1}{\ln(1+t)}$ by t and rearranging reduce to

$$\sum_{i=0}^{m}(-1)^{m+i}i!s(m,i)t^{i+1} = m!\left[\frac{e^{m/t}}{(e^{1/t}-1)^{m+1}} + \int_1^\infty \frac{1}{[\ln(u-1)]^2+\pi^2}\frac{e^{m/t}\,\mathrm{d}\,u}{(e^{1/t}-1+u)^{m+1}}\right].$$

Differentiating $1 \le k \le m+1$ times with respect to t on both sides of the above equation creates

$$\sum_{i=k-1}^{m}(-1)^{m+i}i!s(m,i)\frac{(i+1)!}{(i-k+1)!}t^{i-k+1} \\ = m!\left\{\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left[\frac{e^{m/t}}{(e^{1/t}-1)^{m+1}}\right] + \int_1^\infty \frac{1}{[\ln(u-1)]^2+\pi^2}\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left[\frac{e^{m/t}}{(e^{1/t}-1+u)^{m+1}}\right]\mathrm{d}\,u\right\}.$$

Further letting $t \to 1$ leads to Theorem 4.2. □

Remark 4.1. For some new results about Bernoulli and Stirling numbers of the first and second kinds, please refer to [4, 11, 15] and closely related references therein.

Remark 4.2. This is a slightly modified version of the preprint [9] which has been referenced in [11, 16].

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