Filomat 28:2 (2014), 329–351
DOI 10.2298/FIL1402329L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

q-Extensions of Some Results Involving the Luo-Srivastava Generalizations of the Apostol-Bernoulli and Apostol-Euler Polynomials

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Abstract. Carlitz firstly defined the q-Bernoulli and q-Euler polynomials [*Duke Math. J.*, **15** (1948), 987–1000]. Recently, M. Cenkci and M. Can [*Adv. Stud. Contemp. Math.*, **12** (2006), 213–223], J. Choi, P. J. Anderson and H. M. Srivastava [*Appl. Math. Comput.*, **199** (2008), 723–737] further defined the q-Apostol-Bernoulli and q-Apostol-Euler polynomials. In this paper, we show the generating functions and basic properties of the q-Apostol-Bernoulli and q-Apostol-Euler polynomials, and obtain some relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials which are the corresponding q-extensions of some known results. Some formulas in series of q-Stirling numbers of the second kind are also considered.

1. Introduction, definitions and motivation

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{C}$ denotes the set of complex numbers.

The falling and rising factorial are defined by

$$\{n\}_0 = 1, \quad \{n\}_k = n(n-1)\cdots(n-k+1),$$
$$(n)_0 = 1, \quad (n)_k = n(n+1)\cdots(n+k-1) \quad (n, k \in \mathbb{N}),$$

respectively.

2010 *Mathematics Subject Classification*. Primary 05A30; Secondary 11B68, 33D15, 11B73

Keywords. Bernoulli and Euler polynomials (of higher order), Apostol-Bernoulli and Apostol-Euler polynomials (of higher order), generating function, q-extension, q-Bernoulli and q-Euler polynomials (of higher order), q-Apostol-Bernoulli and q-Apostol-Euler polynomials (of higher order), q-Stirling numbers of the second kind.

Received: 4 April 2013; Accepted: 14 May 2013

Communicated by H. M. Srivastava

Research supported by PhD Program Scholarship Fund of ECNU 2009 of China under Grant # 2009041, Natural Science Foundation Project of Chongqing under Grant CSTC2011JJA00024, Research Project of Science and Technology of Chongqing Education Commission under Grant KJ120625 and Fund of Chongqing Normal University under Grant 10XLR017 and 2011XLZ07.

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The q-shifted factorial are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k) \quad (n \in \mathbb{N}),$$

$$(a;q)_\infty = (1-a)(1-aq)\cdots(1-aq^n)\cdots = \prod_{k=0}^{\infty}(1-aq^k) \quad (|q|<1;\ a,q \in \mathbb{C}).$$

Clearly,

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

The q-numbers, q-numbers factorial and q-numbers shifted factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \qquad [0]_q! = 1, \quad [n]_q! = [1]_q[2]_q\cdots[n]_q \quad (n \in \mathbb{N});$$

$$([a]_q)_n = [a]_q[a+1]_q\cdots[a+n-1]_q \quad (n \in \mathbb{N},\ a \in \mathbb{C})$$

respectively. Clearly,

$$\lim_{q\to1}[a]_q = a, \quad \lim_{q\to1}[n]_q! = n!, \quad \lim_{q\to1}([a]_q)_n = (a)_n.$$

The q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (z,q \in \mathbb{C};\ |z|<1, \left|q\right|<1). \tag{1.1}$$

When $a = q^\alpha$ ($\alpha \in \mathbb{C}$), then the formula (1.1) becomes the following form:

$$\frac{1}{(z;q)_\alpha} = \frac{(q^\alpha z;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(q^\alpha;q)_n}{(q;q)_n}z^n := \sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}z^n \tag{1.2}$$

$$(z,q,\alpha \in \mathbb{C};\ |z|<1, \left|q\right|<1).$$

The above q-standard notations can be found in [1] and [14].

Some interesting extensions of the classical Bernoulli polynomials and numbers were first investigated by Apostol [2, p. 165, Eq. (3.1)] and (more recently) by Srivastava [35, p. 83-84]. We begin by recalling here the Apostol's definitions as follows:

Definition 1.1 (Apostol [2]; see also Srivastava [35]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ in x are defined by means of the generating function*

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \tag{1.3}$$

$$(|t|<2\pi \text{ when } \lambda = 1;\ |t| < \left|\log\lambda\right| \text{ when } \lambda \neq 1)$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \qquad \textit{and} \qquad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \tag{1.4}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers (in fact, it is a function in λ).

Recently, Luo and Srivastava further extended the Apostol-Bernoulli polynomials and Apostol-Euler polynomials as follows (for convenience , we also say the *Apostol-type polynomials*):

Definition 1.2 (*cf.* Luo and Srivastava [26]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of higher order in x are defined by means of the generating function:*

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \tag{1.5}$$

$$(|t|<2\pi \text{ when } \lambda=1;\ |t|<\left|\log\lambda\right| \text{ when } \lambda\neq 1),$$

with, of course,

$$\begin{aligned} &B_n^{(\alpha)}(x):=\mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda):=\mathcal{B}_n^{(\alpha)}(0;\lambda),\\ &\mathcal{B}_n(x;\lambda):=\mathcal{B}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{B}_n(\lambda):=\mathcal{B}_n^{(1)}(\lambda), \end{aligned} \tag{1.6}$$

where $\mathcal{B}_n(\lambda)$, $\mathcal{B}_n^{(\alpha)}(\lambda)$, $\mathcal{B}_n(x;\lambda)$ and $B_n^{(\alpha)}(x)$ denote the so-called Apostol-Bernoulli numbers, Apostol-Bernoulli numbers of higher order (in fact, they are the functions in λ), Apostol-Bernoulli polynomials and Bernoulli polynomials of higher order respectively.

Remark 1.3. *When $\lambda\neq 1$ in* (1.5)*, the order α should tacitly be restricted to nonnegative integer values.*

Definition 1.4 (*cf.* Luo [16]). *The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of higher order in x are defined by means of the generating function:*

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \qquad \left(|t|<\left|\log(-\lambda)\right|\right), \tag{1.7}$$

with, of course,

$$\begin{aligned} &E_n^{(\alpha)}(x):=\mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda):=2^n\mathcal{E}_n^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right),\\ &\mathcal{E}_n(x;\lambda):=\mathcal{E}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{E}_n(\lambda):=2^n\mathcal{E}_n\left(\frac{1}{2};\lambda\right), \end{aligned} \tag{1.8}$$

where $\mathcal{E}_n(\lambda)$, $\mathcal{E}_n^{(\alpha)}(\lambda)$, $\mathcal{E}_n(x;\lambda)$ and $E_n^{(\alpha)}(x)$ ($n\in\mathbb{N}_0$) denote the so-called Apostol-Euler numbers, Apostol-Euler numbers of higher order (in fact, they are the functions in λ), Apostol-Euler polynomials and Euler polynomials of higher order respectively.

1948, Carlitz firstly extended the classical Bernoulli and Euler numbers and polynomials (of higher order) as the q-Bernoulli and q-Euler numbers and polynomials (of higher order)(see, [5–7]).

Recently, Cenkci and Can [9] further defined the q-extensions of Apostol-Bernoulli numbers and polynomials. Subsequently, J. Choi, P. J. Anderson and H. M. Srivastava [11] gave the following q-extensions of Apostol-Bernoulli and Apostol-Euler polynomials of higher order:

Definition 1.5. *For $q,\ \alpha,\ \lambda\in\mathbb{C}$; $|q|<1$, the q-Apostol-Bernoulli numbers and polynomials of higher order in q^x are respectively defined by means of the generating function*

$$U_{\lambda;q}^{(\alpha)}(t)=(-t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}\lambda^n q^n e^{[n]_q t}=\sum_{n=0}^{\infty}\mathcal{B}_{n;q}^{(\alpha)}(\lambda)\frac{t^n}{n!}, \tag{1.9}$$

$$U_{x;\lambda;q}^{(\alpha)}(t)=(-t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}\lambda^n q^{n+x} e^{[n+x]_q t}=\sum_{n=0}^{\infty}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)\frac{t^n}{n!}. \tag{1.10}$$

Obviously, we have

$$\mathcal{B}_{n;q}^{(\alpha)}(\lambda)=\mathcal{B}_{n;q}^{(\alpha)}(0;\lambda),\quad B_{n;q}^{(\alpha)}=B_{n;q}^{(\alpha)}(0),$$

$$\lim_{q\to 1}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)=\mathcal{B}_n^{(\alpha)}(x;\lambda),\qquad \lim_{q\to 1}B_{n;q}^{(\alpha)}(x)=B_n^{(\alpha)}(x),\qquad \lim_{q\to 1}B_{n;q}^{(\alpha)}=B_n^{(\alpha)},$$

where $B_{n;q}^{(\alpha)} := \mathcal{B}_{n;q}^{(\alpha)}(1)$ *and* $B_{n;q}^{(\alpha)}(x) := \mathcal{B}_{n;q}^{(\alpha)}(x;1)$ *denote the q-Bernoulli numbers and polynomials of higher order respectively;* $B_{n;q} := B_{n;q}^{(1)}$ *and* $B_{n;q}(x) := B_{n;q}^{(1)}(x)$ *denote the q-Bernoulli numbers and polynomials respectively.*

Definition 1.6. *For* $q,\ \alpha,\ \lambda \in \mathbb{C};\ |q| < 1$, *the q-Apostol-Euler numbers and polynomials of higher order in* q^x *are respectively defined by means of the generating function*

$$V_{\lambda;q}^{(\alpha)}(t) = 2^{\alpha} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+\frac{\alpha}{2}} e^{2[n+\frac{\alpha}{2}]_q t} = \sum_{n=0}^{\infty} \mathcal{E}_{n;q}^{(\alpha)}(\lambda)\frac{t^n}{n!}, \tag{1.11}$$

$$V_{x;\lambda;q}^{(\alpha)}(t) = 2^{\alpha} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)\frac{t^n}{n!}. \tag{1.12}$$

Obviously,

$$\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = 2^n \mathcal{E}_{n;q}^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right), \quad E_{n;q}^{(\alpha)} = 2^n E_{n;q}^{(\alpha)}\left(\frac{\alpha}{2}\right),$$
$$\lim_{q\to 1} \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \mathcal{E}_n^{(\alpha)}(x;\lambda), \quad \lim_{q\to 1} E_{n;q}^{(\alpha)}(x) = E_n^{(\alpha)}(x), \quad \lim_{q\to 1} E_{n;q}^{(\alpha)} = E_n^{(\alpha)},$$

where $E_{n;q}^{(\alpha)} := \mathcal{E}_{n;q}^{(\alpha)}(1)$ *and* $E_{n;q}^{(\alpha)}(x) := \mathcal{E}_{n;q}^{(\alpha)}(x;1)$ *denote q-Euler numbers and polynomials of higher order respectively;* $E_{n;q} := \mathcal{E}_{n;q}^{(1)}(1)$ *and* $E_{n;q}(x) := \mathcal{E}_{n;q}^{(1)}(x;1)$ *denote q-Euler numbers and polynomials respectively.*

On the subject of the Apostol type polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature (see [3, 9–12, 15–30, 32–43]; see also the references cited in each of these works).

Remark 1.7. *Throughout this paper, we take the principal value of the logarithm* $\log\lambda$, *i.e.,* $\log\lambda = \log|\lambda| + i\arg\lambda\ (-\pi < \arg\lambda \le \pi)$ *when* $\lambda \neq 1$*; We choose* $\log 1 = 0$ *when* $\lambda = 1$.

The aim of this paper is to derive the generating functions and basic properties of q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order, and to research some relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order. We show some q-extensions of some results of Luo and Srivastava [27] (below Theorem A), Srivastava and Pintér [43] (below Theorem B), Cheon [8] (below (4.5)). Furthermore, other formulas involving the q-Stirling numbers of the second kind are also given.

The paper is organized as follows: In the first section we introduce some notation and rewrite some definitions. In the second section we derive the generating functions of the q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order. In the third section we display the basic properties of q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order. In the fourth section we investigate some relationships between these q-polynomials based on some fairly standard techniques for series rearrangement. In the fifth section we give the formulas in terms of the Carlitz's q-Stirling numbers of the second kind. We provide some new and interesting formulas for the Apostol-Bernoulli and Apostol-Euler polynomials in the sixth section.

2. The generating functions of q-Apostol-Bernoulli and q-Apostol-Euler polynomials

In the present section, by Definition 1.5 and Definition 1.6 we can derive the generating functions and the closed formulas for q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order in order to prove some basic properties in Section 3.

By (1.10) and noting that q-binomial theorem (1.2) yields that

$$\begin{aligned} U^{(\alpha)}_{x;\lambda;q}(t) &= (-t)^{\alpha} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} \lambda^n q^{n+x} e^{[n+x]_q t} \\ &= (-t)^{\alpha} e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} \lambda^n q^{n+x} e^{-\frac{q^{n+x}}{1-q} t} \\ &= (-t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(1-q)^k} \frac{t^k}{k!} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (\lambda q^{k+1})^n \\ &= (-t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!}. \end{aligned} \tag{2.1}$$

Therefore, we obtain the generating function of $\mathcal{B}^{(\alpha)}_{n;q}(x;\lambda)$ as follows:

$$U^{(\alpha)}_{x;\lambda;q}(t) = (-t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{B}^{(\alpha)}_{n;q}(x;\lambda) \frac{t^n}{n!}. \tag{2.2}$$

Clearly, by setting $x = 0$ in (2.2) we have the generating function of $\mathcal{B}^{(\alpha)}_{n;q}(\lambda)$:

$$U^{(\alpha)}_{\lambda;q}(t) = (-t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{B}^{(\alpha)}_{n;q}(\lambda) \frac{t^n}{n!}. \tag{2.3}$$

When $\lambda = 1$ then (2.2) and (2.3) become the generating functions of $B^{(\alpha)}_{n;q}(x)$ and $B^{(\alpha)}_{n;q}$ respectively.

Setting $\alpha = \ell \in \mathbb{N}_0$ in (2.2) and (2.3), via simple calculation, we can get the following closed formulas:

$$\mathcal{B}^{(\ell)}_{n;q}(x;\lambda) = \frac{(-1)^{\ell}\{n\}_{\ell}}{(1-q)^{n-\ell}} \sum_{k=0}^{n-\ell} \binom{n-\ell}{k} \frac{(-1)^k q^{(k+1)x}}{(\lambda q^{k+1};q)_{\ell}} \tag{2.4}$$

and

$$\mathcal{B}^{(\ell)}_{n;q}(\lambda) = \frac{(-1)^{\ell}\{n\}_{\ell}}{(1-q)^{n-\ell}} \sum_{k=0}^{n-\ell} \binom{n-\ell}{k} \frac{(-1)^k}{(\lambda q^{k+1};q)_{\ell}}. \tag{2.5}$$

Remark 2.1. *The special cases of* (2.4) *and* (2.5) *by putting* $\lambda = 1$, $\ell = 1$ *are just the Carlitz's results (4.7) and (4.11) of [5, p. 992] respectively.*

Similarly, we obtain the following generating functions of q-Apostol-Euler polynomials by using (1.2) and (1.12).

$$V^{(\alpha)}_{x;\lambda;q}(t) = 2^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{E}^{(\alpha)}_{n;q}(x;\lambda) \frac{t^n}{n!}. \tag{2.6}$$

Clearly, by setting $x = \frac{\alpha}{2}$ and $t \mapsto 2t$ in (2.6) and noting that $\mathcal{E}^{(\alpha)}_{n;q}(\lambda) = 2^n \mathcal{E}^{(\alpha)}_{n;q}(\frac{\alpha}{2};\lambda)$, we obtain the generating function of q-Apostol-Euler numbers $\mathcal{E}^{(\alpha)}_{n;q}(\lambda)$ as follows:

$$V^{(\alpha)}_{\lambda;q}(t) = 2^{\alpha} e^{\frac{2t}{1-q}} \sum_{k=0}^{\infty} \frac{(-2)^k q^{\frac{(k+1)\alpha}{2}}}{(-\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{E}^{(\alpha)}_{n;q}(\lambda) \frac{t^n}{n!}. \tag{2.7}$$

Putting $\lambda = 1$ in (2.6) and (2.7), we can deduce the generating functions of $E_{n;q}^{(\alpha)}(x)$ and $E_{n;q}^{(\alpha)}$ respectively.

By (2.6) and (2.7), we readily derive the following closed formulas:

$$\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \frac{2^\alpha}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1};q)_\alpha} \tag{2.8}$$

and

$$\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = \frac{2^{n+\alpha}}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{\frac{(k+1)\alpha}{2}}}{(-\lambda q^{k+1};q)_\alpha}. \tag{2.9}$$

Remark 2.2. *The special cases of* (2.8) *and* (2.9) *by setting* $\lambda = 1,\ \alpha = 1$ *are some analogues of Carlitz's numbers* ϵ_m *and polynomials* $\epsilon_m(x)$ *in [5, Eqs. (8.14) and (8.17)] respestively.*

3. The basic properties of q-Apostol-Bernoulli and q-Apostol-Euler polynomials

The following elementary properties of the q-Apostol-Bernoulli and q-Apostol-Euler polynomials are readily derived from Definition 1.5 and Definition 1.6. We, therefore, choose to omit the details involved.

Proposition 3.1. *For* $n, \ell \in \mathbb{N};\ \alpha, \lambda \in \mathbb{C}$,

$$\begin{aligned} &\mathcal{B}_{n;q}^{(\alpha)}(\lambda) = \mathcal{B}_{n;q}^{(\alpha)}(0;\lambda), \qquad \mathcal{B}_{n;q}^{(0)}(x;\lambda) = q^x[x]_q^n,\\ &\mathcal{B}_{0;q}^{(\alpha)}(x;\lambda) = \mathcal{B}_{0;q}^{(\alpha)}(\lambda) = \delta_{\alpha,0}, \qquad \mathcal{B}_{n;q}^{(\ell)}(x;\lambda) = 0 \quad (0 \leqq n \leqq \ell - 1). \end{aligned} \tag{3.1}$$

$\delta_{n,k}$ *being the Kronecker's symbol.*

Proposition 3.2. *A expansion for the q-Apostol-Bernoulli polynomials of higher order*

$$\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k;q}^{(\alpha)}(\lambda)q^{(k-\alpha+1)x}[x]_q^{n-k}. \tag{3.2}$$

Proposition 3.3 (difference equation).

$$\lambda q^{\alpha-1}\mathcal{B}_{n;q}^{(\alpha)}(x+1;\lambda) - \mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = n\mathcal{B}_{n-1;q}^{(\alpha-1)}(x;\lambda) \quad (n \geqq 1). \tag{3.3}$$

Proposition 3.4 (differential relationship).

$$\frac{\partial}{\partial x}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = \mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)\log q + n\frac{\log q}{q-1}q^x\mathcal{B}_{n-1;q}^{(\alpha)}(x;\lambda q). \tag{3.4}$$

Proposition 3.5 (addition theorem).

$$\mathcal{B}_{n;q}^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k;q}^{(\alpha)}(x;\lambda)q^{(k-\alpha+1)y}[y]_q^{n-k}. \tag{3.5}$$

Proposition 3.6 (theorem of complement).

$$\mathcal{B}_{n;q}^{(\ell)}(\ell - x;\lambda) = \frac{(-1)^n}{\lambda^\ell}q^{\ell-n-\binom{\ell}{2}}\mathcal{B}_{n;q^{-1}}^{(\ell)}(x;\lambda^{-1}), \tag{3.6}$$

$$\mathcal{B}_{n;q}^{(\ell)}(\ell + x;\lambda) = \frac{(-1)^n}{\lambda^\ell}q^{\ell-n-\binom{\ell}{2}}\mathcal{B}_{n;q^{-1}}^{(\ell)}(-x;\lambda^{-1}). \tag{3.7}$$

Proposition 3.7 (two recursive formulas).

$$(n-\alpha)\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = n[x]_q\mathcal{B}_{n-1;q}^{(\alpha)}(x;\lambda) - \lambda[\alpha]_q q^x \mathcal{B}_{n;q}^{(\alpha+1)}(x+1;\lambda), \tag{3.8}$$

$$[\alpha]_q q^{x-\alpha}\mathcal{B}_{n;q}^{(\alpha+1)}(x;\lambda) = n\left([x]_q - [\alpha]_q q^{x-\alpha}\right)\mathcal{B}_{n-1;q}^{(\alpha)}(x;\lambda) + (\alpha-n)\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda). \tag{3.9}$$

Proposition 3.8. *For $n, \in \mathbb{N};\ \alpha, \lambda \in \mathbb{C}$,*

$$\begin{aligned} &\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = 2^n\mathcal{E}_{n;q}^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right), \qquad \mathcal{E}_{n;q}^{(0)}(x;\lambda) = q^x[x]_q^n, \\ &\mathcal{E}_{0;q}^{(\alpha)}(\lambda) = \frac{(2\sqrt{q})^\alpha}{(-\lambda q;q)_\alpha}, \qquad \mathcal{E}_{0;q}^{(\alpha)}(x;\lambda) = \frac{2^\alpha q^x}{(-\lambda q;q)_\alpha}. \end{aligned} \tag{3.10}$$

Proposition 3.9. *The formula of the q-Apostol-Euler polynomials of higher order*

$$\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}2^{-k}\mathcal{E}_{k;q}^{(\alpha)}(\lambda)q^{(k+1)(x-\frac{\alpha}{2})}\left[x-\frac{\alpha}{2}\right]_q^{n-k}. \tag{3.11}$$

Proposition 3.10 (difference equation).

$$\lambda q^{\alpha-1}\mathcal{E}_{n;q}^{(\alpha)}(x+1;\lambda) + \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = 2\mathcal{E}_{n;q}^{(\alpha-1)}(x;\lambda) \quad (n \geqq 0). \tag{3.12}$$

Proposition 3.11 (differential relationship).

$$\frac{\partial}{\partial x}\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)\log q + n\frac{\log q}{q-1}q^x\mathcal{E}_{n-1;q}^{(\alpha)}(x;\lambda q). \tag{3.13}$$

Proposition 3.12 (integral formula).

$$\int_a^b q^x\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda q)\,\mathrm{d}x = \frac{1-q}{n+1}\int_a^b \mathcal{E}_{n+1;q}^{(\alpha)}(x;\lambda)\,\mathrm{d}x + \frac{q-1}{\log q}\,\frac{\mathcal{E}_{n+1;q}^{(\alpha)}(b;\lambda) - \mathcal{E}_{n+1;q}^{(\alpha)}(a;\lambda)}{n+1}. \tag{3.14}$$

Proposition 3.13 (addition theorem).

$$\mathcal{E}_{n;q}^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k;q}^{(\alpha)}(x;\lambda)q^{(k+1)y}[y]_q^{n-k}. \tag{3.15}$$

Proposition 3.14 (theorem of complement).

$$\mathcal{E}_{n;q}^{(\alpha)}(\alpha-x;\lambda) = \frac{(-1)^n}{\lambda^\alpha q^{\binom{\alpha}{2}+n}}\mathcal{E}_{n;q^{-1}}^{(\alpha)}(x;\lambda^{-1}), \tag{3.16}$$

$$\mathcal{E}_{n;q}^{(\alpha)}(\alpha+x;\lambda) = \frac{(-1)^n}{\lambda^\alpha q^{\binom{\alpha}{2}+n}}\mathcal{E}_{n;q^{-1}}^{(\alpha)}(-x;\lambda^{-1}). \tag{3.17}$$

Proposition 3.15 (two recursive formulas).

$$\mathcal{E}_{n+1;q}^{(\alpha)}(x;\lambda) = [x]_q\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) - \frac{\lambda}{2}[\alpha]_q q^x\mathcal{E}_{n;q}^{(\alpha+1)}(x+1;\lambda), \tag{3.18}$$

$$[\alpha]_q q^{x-\alpha}\mathcal{E}_{n;q}^{(\alpha+1)}(x;\lambda) = 2\mathcal{E}_{n+1;q}^{(\alpha)}(x;\lambda) + 2\left([\alpha]_q q^{x-\alpha} - [x]_q\right)\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda). \tag{3.19}$$

Remark 3.16. *The Proposition 3.1–Proposition 3.7 are the q-extensions of the basic properties for Apostol-Bernoulli polynomials of higher order (see, [26, p. 301, Eqs. (55)–(63)]).*

When $\lambda = 1$, the Proposition 3.1–Proposition 3.7 will become the corresponding properties for the q-Bernoulli numbers and polynomials of higher order.

When $\lambda = 1,\ \alpha = 1$ or $\ell = 1$, the Proposition 3.1–Proposition 3.7 will become the corresponding basic properties of Carlitz's numbers η_m and polynomials $\eta_m(x)$ in [5, p. 991–993].

Remark 3.17. *The Proposition 3.8–Proposition 3.15 are the q-extensions of the basic properties for Apostol-Euler polynomials of higher order (see, [16, p. 918–919, Eqs. (3)–(13)]).*

When $\lambda = 1$, the Proposition 3.8–Proposition 3.15 will become the corresponding proerties for the q-Euler numbers and polynomials of higher order.

When $\lambda = 1$, $\alpha = 1$ or $\ell = 1$, the Proposition 3.8–Proposition 3.15 will become some analogues of the basic properties of Carlitz's numbers ϵ_m and polynomials $\epsilon_m(x)$ in [5, p. 998–1000].

4. Some explicit relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order

In this section we shall investigate some explicit relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials based on the techniques for series rearrangement.

We now begin by recalling some earlier results of Luo and Srivastava (see, [27]) given by Theorem A below.

Theorem A. For $n \in \mathbb{N}_0;\ \alpha \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \{-1\}$, the following relationships

$$\mathcal{B}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \left[\mathcal{B}_k^{(\alpha)}(y;\lambda) + \frac{k}{2}\mathcal{B}_{k-1}^{(\alpha-1)}(y;\lambda) \right] \mathcal{E}_{n-k}(x;\lambda), \tag{4.1}$$

$$\mathcal{E}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[\mathcal{E}_{k+1}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1}^{(\alpha)}(y;\lambda) \right] \mathcal{B}_{n-k}(x;\lambda) + \frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^{\alpha} \mathcal{B}_{n+1}(x;\lambda) \tag{4.2}$$

hold true.

The special cases of Theorem A for $\lambda = 1$ are just the following elegant results of Srivastava and Á. Pintér [43]:

Theorem B. For $n \in \mathbb{N}_0;\ \alpha \in \mathbb{C}$, the following relationships

$$B_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left[B_k^{(\alpha)}(y) + \frac{k}{2} B_{k-1}^{(\alpha-1)}(y) \right] E_{n-k}(x), \tag{4.3}$$

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[E_{k+1}^{(\alpha-1)}(y) - E_{k+1}^{(\alpha)}(y) \right] B_{n-k}(x) \tag{4.4}$$

hold true.

If further putting $\alpha = 1$ in (4.3) of Theorem B and then letting $y \to 0$, in view of that $B_n^{(0)}(x) = x^n$ and $B_1 = -\frac{1}{2}$, gives us the following Cheon's main result [8]:

$$B_n(x) = \sum_{\substack{k=0 \\ (k\neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) \qquad (n \in \mathbb{N}_0). \tag{4.5}$$

In order to obtain the main results of this paper we need the following facts and lemmas.

Taking $y = 1$ in (3.5), we get

$$\mathcal{B}_{n;q}^{(\alpha)}(x+1;\lambda) = \sum_{k=0}^{n} \binom{n}{k} q^{k-\alpha+1} \mathcal{B}_{k;q}^{(\alpha)}(x;\lambda). \tag{4.6}$$

It follows from (3.3) and (4.6) that

$$\mathcal{B}_{n;q}^{(\alpha-1)}(x;\lambda)=\frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}q^k\mathcal{B}_{k;q}^{(\alpha)}(x;\lambda)-\mathcal{B}_{n+1;q}^{(\alpha)}(x;\lambda)\right]\qquad(n\in\mathbb{N}_0),\tag{4.7}$$

which, in the special case when $\alpha=1$ and noting that $\mathcal{B}_{n;q}^{(0)}(x;\lambda)=q^x[x]_q^n$, we find the following explicit expansion:

$$q^x[x]_q^n=\frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}q^k\mathcal{B}_{k;q}(x;\lambda)-\mathcal{B}_{n+1;q}(x;\lambda)\right],\tag{4.8}$$

which is an q-extension of the known expansion [27, p. 634, Eq. (29)]:

$$x^n=\frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_k(x;\lambda)-\mathcal{B}_{n+1}(x;\lambda)\right].\tag{4.9}$$

Further, setting $\lambda=1$ in (4.8), we easily obtain the following expansion:

$$q^x[x]_q^n=\frac{1}{n+1}\left[\sum_{k=0}^{n}\binom{n+1}{k}q^kB_{k;q}(x)-(1-q^{n+1})B_{n+1;q}(x)\right],\tag{4.10}$$

which is an q-extension of the familiar expansion (*e.g.*, [31, p. 26]):

$$x^n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x).\tag{4.11}$$

It is obvious that

$$[my]_q=[y]_{q^m}[m]_q\tag{4.12}$$

From (3.5) and (4.12) we have

$$\begin{aligned}\mathcal{B}_{n;q^m}^{(\alpha)}(x+y;\lambda)&=\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)[y]_{q^m}^{n-k}\\&=[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)[my]_q^{n-k}.\end{aligned}$$

Upon setting $y=\frac{1}{m}$, we obtain the following formula:

$$\sum_{k=0}^{n}\binom{n}{k}q^{k-\alpha+1}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)=[m]_q^n\mathcal{B}_{n;q^m}^{(\alpha)}\left(x+\frac{1}{m};\lambda\right).\tag{4.13}$$

We define the following polynomials in q^x:

$$\mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda)=\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda).\tag{4.14}$$

Clearly, we have

$$\lim_{q\to1}\mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda)=m^n\mathcal{B}_n^{(\alpha)}\left(x+\frac{1}{m};\lambda\right),\tag{4.15}$$

$$\mathcal{B}_{n;q^m;\frac{1}{m}}^{(\alpha)}(x+1;\lambda)=[m]_q^n\mathcal{B}_{n;q^m}^{(\alpha)}\left(x+\frac{1}{m};\lambda\right),\tag{4.16}$$

$$\mathcal{B}_{n;q}^{(\alpha)}(x+1;\lambda)=\mathcal{B}_{n;q;1}^{(\alpha)}(x+1;\lambda),\qquad\mathcal{B}_{n;q^m;y}(x+1;\lambda)=\mathcal{B}_{n;q^m;y}^{(1)}(x+1;\lambda),$$

$$B_{n;q^m;y}^{(\alpha)}(x+1)=\mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;1),\qquad B_{n;q^m;y}(x+1)=\mathcal{B}_{n;q^m;y}(x+1;1).$$

It is easy to see that the equations (4.13) and (4.14) are q-extensions of the equation (see, [27, p. 634, Eq.(26)] for $x \longleftrightarrow y, y = \frac{1}{m}$)

$$\sum_{k=0}^{n} \binom{n}{k} m^k \mathcal{B}_k^{(\alpha)}(x;\lambda) = m^n \mathcal{B}_n^{(\alpha)}\left(x + \frac{1}{m};\lambda\right). \tag{4.17}$$

The following special values of $\mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda)$ are easily obtained from (4.14) by simple computation.

$$\mathcal{B}_{n;q^m;y}^{(0)}(x;\lambda) = q^{m(x+y-1)}(1 + q^{my}[mx - m]_q)^n, \tag{4.18}$$

$$\mathcal{B}_{0;q^m;y}^{(\alpha)}(x;\lambda) = q^{m(x+y-1)}\delta_{\alpha,0},\ \mathcal{B}_{n;q^m;y}^{(\ell)}(x;\lambda) = 0 \quad (0 \leqq n \leqq \ell - 1), \tag{4.19}$$

where $\delta_{n,k}$ denotes the Kronecker's symbol.

The polynomials $\mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda)$ satisfies the following difference relationship.

Lemma 4.1. *For $n \geqq 1$,*

$$\lambda q^{m(\alpha-1)} \mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda) - \mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda) = n[m]_q \mathcal{B}_{n-1;q^m;y}^{(\alpha-1)}(x;\lambda). \tag{4.20}$$

Proof. By (4.14) and applying (3.3), we obtain

$$\begin{aligned}
&\lambda q^{m(\alpha-1)} \mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda) - \mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} q^{m(k-\alpha+1)y}[m]_q^k \Big[\lambda q^{m(\alpha-1)} \mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda) - \mathcal{B}_{k;q^m}^{(\alpha)}(x-1;\lambda)\Big] \\
&\quad = \sum_{k=0}^{n} k \binom{n}{k} q^{m(k-\alpha+1)y}[m]_q^k \mathcal{B}_{k-1;q^m}^{(\alpha-1)}(x-1;\lambda) \\
&\quad = n \sum_{k=0}^{n-1} \binom{n-1}{k} q^{m(k-\alpha+2)y}[m]_q^{k+1} \mathcal{B}_{k;q^m}^{(\alpha-1)}(x-1;\lambda) \\
&\quad = n[m]_q \mathcal{B}_{n-1;q^m;y}^{(\alpha-1)}(x-1;\lambda).
\end{aligned}$$

Hence, the formula (4.20) follows. □

On the other hand, if setting $y = 1$ in (3.15), we have

$$\mathcal{E}_{n;q}^{(\alpha)}(x+1;\lambda) = \sum_{k=0}^{n} \binom{n}{k} q^{k+1} \mathcal{E}_{k;q}^{(\alpha)}(x;\lambda). \tag{4.21}$$

It follows from (3.12) and (4.21) that

$$\mathcal{E}_{n;q}^{(\alpha-1)}(x;\lambda) = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} q^{k+\alpha} \mathcal{E}_{k;q}^{(\alpha)}(x;\lambda) + \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)\right] \qquad (n \in \mathbb{N}_0) \tag{4.22}$$

which, in the special case when $\alpha = 1$ and noting that $\mathcal{E}_{n;q}^{(0)}(x;\lambda) = q^x[x]_q^n$, we arrive at the following explicit expansion:

$$q^x[x]_q^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} q^{k+1} \mathcal{E}_{k;q}(x;\lambda) + \mathcal{E}_{n;q}(x;\lambda)\right], \tag{4.23}$$

which is an q-extension of the known expansion (see, [27, p. 635, Eq. (32)])

$$x^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k(x;\lambda) + \mathcal{E}_n(x;\lambda)\right]. \tag{4.24}$$

Further, setting $\lambda = 1$ in (4.23), we easily obtain the following expansion:

$$q^x[x]_q^n = \frac{1}{2}\left[\sum_{k=0}^{n} \binom{n}{k} q^{k+1} E_{k;q}(x) + E_{n;q}(x)\right], \tag{4.25}$$

which is an q-extension of the well-known expansion (*e.g.*, [43, p. 378, Eq. (29)]):

$$x^n = \frac{1}{2}\left[\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x)\right]. \tag{4.26}$$

From (3.15) and (4.12) we have

$$\begin{aligned}\mathcal{E}_{n;q^m}^{(\alpha)}(x+y;\lambda) &= \sum_{k=0}^{n} \binom{n}{k} q^{m(k+1)y} \mathcal{E}_{k;q^m}^{(\alpha)}(x;\lambda)[y]_{q^m}^{n-k} \\ &= [m]_q^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{m(k+1)y} [m]_q^k \mathcal{E}_{k;q^m}^{(\alpha)}(x;\lambda)[my]_q^{n-k}.\end{aligned}$$

If we set $y = \frac{1}{m}$, we obtain the following formula:

$$\sum_{k=0}^{n} \binom{n}{k} q^{k+1} [m]_q^k \mathcal{E}_{k;q^m}^{(\alpha)}(x;\lambda) = [m]_q^n \mathcal{E}_{n;q^m}^{(\alpha)}\left(x + \frac{1}{m};\lambda\right). \tag{4.27}$$

We define the following polynomials in q^x:

$$\mathcal{E}_{n;q^m;y}^{(\alpha)}(x+1;\lambda) = \sum_{k=0}^{n} \binom{n}{k} q^{m(k+1)y} [m]_q^k \mathcal{E}_{k;q^m}^{(\alpha)}(x;\lambda). \tag{4.28}$$

Clearly, we have

$$\lim_{q\to 1} \mathcal{E}_{n;q^m;y}^{(\alpha)}(x+1;\lambda) = m^n \mathcal{E}_n^{(\alpha)}\left(x + \frac{1}{m};\lambda\right), \tag{4.29}$$

$$\mathcal{E}_{n;q^m;\frac{1}{m}}^{(\alpha)}(x+1;\lambda) = [m]_q^n \mathcal{E}_{n;q^m}^{(\alpha)}\left(x + \frac{1}{m};\lambda\right), \tag{4.30}$$

$$\mathcal{E}_{n;q}^{(\alpha)}(x+1;\lambda) = \mathcal{E}_{n;q;1}^{(\alpha)}(x+1;\lambda), \qquad \mathcal{E}_{n;q^m;y}(x+1;\lambda) = \mathcal{E}_{n;q^m;y}^{(1)}(x+1;\lambda),$$

$$E_{n;q^m;y}^{(\alpha)}(x+1) = \mathcal{E}_{n;q^m;y}^{(\alpha)}(x+1;1), \qquad E_{n;q^m;y}(x+1) = \mathcal{E}_{n;q^m;y}(x+1;1).$$

It is easy to observe that the equations (4.27) and (4.28) are q-extensions of the equation (see, [27, p. 634, Eq.(27)] for $x \longleftrightarrow y, y = \frac{1}{m}$)

$$\sum_{k=0}^{n} \binom{n}{k} m^k \mathcal{E}_k^{(\alpha)}(x;\lambda) = m^n \mathcal{E}_n^{(\alpha)}\left(x + \frac{1}{m};\lambda\right). \tag{4.31}$$

The following special values of $\mathcal{E}_{n;q^m;y}^{(\alpha)}(x;\lambda)$ are easily obtained from (4.28).

$$\mathcal{E}_{n;q^m;y}^{(0)}(x;\lambda) = q^{m(x+y-1)}(1 + q^{my}[mx-m]_q)^n, \tag{4.32}$$

$$\mathcal{E}_{0;q^m;y}^{(\alpha)}(x;\lambda) = \frac{2^\alpha q^{m(x+y-1)}}{(-\lambda q^m;q^m)_\alpha}. \tag{4.33}$$

Similarly, by (3.12) and (4.28) the polynomials $\mathcal{E}_{n;q^m;y}^{(\alpha)}(x;\lambda)$ in q^x also satisfy the following difference relationship:

Lemma 4.2. *For $n \geqq 0$,*

$$\lambda q^{m(\alpha-1)}\mathcal{E}_{n;q^m;y}^{(\alpha)}(x+1;\lambda)+\mathcal{E}_{n;q^m;y}^{(\alpha)}(x;\lambda)=2\mathcal{E}_{n;q^m;y}^{(\alpha-1)}(x;\lambda). \tag{4.34}$$

Next, by making use of the above formulas and results, we now prove the following formulas of the q-Apostol-Bernoulli polynomials of higher order.

Theorem 4.3. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \alpha,\ \lambda \in \mathbb{C}$, the following relationship:*

$$\begin{aligned}\mathcal{B}_{n;q^m}^{(\alpha)}(x+y;\lambda)=&\frac{1}{2[m]_q^n}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\\&+q^{n-k-m(y+\alpha-1)+1}\Big[\mathcal{B}_{k;q^m;y}^{(\alpha)}(x;\lambda)+k[m]_q\mathcal{B}_{k-1;q^m;y}^{(\alpha-1)}(x;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(my;\lambda)\end{aligned} \tag{4.35}$$

holds true between the q-Apostol-Bernoulli polynomials of higher order and q-Apostol-Euler polynomials.

Proof. First replacing q by q^m in (3.5), and then applying the relation (4.12) and making the suitable substitution in (4.23), we obtain

$$\begin{aligned}\mathcal{B}_{n;q^m}^{(\alpha)}(x+y;\lambda)=&[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)[my]_q^{n-k}\\
=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\left[\lambda\sum_{j=0}^{n-k}\binom{n-k}{j}q^{j+1}\mathcal{E}_{j;q}(my;\lambda)+\mathcal{E}_{n-k;q}(my;\lambda)\right]\\
=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\mathcal{E}_{n-k;q}(my;\lambda)\\
&+\frac{1}{2}\lambda[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\sum_{j=0}^{n-k}\binom{n-k}{j}q^{j+1}\mathcal{E}_{j;q}(my;\lambda)\\
=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\mathcal{E}_{n-k;q}(my;\lambda)\\
&+\frac{1}{2}\lambda[m]_q^{-n}\sum_{j=0}^{n}\binom{n}{j}q^{j+1}\mathcal{E}_{j;q}(my;\lambda)\sum_{k=0}^{n-j}\binom{n-j}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\\
=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)\mathcal{E}_{n-k;q}(my;\lambda)\\
&+\frac{1}{2}\lambda[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k+1}\mathcal{E}_{n-k;q}(my;\lambda)\sum_{j=0}^{k}\binom{k}{j}q^{m(j-\alpha)y}[m]_q^j\mathcal{B}_{j;q^m}^{(\alpha)}(x;\lambda)\\
=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}\left[q^{m(k-\alpha)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)+\lambda q^{n-k-my+1}\mathcal{B}_{k;q^m;y}^{(\alpha)}(x+1;\lambda)\right]\mathcal{E}_{n-k;q}(my;\lambda).\end{aligned}$$

In the above process we have inverted the order of summation and applied the following elementary combinatorial identity:

$$\binom{m}{l}\binom{l}{n}=\binom{m}{n}\binom{m-n}{m-l}. \tag{4.36}$$

Finally, in light of the difference relationship (4.20) of Lemma 4.1, we obtain the assertion (4.35) at once. This proof is complete. □

In view of the symmetry of x, y in Theorem 4.3, the formula (4.35) can be rewritten in the following form:

$$\begin{aligned}\mathcal{B}_{n;q^m}^{(\alpha)}(x+y;\lambda) =& \frac{1}{2[m]_q^n}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{m(k-\alpha)x}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(y;\lambda) + q^{n-k-m(x+\alpha-1)+1}\\ &\times\Big[\mathcal{B}_{k;q^m;x}^{(\alpha)}(y;\lambda) + k[m]_q\mathcal{B}_{k-1;q^m;x}^{(\alpha-1)}(y;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(mx;\lambda).\end{aligned} \tag{4.37}$$

It follows from (4.37) that we give the following corollaries which are the corresponding q-extensions for some well-known results.

Setting $m = 1$ in (4.37), we obtain

Corollary 4.4. *For $n \in \mathbb{N}_0;\ \alpha,\ \lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned}\mathcal{B}_{n;q}^{(\alpha)}(x+y;\lambda) =& \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{(k-\alpha)x}\mathcal{B}_{k;q}^{(\alpha)}(y;\lambda)\\ &+ q^{n-k-x-\alpha+2}\Big[\mathcal{B}_{k;q;x}^{(\alpha)}(y;\lambda) + k\mathcal{B}_{k-1;q;x}^{(\alpha-1)}(y;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(x;\lambda)\end{aligned} \tag{4.38}$$

holds true.

Obviously, the formula (4.38) when $q \to 1$ reduces to (4.1) of Theorem A. Therefore, the formula (4.38) is just an q-extension of the *main* formula (4.1) of Luo and Srivastava [27, p. 636, Theorem 1].

Further, we set $y = 0$ in (4.38), we deduce that

$$\begin{aligned}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) =& \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{(k-\alpha)x}\mathcal{B}_{k;q}^{(\alpha)}(\lambda)\\ &+ q^{n-k-x-\alpha+2}\Big[\mathcal{B}_{k;q;x}^{(\alpha)}(0;\lambda) + k\mathcal{B}_{k-1;q;x}^{(\alpha-1)}(0;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(x;\lambda),\end{aligned} \tag{4.39}$$

which is just an q-extension of the formula of Luo and Srivastava (see, [27, p. 637, Eq. (49)]):

$$\mathcal{B}_{n}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\Big[\mathcal{B}_{k}^{(\alpha)}(\lambda) + \frac{k}{2}\mathcal{B}_{k-1}^{(\alpha-1)}(\lambda)\Big]\mathcal{E}_{n-k}(x;\lambda). \tag{4.40}$$

Putting $\lambda = 1$ in (4.37), we have

Corollary 4.5. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned}B_{n;q^m}^{(\alpha)}(x+y) =& \frac{1}{2[m]_q^n}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{m(k-\alpha)x}[m]_q^k B_{k;q^m}^{(\alpha)}(y)\\ &+ q^{n-k-m(x+\alpha-1)+1}\Big[B_{k;q^m;x}^{(\alpha)}(y) + k[m]_q B_{k-1;q^m;x}^{(\alpha-1)}(y)\Big]\Big]E_{n-k;q}(mx)\end{aligned} \tag{4.41}$$

holds true between the q-Bernoulli polynomials of higher order and q-Euler polynomials.

In particular, setting $\lambda = 1$ in (4.38) or $m = 1$ in (4.41), we thus arrive at the following corollary.

Corollary 4.6. [28, p. 249, Theorem 1, Eq. (3.1)] *For $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned} B_{n;q}^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-\alpha)x} B_{k;q}^{(\alpha)}(y) + \frac{1}{2} q^{n-k-x-\alpha+2} B_{k;q;x}^{(\alpha)}(y) \\ + q^{n-k-x-\alpha+2} \frac{k}{2} B_{k-1;q;x}^{(\alpha-1)}(y) \Big] E_{n-k;q}(x) \end{aligned} \tag{4.42}$$

holds true.

It is obvious that the formula (4.42) when $q \to 1$ reduces to (4.3) of Theorem B. Hence, the formula (4.42) is indeed an q-extension of the *main* formula (4.3) of Srivastava and Á. Pintér (see, [43, p. 379, Theorem 1]).

Setting $\alpha = 1$ in (4.37) and noting that (4.18), we have

Corollary 4.7. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned} \mathcal{B}_{n;q^m}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{m(k-1)x} [m]_q^{k-n} \mathcal{B}_{k;q^m}(y;\lambda) + \frac{1}{2} q^{n-k-mx+1} [m]_q^{-n} \mathcal{B}_{k;q^m;x}(y;\lambda) \\ + \frac{k}{2} [m]_q^{1-n} q^{n-k-m+my+1} (1 + q^{mx}[my-m]_q)^{k-1} \Big] \mathcal{E}_{n-k;q}(mx;\lambda) \end{aligned} \tag{4.43}$$

holds true between the q-Apostol-Bernoulli polynomials and q-Apostol-Euler polynomials.

Further, putting $m = 1$ in (4.43), we get

$$\begin{aligned} \mathcal{B}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-1)x} \mathcal{B}_{k;q}(y;\lambda) + \frac{1}{2} q^{n-k-x+1} \mathcal{B}_{k;q;x}(y;\lambda) \\ + \frac{k}{2} q^{n-k+y} (1 + q^{x}[y-1]_q)^{k-1} \Big] \mathcal{E}_{n-k;q}(x;\lambda), \end{aligned} \tag{4.44}$$

which is an q-extension of the known formula (see, [27, p. 638, Eq. (54)])

$$\mathcal{B}_n(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \left[\mathcal{B}_k(y;\lambda) + \frac{k}{2} y^{k-1} \right] \mathcal{E}_{n-k}(x;\lambda). \tag{4.45}$$

Letting $y \to 0$ in (4.44), we obtain

Corollary 4.8. *For $n \in \mathbb{N}_0$; $\lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned} \mathcal{B}_{n;q}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-1)x} \mathcal{B}_{k;q}(\lambda) + \frac{1}{2} q^{n-k-x+1} \mathcal{B}_{k;q;x}(0;\lambda) \\ + \frac{k}{2} q^{n-k} (1 - q^{x-1})^{k-1} \Big] \mathcal{E}_{n-k;q}(x;\lambda) \end{aligned} \tag{4.46}$$

holds true.

When $q \to 1$, the formula (4.46) reduces to the following form (see, [27, p. 637, Eq. (51)]):

$$\mathcal{B}_n(x;\lambda) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \binom{n}{k} \mathcal{B}_k(\lambda) \mathcal{E}_{n-k}(x;\lambda) + n \left[\mathcal{B}_1(\lambda) + \frac{1}{2} \right] \mathcal{E}_{n-1}(x;\lambda) \tag{4.47}$$

$(\lambda \in \mathbb{C},\ n \in \mathbb{N}_0)$.

Therefore, the formula (4.46) is an q-extension of (4.47).

When $\lambda = 1$, the formula (4.43) reduces to the following known result:

Corollary 4.9. [24, p. 11, Eq. (3.1)] *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned} B_{n;q^m}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \Big[&\frac{1}{2} q^{m(k-1)x} [m]_q^{k-n} B_{k;q^m}(y) + \frac{1}{2} q^{n-k-mx+1} [m]_q^{-n} B_{k;q^m;x}(y) \\ &+ \frac{k}{2} [m]_q^{1-n} q^{n-k-m+my+1} (1 + q^{mx}[my-m]_q)^{k-1} \Big] E_{n-k;q}(mx) \end{aligned} \tag{4.48}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

If we take $\lambda = 1$ in (4.46), we have

Corollary 4.10. [24, p. 13, Eq. (3.8)] *For $n \in \mathbb{N}_0$, the following relationship*

$$B_{n;q}(x) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-1)x} B_{k;q} + \frac{1}{2} q^{n-k-x+1} B_{k;q;x}(0) + \frac{k}{2} q^{n-k} (1 - q^{x-1})^{k-1} \Big] E_{n-k;q}(x) \tag{4.49}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

It is easy to verify that the formula (4.49) is an q-extension of the Cheon's *main* result (4.5) (see, [8, p. 368, Theorem 3]).

In the same manner, we can obtain the following theorem.

Theorem 4.11. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \alpha \in \mathbb{C},\ \lambda \in \mathbb{C} \setminus \{-1\}$, the following relationship*

$$\begin{aligned} \mathcal{E}_{n;q^m}^{(\alpha)}(x+y;\lambda) =& [m]_q^{-n} \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \Big[q^{n-k-m(x+\alpha-1)} \Big[2\mathcal{E}_{k+1;q^m;x}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1;q^m;x}^{(\alpha)}(y;\lambda) \Big] \\ &- [m]_q^{k+1} q^{m(k+1)x} \mathcal{E}_{k+1;q^m}^{(\alpha)}(y;\lambda) \Big] \mathcal{B}_{n-k;q}(mx;\lambda) + \frac{2^\alpha q^{my}(\lambda q^{n+1}-1)}{(n+1)[m]_q^n(-\lambda q^m;q^m)_\alpha} \mathcal{B}_{n+1;q}(mx;\lambda) \end{aligned} \tag{4.50}$$

holds true between the q-Apostol-Euler polynomials of higher order and q-Apostol-Bernoulli polynomials.

In the following we give some interesting special cases of (4.50).

Setting $\lambda = 1$ in (4.50), we have

Corollary 4.12. *For $n \in \mathbb{N}_0;\ \alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned} E_{n;q^m}^{(\alpha)}(x+y) =& [m]_q^{-n} \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \Big[q^{n-k-m(x+\alpha-1)} \Big[2E_{k+1;q^m;x}^{(\alpha-1)}(y) - E_{k+1;q^m;x}^{(\alpha)}(y) \Big] \\ &- [m]_q^{k+1} q^{m(k+1)x} E_{k+1;q^m}^{(\alpha)}(y) \Big] B_{n-k;q}(mx) + \frac{2^\alpha q^{my}(q^{n+1}-1)}{(n+1)[m]_q^n(-q^m;q^m)_\alpha} B_{n+1;q}(mx) \end{aligned} \tag{4.51}$$

holds true between the q-Apostol-Euler polynomials of higher order and q-Apostol-Bernoulli polynomials.

Taking $m = 1$ in (4.50) we get

Corollary 4.13. *For $n \in \mathbb{N}_0;\ \alpha \in \mathbb{C},\ \lambda \in \mathbb{C} \setminus \{-1\}$, the following relationship*

$$\begin{aligned} \mathcal{E}_{n;q}^{(\alpha)}(x+y;\lambda) =& \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \Big[q^{n-k-x-\alpha+1} \Big[2\mathcal{E}_{k+1;q;x}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1;q;x}^{(\alpha)}(y;\lambda) \Big] \\ &- q^{(k+1)x} \mathcal{E}_{k+1;q}^{(\alpha)}(y;\lambda) \Big] \mathcal{B}_{n-k;q}(x;\lambda) + \frac{2^\alpha q^{y}(\lambda q^{n+1}-1)}{(n+1)(-\lambda q;q)_\alpha} \mathcal{B}_{n+1;q}(x;\lambda) \end{aligned} \tag{4.52}$$

holds true between the q-Apostol-Euler polynomials of higher order and q-Apostol-Bernoulli polynomials.

The formula (4.52) when $q \to 1$ reduces to (4.2) of Theorem A. Therefore, the formula (4.52) is an q-extension of the *main* formula (4.2) of Luo and Srivastava (see, [27, p. 638, Theorem 2]).

Further, we put $x = 0$ in (4.52) and then replace y by x, we deduce that

$$\begin{aligned}\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\Big[q^{n-k-\alpha+1}\Big[2\mathcal{E}_{k+1;q;0}^{(\alpha-1)}(x;\lambda) - \mathcal{E}_{k+1;q;0}^{(\alpha)}(x;\lambda)\Big] \\ - \mathcal{E}_{k+1;q}^{(\alpha)}(x;\lambda)\Big]\mathcal{B}_{n-k;q}(\lambda) + \frac{2^{\alpha}q^{x}(\lambda q^{n+1}-1)}{(n+1)(-\lambda q;q)_{\alpha}}\mathcal{B}_{n+1;q}(\lambda),\end{aligned} \tag{4.53}$$

which is an q-extension of the formula of Luo and Srivastava (see, [27, p. 638, Eq. (63)]):

$$\mathcal{E}_{n}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\Big[\mathcal{E}_{k+1}^{(\alpha-1)}(x;\lambda) - \mathcal{E}_{k+1}^{(\alpha)}(x;\lambda)\Big]\mathcal{B}_{n-k}(\lambda) + \frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^{\alpha}\mathcal{B}_{n+1}(\lambda). \tag{4.54}$$

If we put $x = \dfrac{\alpha}{2}$ in (4.53) and note that $\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = 2^{n}\mathcal{E}_{n;q}^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right)$, we derive that

$$\begin{aligned}\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = \sum_{k=0}^{n} \frac{2^{n-k-1}}{k+1}\binom{n}{k}\Big[q^{n-k-\alpha+1}\Big[2^{k+2}\mathcal{E}_{k+1;q;0}^{(\alpha-1)}\left(\frac{\alpha}{2};\lambda\right) - \mathcal{E}_{k+1;q;0}^{(\alpha)}(\lambda)\Big] - \mathcal{E}_{k+1;q}^{(\alpha)}(\lambda)\Big]\mathcal{B}_{n-k;q}(\lambda) \\ + \frac{2^{\alpha}q^{\frac{\alpha}{2}}(\lambda q^{n+1}-1)}{(n+1)(-\lambda q;q)_{\alpha}}\mathcal{B}_{n+1;q}(\lambda),\end{aligned} \tag{4.55}$$

which is an q-extension of the formula of Luo and Srivastava (see, [27, p. 638, Eq. (63)]):

$$\mathcal{E}_{n}^{(\alpha)}(\lambda) = \sum_{k=0}^{n} \frac{2^{n-k}}{k+1}\binom{n}{k}\Big[2^{k+1}\mathcal{E}_{k+1}^{(\alpha-1)}\left(\frac{\alpha}{2};\lambda\right) - \mathcal{E}_{k+1}^{(\alpha)}(\lambda)\Big]\mathcal{B}_{n-k}(\lambda) + \frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^{\alpha}\mathcal{B}_{n+1}(\lambda). \tag{4.56}$$

If we put $\alpha = 1$ and $\lambda = 1$ in (4.55), we have

$$\begin{aligned}E_{n;q} = \sum_{k=0}^{n} \frac{2^{n-k-1}}{k+1}\binom{n}{k}\Big[q^{n-k-\alpha+1}\Big[2^{k+2}E_{k+1;q;0}^{(0)}\left(\frac{1}{2}\right) - E_{k+1;q;0}\Big] - E_{k+1;q}\Big]B_{n-k;q} \\ + \frac{2q^{\frac{1}{2}}(q^{n+1}-1)}{(n+1)(\lambda+1)}B_{n+1;q},\end{aligned} \tag{4.57}$$

which is an q-extension of the familiar formula (see, [27, p. 638, Eqs. (65)]):

$$E_{n} = \sum_{k=0}^{n} \frac{2^{n-k}}{k+1}\binom{n}{k}(1-E_{k+1})B_{n-k}. \tag{4.58}$$

Putting $\lambda = 1$ in (4.52), we arrive at

Corollary 4.14. [28, p. 249, Theorem 1, Eq. (3.2)] *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned}E_{n;q}^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\Big[q^{n-k-x-\alpha+1}\Big[2E_{k+1;q;x}^{(\alpha-1)}(y) - E_{k+1;q;x}^{(\alpha)}(y)\Big] \\ - q^{(k+1)x}E_{k+1;q}^{(\alpha)}(y)\Big]B_{n-k;q}(x) + \frac{2^{\alpha}q^{y}(q^{n+1}-1)}{(n+1)(-q;q)_{\alpha}}B_{n+1;q}(x)\end{aligned} \tag{4.59}$$

holds true between the q-Euler polynomials of higher order and q-Bernoulli polynomials.

The formula (4.59) reduces to (4.4) of Theorem B when $q \to 1$. Hence, the formula (4.59) is an q-extension of the *main* formula (4.4) of Srivastava and Á. Pintér (see, [43, p. 380, Theorem 2]).

Letting $\alpha = 1$ in (4.50) and noting that (4.32), we have

Corollary 4.15. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\lambda \in \mathbb{C} \setminus \{-1\}$, the following relationship*

$$\begin{aligned}\mathcal{E}_{n;q^m}(x+y;\lambda) =&[m]_q^{-n}\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k+m(y-1)}(1+q^{mx}[my-m]_q)^{k+1}\\ &- q^{n-k-mx}\mathcal{E}_{k+1;q^m;x}(y;\lambda) - [m]_q^{k+1}q^{m(k+1)x}\mathcal{E}_{k+1;q^m}(y;\lambda)\Big]\mathcal{B}_{n-k;q}(mx;\lambda)\\ &+\frac{2q^{my}(\lambda q^{n+1}-1)}{(n+1)[m]_q^n(\lambda q^m+1)}\mathcal{B}_{n+1;q}(mx;\lambda)\end{aligned} \tag{4.60}$$

holds true between the q-Apostol-Euler polynomials and q-Apostol-Bernoulli polynomials.

Setting $m = 1$, the formula (4.60) becomes that

$$\begin{aligned}\mathcal{E}_{n;q}(x+y;\lambda) =&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k+y-1}(1+q^{x}[y-1]_q)^{k+1}\\ &- q^{n-k-x}\mathcal{E}_{k+1;q;x}(y;\lambda) - q^{(k+1)x}\mathcal{E}_{k+1;q}(y;\lambda)\Big]\mathcal{B}_{n-k;q}(x;\lambda)\\ &+\frac{2q^{y}(\lambda q^{n+1}-1)}{(n+1)(\lambda q+1)}\mathcal{B}_{n+1;q}(x;\lambda),\end{aligned} \tag{4.61}$$

which is an q-extension of the formula of Luo and Srivastava [27, p. 638, Eq. (56)]:

$$\mathcal{E}_n(x+y;\lambda) = \sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[y^{k+1} - \mathcal{E}_{k+1}(y;\lambda)\right]\mathcal{B}_{n-k}(x;\lambda) + \frac{2(\lambda-1)}{(n+1)(\lambda+1)}\mathcal{B}_{n+1}(x;\lambda).$$

Setting $y = 0$ in (4.61), we have

$$\begin{aligned}\mathcal{E}_{n;q}(x;\lambda) =&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k-1}(1-q^{x-1})^{k+1} - q^{n-k-x}\mathcal{E}_{k+1;q;x}(0;\lambda) - q^{(k+1)x}\mathcal{E}_{k+1;q}(0;\lambda)\Big]\mathcal{B}_{n-k;q}(x;\lambda)\\ &+\frac{2(\lambda q^{n+1}-1)}{(n+1)(\lambda q+1)}\mathcal{B}_{n+1;q}(x;\lambda),\end{aligned} \tag{4.62}$$

is an q-extension of the formula of Luo and Srivastava [27, p. 638, Eq. (57)]:

$$\mathcal{E}_n(x;\lambda) = -\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\mathcal{E}_{k+1}(0;\lambda)\mathcal{B}_{n-k}(x;\lambda) + \frac{2(\lambda-1)}{(n+1)(\lambda+1)}\mathcal{B}_{n+1}(x;\lambda).$$

Taking $\lambda = 1$ in (4.60), we have

Corollary 4.16. [24, p. 13, Eq. (3.10)] *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned}E_{n;q^m}(x+y) =&[m]_q^{-n}\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k+m(y-1)}(1+q^{mx}[my-m]_q)^{k+1}\\ &- q^{n-k-mx}E_{k+1;q^m;x}(y) - [m]_q^{k+1}q^{m(k+1)x}E_{k+1;q^m}(y)\Big]B_{n-k;q}(mx)\\ &+\frac{2q^{my}(q^{n+1}-1)}{(n+1)[m]_q^n(q^m+1)}B_{n+1;q}(mx)\end{aligned} \tag{4.63}$$

holds true between the q-Euler polynomials and q-Bernoulli polynomials.

Setting $y = 0$ in (4.63), we deduce that (see, [24, p. 13, Eq. (3.13)]):

$$\begin{aligned} E_{n;q^m}(x) =&[m]_q^{-n}\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k-m}(1-q^{m(x-1)}[m]_q)^{k+1}-q^{n-k-mx}E_{k+1;q^m;x}(0) \\ &-[m]_q^{k+1}q^{m(k+1)x}E_{k+1;q^m}(0)\Big]B_{n-k;q}(mx)+\frac{2(q^{n+1}-1)}{(n+1)[m]_q^n(q^m+1)}B_{n+1;q}(mx). \end{aligned} \tag{4.64}$$

By setting $m = 1$ in (4.64) we deduce that (see, [24, p. 13, Eq. (3.16)]):

$$\begin{aligned} E_{n;q}(x) =&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k-1}(1-q^{x-1})^{k+1}-q^{n-k-x}E_{k+1;q;x}(0)-q^{(k+1)x}E_{k+1;q}(0)\Big]B_{n-k;q}(x) \\ &+\frac{2(q^{n+1}-1)}{(n+1)(q+1)}B_{n+1;q}(x). \end{aligned} \tag{4.65}$$

5. Some formulas involving the q-Stirling numbers of the second kind

In this section we provide some formulas for the q-Apostol-Bernoulli and q-Apostol-Euler polynomials in series of the q-Stirling numbers of the second kind. Some interesting special cases are also considered. We know that the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k},$$

which satisfies the following relationships:

$$\begin{aligned} &\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \quad (0 \leq k \leq n), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \quad (n < k), \\ &\begin{bmatrix} x \\ k \end{bmatrix}_q = \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_q + q^k\begin{bmatrix} x-1 \\ k \end{bmatrix}_q \qquad (n,k \in \mathbb{N};\ x \in \mathbb{C}). \end{aligned}$$

We recall that the Stirling numbers of the second kind $S(n,k)$ are defined by means of the following expansion (see, [13, p. 207, Theorem B])

$$x^n = \sum_{k=0}^{n}\binom{x}{k}k!S(n,k). \tag{5.1}$$

So that

$$S(n,0) = \delta_{n,0} \qquad S(n,1) = S(n,n) = 1 \qquad S(n,n-1) = \binom{n}{2},$$

where $\delta_{m,n}$ denotes the Kronecker's symbol.

In 1948, Carlitz firstly gave an q-extension of the Stirling numbers of the second kind, i.e., the so-called q-Stirling numbers of the second kind $S_q(n,k)$ are defined by (see, [5, p. 989, Eq. (3.1)])

$$[x]_q^n = \sum_{k=0}^{n}S_q(n,k)[k]_q!\begin{bmatrix} x \\ k \end{bmatrix}_q q^{\binom{k}{2}}. \tag{5.2}$$

Carlitz also showed that the q-Stirling numbers of the second kind $S_q(n,k)$ satisfy the following relationships (see, [5, p. 990, Eq. (3.2) and (3.5)]):

$$S_q(n+1,k) = S_q(n,k-1) + [k]_q S_q(n,k),$$
$$S_q(n,k) = (q-1)^{k-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \begin{bmatrix} j \\ k \end{bmatrix}_q .$$

Obviously,

$$S_q(n,0) = \delta_{n,0} \qquad S_q(n,1) = S_q(n,n) = 1 \qquad S_q(n,n-1) = \frac{n-[n]_q}{1-q}.$$

Noting that (4.12) and making the appropriate substitution in (5.2) into the right side of the formulas (3.5) and (3.15) respectively, we obtain Theorem 5.1 below.

Theorem 5.1. *For $\alpha,\ \lambda \in \mathbb{C}$; $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the following relationships*

$$\mathcal{B}^{(\alpha)}_{n;q^m}(x+y;\lambda) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{m(j-\alpha+1)x+\binom{k}{2}} [m]_q^{j-n} \mathcal{B}^{(\alpha)}_{j;q^m}(y;\lambda) S_q(n-j,k), \tag{5.3}$$

$$\mathcal{E}^{(\alpha)}_{n;q^m}(x+y;\lambda) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{m(j+1)x+\binom{k}{2}} [m]_q^{j-n} \mathcal{E}^{(\alpha)}_{j;q^m}(y;\lambda) S_q(n-j,k) \tag{5.4}$$

hold true between the q-Apostol-type polynomials of higher order and q-Stirling numbers of the second kind.

Setting $m = 1$ in (5.3) and (5.4) of Theorem 5.1, we then obtain the following corollary:

Corollary 5.2. *For $\alpha,\ \lambda \in \mathbb{C}$; $n \in \mathbb{N}_0$, the following relationships*

$$\mathcal{B}^{(\alpha)}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{(j-\alpha+1)x+\binom{k}{2}} \mathcal{B}^{(\alpha)}_{j;q}(y;\lambda) S_q(n-j,k), \tag{5.5}$$

$$\mathcal{E}^{(\alpha)}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{(j+1)x+\binom{k}{2}} \mathcal{E}^{(\alpha)}_{j;q}(y;\lambda) S_q(n-j,k) \tag{5.6}$$

hold true.

It is easy to verify that the formulas (5.5) and (5.6) are respectively the q-extensions of the corresponding formulas (75) and (76) of [27, p. 641].

By setting $\lambda = 1$ and $m = 1$ in (5.3), and taking $\lambda = 1$ and $m = 1$ in (5.4), we have

Corollary 5.3. [28, p. 253, Theorem 3, Eq. (4.11) and (4.12)] *For $\alpha \in \mathbb{C}$; $n \in \mathbb{N}_0$, the following relationships*

$$B^{(\alpha)}_{n;q}(x+y) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{(j-\alpha+1)x+\binom{k}{2}} B^{(\alpha)}_{j;q}(y) S_q(n-j,k), \tag{5.7}$$

$$E^{(\alpha)}_{n;q}(x+y) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{(j+1)x+\binom{k}{2}} E^{(\alpha)}_{j;q}(y) S_q(n-j,k) \tag{5.8}$$

hold true.

Letting $q \to 1$ in (5.7) and (5.8), we obtain the corresponding formulas of Bernoulli and Euler polynomials of higher order.

Setting $\lambda = 1$ and $\alpha = 1$ in (5.3) and (5.4) of Theorem 5.1, then we obtain the following corollary:

Corollary 5.4. [24, p. 14, Eq. (4.5) and (4.6)] *For $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, the following relationships*

$$B_{n;q^m}(x+y) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{mjx+\binom{k}{2}} [m]_q^{j-n} B_{j;q^m}(y) S_q(n-j,k), \tag{5.9}$$

$$E_{n;q^m}(x+y) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{m(j+1)x+\binom{k}{2}} [m]_q^{j-n} E_{j;q^m}(y) S_q(n-j,k) \tag{5.10}$$

hold true.

6. Further observations and consequences

In this section we give some new and interesting formulas of the Apostol-Bernoulli and Apostol-Euler polynomials of higher order based on the corresponding formulas in Section 4.

Letting $q \to 1$ in (4.37) and (4.50) and noting that (4.15) and (4.29), we obtain the following interesting formulas for Apostol-Bernoulli and Apostol-Euler polynomials of higher order.

Theorem 6.1. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$; $\lambda \in \mathbb{C} \setminus \{-1\}$, the following relationships:*

$$\mathcal{B}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[\mathcal{B}_k^{(\alpha)}(y;\lambda) + \mathcal{B}_k^{(\alpha)}\left(y + \frac{1-m}{m};\lambda\right) + k\mathcal{B}_{k-1}^{(\alpha-1)}\left(y + \frac{1-m}{m};\lambda\right) \right] \mathcal{E}_{n-k}(mx;\lambda), \tag{6.1}$$

$$\begin{aligned} \mathcal{E}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \left[2\mathcal{E}_{k+1}^{(\alpha-1)}\left(y + \frac{1-m}{m};\lambda\right) - \mathcal{E}_{k+1}^{(\alpha)}\left(y + \frac{1-m}{m};\lambda\right) - \mathcal{E}_{k+1}^{(\alpha)}(y;\lambda) \right] \mathcal{B}_{n-k}(mx;\lambda) \\ + \frac{\lambda-1}{m^n(n+1)} \left(\frac{2}{\lambda+1}\right)^{\alpha} \mathcal{B}_{n+1}(mx;\lambda) \end{aligned} \tag{6.2}$$

hold true.

Clearly, the above formulas (6.1) and (6.2) are the corresponding extensions of the formulas (4.1) and (4.2) of Theorem A.

If we set $\lambda = 1$ in (6.1) and (6.2), we obtain the following Corollary.

Corollary 6.2. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$, the following relationships*

$$B_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[B_k^{(\alpha)}(y) + B_k^{(\alpha)}\left(y + \frac{1-m}{m}\right) + kB_{k-1}^{(\alpha-1)}\left(y + \frac{1-m}{m}\right) \right] E_{n-k}(mx), \tag{6.3}$$

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \left[2E_{k+1}^{(\alpha-1)}\left(y + \frac{1-m}{m}\right) - E_{k+1}^{(\alpha)}\left(y + \frac{1-m}{m}\right) - E_{k+1}^{(\alpha)}(y) \right] B_{n-k}(mx) \tag{6.4}$$

hold true.

Obviously, the above formulas (6.3) and (6.4) are the corresponding extensions of the formulas (4.3) and (4.4) of Theorem B.

Taking $\alpha = 1$ in (6.1) and (6.2) of Theorem 6.1, we have

Corollary 6.3. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\lambda \in \mathbb{C} \setminus \{-1\}$, the following relationships*

$$\mathcal{B}_n(x+y;\lambda) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[\mathcal{B}_k(y;\lambda) + \mathcal{B}_k\left(y + \frac{1-m}{m};\lambda\right) + k\left(y + \frac{1-m}{m}\right)^{k-1} \right] \mathcal{E}_{n-k}(mx;\lambda), \tag{6.5}$$

$$\begin{aligned} \mathcal{E}_n(x+y;\lambda) = \sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \Bigg[2\left(y + \frac{1-m}{m}\right)^{k+1} - \mathcal{E}_{k+1}\left(y + \frac{1-m}{m};\lambda\right) \\ - \mathcal{E}_{k+1}(y;\lambda) \Bigg] \mathcal{B}_{n-k}(mx;\lambda) + \frac{2}{m^n(n+1)} \frac{\lambda-1}{\lambda+1} \mathcal{B}_{n+1}(mx;\lambda) \end{aligned} \tag{6.6}$$

hold true.

Setting $\lambda = 1$ in (6.5) and (6.6), we obtain the following new and interesting formulas respectively.

$$B_n(x+y) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[B_k(y) + B_k\left(y + \frac{1-m}{m}\right) + k\left(y + \frac{1-m}{m}\right)^{k-1} \right] E_{n-k}(mx), \tag{6.7}$$

$$E_n(x+y) = \sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \left[2\left(y + \frac{1-m}{m}\right)^{k+1} - E_{k+1}\left(y + \frac{1-m}{m}\right) - E_{k+1}(y) \right] B_{n-k}(mx). \tag{6.8}$$

Obviously, the formulas (6.7) and (6.8) are respectively the extensions of the formulas of Srivastava and Á. Pintér (see, [43]):

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left[B_k(y) + \frac{k}{2} y^{k-1} \right] E_{n-k}(x), \tag{6.9}$$

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left[y^{k+1} - E_{k+1}(y) \right] B_{n-k}(x). \tag{6.10}$$

Acknowledgements

Dedicated to Professor Hari M. Srivastava on the Occasion of his 73rd Birth Anniversary

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