Filomat 28:2 (2014), 257–262
DOI 10.2298/FIL1402257S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# A new common fixed point theorem for Suzuki-Meir-Keeler contractions

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**Abstract.** In this paper, we extend a recent Meir-Keeler type fixed point theorem of Suzuki (2008) to a pair of maps on a metric space.

## 1. Introduction

The following important result due to Meir and Keeler [6] is a generalization of the classical Banach contraction theorem.

**Theorem 1.1.** *A selfmap $S$ of a complete metric space $(X, d)$ satisfying the condition:*

*for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Sx, Sy) < \varepsilon, \tag{MK.1}$$

*possesses a unique fixed point.*

A map $S$ satisfying the condition (MK.1) is popularly called Meir-Keeler contraction (see, for instance, [9]). The elegant technique employed to prove Theorem 1.1 attracted several authors to work along these lines and subsequently Theorem 1.1 was generalized and extended in various ways (see, for instance, Ćirić [1], Jachymski [2], Kikkawa and Suzuki [3], Kuczma et al. [4], Lim [5], Park and Rhoades [7], Rhoades et al. [8], Singh et al. [9], Suzuki [10] and references of [2] and [9]).
Entirely different and an ingenious approach to generalize Theorem 1.1 was adopted by Suzuki [10] to obtain the following result.

**Theorem 1.2.** *Let $S$ be a selfmap of a complete metric space $(X, d)$ such that*

*for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,*

$$\frac{1}{2}d(x, Sx) < d(x, y) \quad \text{and} \quad d(x, y) < \varepsilon + \delta \text{ imply } d(Sx, Sy) \leq \varepsilon\ ; \tag{S.1}$$

$$\frac{1}{2}d(x, Sx) < d(x, y) \quad \text{implies} \quad d(Sx, Sy) < d(x, y). \tag{S.2}$$

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
*Keywords*. Fixed point, Meir-Keeler contraction, Suzuki-Meir-Keeler contraction
Received: 18 January 2013; Accepted: 23 August 2013
Communicated by Vladimir Rakocevic
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*Then S has a unique fixed point in X.*

The purpose of this paper is to obtain an extension of Theorem 1.2 for a pair of maps on a metric space.

## 2. Main Result

Throughout the paper we denote by $N$ the set of positive integers.

**Theorem 2.1.** *Let $X$ be a complete metric space and let $S, T : X \to X$. Assume that for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,*

$$\frac{1}{2}\min\{d(x, Sx), d(y, Ty)\} < d(x, y) \quad \textit{and} \quad d(x, y) < \varepsilon + \delta \quad \textit{imply} \quad d(Sx, Ty) \le \varepsilon \tag{1}$$

*and*

$$\frac{1}{2}\min\{d(x, Sx), d(y, Ty)\} < d(x, y) \quad \textit{implies} \quad d(Sx, Ty) < d(x, y). \tag{2}$$

*Then there exists a unique element $z \in X$ such that $Sz = z = Tz$, that is, $z$ is the unique common fixed point of $S$ and $T$.*

*Proof.* We assert that

(i) $d(x, Sx) > 0$ implies $d(Sx, TSx) < d(x, Sx)$, $x \in X$, and
(ii) $d(y, Ty) > 0$ implies $d(STy, Ty) < d(y, Ty)$, $y \in X$.

Evidently it is enough to prove (i). For any $x \in X$, if $d(x, Sx) > 0$ then

$$\frac{1}{2}d(x, Sx) < d(x, Sx). \tag{3}$$

Now if

$$d(x, Sx) < d(Sx, TSx) \tag{4}$$

then by (2),

$$d(Sx, TSx) < d(x, Sx), \quad \text{a contradiction to (4).}$$

So $d(Sx, TSx) < d(x, Sx)$. If

$$d(Sx, TSx) = d(x, Sx) \tag{5}$$

then by (2),

$$d(Sx, TSx) < d(x, Sx), \quad \text{a contradiction to (5).}$$

Thus

$$d(Sx, TSx) < d(x, Sx) \tag{6}$$

holds for all $x \in X$ with $Sx \neq x$.
Pick $u_0 \in X$. Construct a sequence $\{u_n\}$ in $X$ such that

$$u_1 = Tu_0,\ u_2 = Su_1, \ldots,\ u_{2n+1} = Tu_{2n}, u_{2n} = Su_{2n-1}, \quad n \in N.$$

If for any $n$, $Su_{2n-1} = u_{2n-1}$, then $u_{2n-1}$ is a fixed point of $S$. So we take $Su_{2n-1} \neq u_{2n-1}$ for all $n \in N$. So by (i),

$$d(Su_{2n-1}, TSu_{2n-1}) < d(u_{2n-1}, Su_{2n-1}),$$

that is,

$$d(u_{2n}, u_{2n+1}) < d(u_{2n-1}, u_{2n}). \tag{7}$$

If for any $n$, $Tu_{2n} = u_{2n}$, then $u_{2n}$ is a fixed point of $T$. So we take $Tu_{2n} \neq u_{2n}$ for all $n \in N$. Therefore by (ii),

$$d(STu_{2n}, Tu_{2n}) < d(u_{2n}, Tu_{2n}),$$

that is,

$$d(u_{2n+2}, u_{2n+1}) < d(u_{2n}, u_{2n+1}). \tag{8}$$

Hence by (7) and (8), we have for all $n \in N$,

$$d(u_n, u_{n+1}) < d(u_{n-1}, u_n). \tag{9}$$

Since the sequence $\{d(u_n, u_{n+1})\}$ is strictly decreasing and bounded below by 0, $\{d(u_n, u_{n+1})\}$ converges to some $\alpha \geq 0$. Assume $\alpha > 0$. Since $\{d(u_n, u_{n+1})\}$ is strictly decreasing,

$$d(u_n, u_{n+1}) > \alpha, \quad n \in N. \tag{10}$$

Now by (7),

$$\frac{1}{2} d(u_{2n}, u_{2n+1}) < d(u_{2n-1}, u_{2n}),$$

that is

$$\frac{1}{2} \min\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\} < d(u_{2n-1}, u_{2n}),$$

that is

$$\frac{1}{2} \min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} < d(u_{2n-1}, u_{2n}). \tag{11}$$

By the definition of $\alpha$, there exists $2n-1 \in N$ such that

$$d(u_{2n-1}, u_{2n}) < \alpha + \delta. \tag{12}$$

Then in view of (1), (11) and (12) yield $d(u_{2n}, u_{2n+1}) \leq \alpha$. This is a contradiction to (10). So our assumption $\alpha > 0$ is wrong. Hence $\alpha = 0$, and we have proved that

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0.$$

Now fix $\varepsilon > 0$. Then there exists

$$\delta \in (0, \varepsilon) \quad \text{(i.e. } \delta > 0) \tag{13}$$

such that by (1),

$$\frac{1}{2} \min\{d(x, Sx), d(y, Ty)\} < d(x, y) \text{ and } d(x, y) < \varepsilon + \delta \quad \text{imply} \quad d(Sx, Ty) \leq \varepsilon.$$

Since $\delta > 0$ and $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$, there exists $p \in N$ such that

$$d(u_n, u_{n+1}) < \delta \quad \text{for all } \; n \geq p. \tag{14}$$

Now we show by induction that

$$d(u_p, u_{p+q}) < \varepsilon + \delta, \quad q \in N. \tag{15}$$

Notice that from (14), $d(u_p, u_{p+1}) < \delta$.
Since $\varepsilon > 0$, we obtain $d(u_p, u_{p+1}) < \delta < \varepsilon + \delta$. Hence (15) holds for $q = 1$.
Now we consider the following two cases:

(a) $d(u_p, u_{p+q}) \leq \varepsilon$, and
(b) $d(u_p, u_{p+q}) > \varepsilon$.

In the case (a), we have by triangle inequality,

$$\begin{aligned} d(u_p, u_{p+q+1}) &\leq d(u_p, u_{p+q}) + d(u_{p+q}, u_{p+q+1}) \\ &\leq \varepsilon + d(u_{p+q}, u_{p+q+1}). \end{aligned}$$

So by (14), $d(u_p, u_{p+q+1}) < \varepsilon + \delta$.
Thus in the case (a), (15) holds for all $q \in N$.
In the case (b), we have

$$\varepsilon < d(u_p, u_{p+q}) < \varepsilon + \delta. \tag{16}$$

Combining (13) and (14), we have

$$d(u_p, u_{p+1}) < \delta < \varepsilon. \tag{17}$$

Combining (16) and (17), we obtain

$$d(u_p, u_{p+1}) < \delta < \varepsilon < d(u_p, u_{p+q}). \tag{18}$$

Clearly from (16), $d(u_p, u_{p+q}) > 0$. So

$$d(u_p, u_{p+q}) < 2d(u_p, u_{p+q}). \tag{19}$$

Combining (18) and (19), we obtain

$$d(u_p, u_{p+1}) < 2d(u_p, u_{p+q}).$$

Therefore

$$\frac{1}{2}\min\{d(u_p, u_{p+1}), d(u_{p+q}, u_{p+q+1})\} < d(u_p, u_{p+q}) \quad \text{and} \quad d(u_p, u_{p+q}) < \varepsilon + \delta.$$

Hence by (1),

$$d(u_{p+1}, u_{p+q+1}) \leq \varepsilon. \tag{20}$$

Using (14) and (20) in the triangle inequality

$$d(u_p, u_{p+q+1}) \leq d(u_p, u_{p+1}) + d(u_{p+1}, u_{p+q+1}),$$

we obtain

$$d(u_p, u_{p+q+1}) < \delta + \varepsilon.$$

Therefore by induction, (15) holds for every $q \in N$ in case (b) as well, and we conclude that $d(u_p, u_{p+q}) < \varepsilon + \delta$ for every $q \in N$. Consequently $\lim_{n\to\infty} \sup_{q>n} d(u_n, u_q) = 0$.

Therefore $\{u_n\}$ is a Cauchy sequence. Since $X$ is complete, the sequence $\{u_n\}$ has a limit in $X$. Call it $z$. Now we show that $z$ is a common fixed point of $S$ and $T$.

Since $u_{2n} \neq Tu_{2n}$ for all $n \in N$, the sequence $\{d(u_{2n}, u_{2n+1})\}$ is strictly decreasing.

If we assume that

$$d(u_{2n}, u_{2n+1}) \geq 2d(u_{2n}, z) \quad \text{and} \quad d(u_{2n+1}, u_{2n+2}) \geq 2d(u_{2n+1}, z)$$

holds for some $n \in N$, then we have

$$\begin{aligned} d(u_{2n}, u_{2n+1}) &\leq d(u_{2n}, z) + d(z, u_{2n+1}) \\ &\leq \frac{d(u_{2n}, u_{2n+1}) + d(u_{2n+1}, u_{2n+2})}{2} \end{aligned}$$

that is, $d(u_{2n}, u_{2n+1}) \leq d(u_{2n+1}, u_{2n+2})$, a contradiction.

So for any $n \in N$, either

(c) $d(u_{2n}, u_{2n+1}) < 2d(u_{2n}, z)$, or
(d) $d(u_{2n+1}, u_{2n+2}) < 2d(u_{2n+1}, z)$.

First we assume that (c) is true and consider the following two cases.

Case 1: If $d(u_{2n}, Tu_{2n}) < d(z, Sz)$, then

$$\frac{1}{2} \min\{d(u_{2n}, Tu_{2n}), d(z, Sz)\} < \frac{1}{2} d(u_{2n}, Tu_{2n}) = \frac{1}{2} d(u_{2n}, u_{2n+1}) < d(u_{2n}, z).$$

So by (2), $d(Tu_{2n}, Sz) < d(u_{2n}, z)$, that is, $d(u_{2n+1}, Sz) < d(u_{2n}, z)$.

Case 2: If $d(z, Sz) < d(u_{2n}, Tu_{2n})$, then

$$\frac{1}{2} \min\{d(z, Sz), d(u_{2n}, Tu_{2n})\} < \frac{1}{2} d(z, Sz) < \frac{1}{2} d(u_{2n}, Tu_{2n}) < d(u_{2n}, z).$$

So by (2), $d(Tu_{2n}, Sz) < d(u_{2n}, z)$, that is, $d(u_{2n+1}, Sz) < d(u_{2n}, z)$.

So in either of the two cases (1) and (2), we obtain

$$d(u_{2n+1}, Sz) < d(u_{2n}, z) \quad \text{holds for all } n \in N.$$

Passing to the limit, this yields $Sz = z$. Analogously, $Tz = z$.

Now we suppose (d) is true. Proceeding as in case (c), one can show that $z$ is fixed point of $T$, and $z$ is a fixed point of $S$ as well.

Thus this is completely proved that $z$ is common fixed point of $S$ and $T$.

Now we prove the uniqueness of the common fixed point. Suppose $y$ is another common fixed point of $S$ and $T$. Then

$$\frac{1}{2} \min\{d(z, Sz), d(y, Ty)\} = 0 < d(z, y) \quad \text{implies} \quad d(Sz, Ty) < d(z, y)\,.$$

That is, $d(z, y) = d(Sz, Ty) < d(z, y)$.

This contradiction proves that $y = z$. □

We remark that Suzuki-Meir-Keeler contraction $S$, viz. $S$ satisfying (S.1) and (S.2) is obtained from the conditions (1) and (2) with $T = S$. Hence we have the following.

**Corollary 2.2.** *Theorem 1.2.*

## References


[1] Lj.B. Ćirić, A new fixed point theorem for contractive mapping, Publ. Inst. Math. (Beograd) (N.S.) 30(44) (1981) 25–27.

[2] J. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, J. Math. Anal. Appl. 194 (1995) 293–303.

[3] M. Kikkawa and Tomonari Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Anal. 69 (2008) 2942–2949.

[4] M. Kuczma, B. Choczewski and R. Ger, Iterative functional equation, Encyclopedia of Mathematics and Applications, Vol. 32, Cambridge University Press, Cambridge, 1990.

[5] Teck-Cheong Lim, On characterizations of Meir-Keeler contractive maps, Nonlinear Anal. 46 (1) (2001) 113–120.

[6] A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969) 326–329.

[7] S. Park and B.E. Rhoades, Meir-Keeler type contractive conditions, Math. Japon. 26 (1981) 13–20.

[8] B.E. Rhoades, S. Park and K.B. Moon, On generalization of Meir-Keeler type contraction maps, J. Math. Anal. Appl. 146 (1990) 482–494.

[9] S.L. Singh, Ashish Kumar and Y.J. Cho, Fixed points of Meir-Keeler type hybrid contractions, Pan Amer. Math. J. 16 (2006) 35–54.

[10] Tomonari Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861–1869.