Filomat 28:2 (2014), 285–303
DOI 10.2298/FIL1402285F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

TETRAVALENT ONE-REGULAR GRAPHS OF ORDER $4p^2$

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Abstract. A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this paper tetravalent one-regular graphs of order $4p^2$, where p is a prime, are classified.

1. Introduction

A graph is *arc-transitive* if its automorphism group acts transitively on the set of its arcs. A graph is *one-regular* if its automorphism group acts regularly on the set of its arcs. Not surprisingly arc-transitive graphs - and one-regular graphs in particular - have received considerable attention over the years, the aim being to obtain structural results and possibly a classification of such graphs of particular orders or satisfying certain additional properties. Research in one-regular graphs is interesting for two reasons, the first being their connection to regular maps, a lively area of research. Namely, the underlying graphs of chiral maps admit one-regular group actions with a cyclic vertex stabilizers (see, for example, [8, 10–12]). Second, one may argue that one-regular graphs are interesting in their own right if one's goal is a description of arc-transitive graphs. For some classes of Cayley graphs, for example, circulants, this has been achieved, whereas for others, such as Cayley graphs of dihedral groups, all 2-arc-transitive graphs have been completely classified [16], but arc-transitivity remains an open problem.

Clearly, a one-regular graph with no isolated vertices is connected, and it is of valency 2 if and only if it is a cycle. The first example of a cubic one-regular graph was constructed by Frucht [21]. Further research in cubic one-regular graphs has been part of a more general project dealing with the investigation of cubic arc-transitive graphs (see [9, 15, 17–20, 31]). Tetravalent one-regular graphs have also received considerable attention. In [4] tetravalent one-regular graphs of prime order were constructed, and in [30] an infinite family of tetravalent one-regular Cayley graphs on alternating groups is given. Tetravalent one-regular circulant graphs were classified in [41], and tetravalent one-regular Cayley graphs on abelian groups were classified in [40]. Next, one may extract a classification of tetravalent one-regular Cayley graphs on dihedral

2010 *Mathematics Subject Classification*. Primary 05C25; Secondary 20B25
Keywords. one-regular graph, symmetric graph, Cayley graph
Received: 01 October 2012; Accepted: 13 September 2013
Communicated by Dragan Stevanović
The first author supported by the National Natural Science Foundation of China (11171020 and 11231008). The second author and the third author supported by P1-0285. The forth author supported by ARRS, proj. mladi raziskovalci, and H.C. Ørsted Postdoctoral Fellowship at DTU (Technical University of Denmark).
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groups from [26, 36, 38]. Let p and q be primes. Clearly every tetravalent one-regular graph of order p is a circulant graph. Also, by [7, 32, 34, 37, 40, 41], every tetravalent one-regular graph of order pq or p^2 is a circulant graph. Furthermore, the classification of tetravalent one-regular graphs of order $2pq$ is given in [43]. The aim of this paper is to classify tetravalent one-regular graphs of order $4p^2$, see Theorem 5.1. (For more results on tetravalent arc-transitive graphs, see [22, 23, 27, 33].)

In the next section we gather various concepts that are needed in the analysis of tetravalent one-regular graphs in Section 4 and in the proof of our main result in Section 5. In Section 3, we give examples of tetravalent one-regular graphs of order $4p^2$, where p is a prime.

2. Preliminaries

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, its edge set, its arc-set and its full automorphism group, respectively. For $u, v \in V(X)$, denote by uv the edge incident to u and v in X. By C_n and K_n we denote the cycle of length n and the complete graph of order n, respectively.

A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. The graph X is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group is vertex-transitive, edge-transitive and arc-transitive, respectively. An arc-transitive graph is also called a *symmetric* graph. An arc-transitive graph X is said to be *one-regular* if $\mathrm{Aut}(X)$ acts regularly on $A(X)$. A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *k-arc-transitive* if it acts transitively on the set of k-arcs, and it is said to be *k-regular* if it is k-arc-transitive and the stabilizer of a k-arc in G is trivial.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. The permutation group $R(G) = \{R(g) \mid g \in G\}$ on G is called the *right regular representation* of G. It is easy to see that $R(G)$ is isomorphic to G, and it is a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [42, Proposition 1.5] proved that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$.

Given a transitive group G acting on a set V, we say that a partition $\mathcal{B}$ of V is *G-invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only G-invariant partitions of V, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. In the latter case we shall refer to a corresponding G-invariant partition as to an *imprimitive block system* of G.

2.1. Group theoretic results

Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, and by $\mathbb{Z}_n^*$ the multiplicative group of units of $\mathbb{Z}_n$. For two groups M and N, $N \leq M$ means that N is a subgroup of M and $N < M$ means that N is a proper subgroup of M.

For a permutation group G on a set Ω and $\alpha \in \Omega$ we let G_α denote the stabilizer of α in G, that is, the subgroup of G fixing the element $\alpha \in \Omega$. The group G is said to be *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and it is said to be *regular* if it is both transitive and semiregular on Ω.

Below we gather various group-theoretic results that are needed in the subsequent sections of this paper. The first one is about transitive abelian permutation groups.

Proposition 2.1. [35, Proposition 4.4] *Every transitive abelian group G on a set Ω is regular.*

For a subgroup H of a group G, let $C_G(H)$ be the centralizer of H in G, and let $N_G(H)$ be the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.2. [25, Chapter I, Theorem 4.5] *Let G be a group and H a subgroup of G. Then the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of H.*

The following result can be extracted from [13, P.285, summary].

Proposition 2.3. [13] *Let $G = \mathrm{PSL}(2,7)$ and let $A = \mathrm{PGL}(2,7)$. Then Sylow 2-subgroups of G and A are, respectively, isomorphic to D_8 and D_{16}. Moreover, all involutions of G are conjugate, and G has no subgroup of order 14.*

The following classical result is due to Wielandt [35, Theorems 3.4].

Proposition 2.4. [35] *Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. Let $w \in \Omega$. If p^m divides the length of the G-orbit containing ω, then p^m also divides the length of the P-orbit containing w.*

2.2. Graph covers

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p\colon V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})}\colon N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The set $\mathrm{fib}_v = p^{-1}(v)$ is a *fibre* of a vertex $v \in V(X)$. The subgroup K of all those automorphisms of $\widetilde{X}$ which fix each of the fibres setwise is called the *group of covering transformations*. If the group of covering transformations is regular on the fibres of $\widetilde{X}$, we say that $\widetilde{X}$ is a *regular K-covering*. We say that $\alpha \in \mathrm{Aut}(X)$ *lifts* to an automorphism of $\widetilde{X}$ if there exists $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$, called the *lift* of α, such that $\widetilde{\alpha}p = p\alpha$.

Let X be a graph and K a finite group. A *K-voltage assignment* of X is a function $\phi\colon A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$, where a^{-1} denotes the reverse arc of the arc a. The values of ϕ are called *voltages*, and K is the *voltage group*. The graph $X \times_\phi K$ derived from a voltage assignment $\phi\colon A(X) \to K$ has vertex set $V(X) \times K$ and edges of the form $(u, g)(v, g\phi(a))$ where $a = (u, v) \in A(X)$ and $g \in K$. Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p\colon X \times_\phi K \to X$. By letting K act on $V(X \times_\phi K)$ as $(u, g')^g = (u, gg')$, $(u, g') \in V(X \times_\phi K)$, one obtains a semiregular subgroup of $\mathrm{Aut}(X \times_\phi K)$, showing that $X \times_\phi K$ can in fact be viewed as a K-covering. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Moreover, Gross and Tucker [24] showed that every regular covering $\widetilde{X}$ of a graph X can in fact be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. (Given a spanning tree T of a graph X, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are all equal to the identity of K.) If $X \times_\phi K \to X$ is a connected K-covering derived from a T-reduced voltage assignment ϕ then the problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\overline{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by $(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha)$, where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on C and C^α, respectively. Note that if K is abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X. The next proposition is a special case of [30, Theorem 4.2].

Proposition 2.5. [30] *Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then, an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

For more results on graph covers we refer the reader to [1, 2, 14, 28, 29].

2.3. Tetravalent arc-transitive graphs

In this subsection we gather known results about tetravalent arc-transitive graphs that will be needed in subsequent sections. The first two propositions can be deduced from [40, Theorem 3.5].

Proposition 2.6. [40] *Let p be a prime, and $G \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$ or $G \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$. Then there exists a tetravalent one-regular Cayley graph on G if and only if $p-1$ is a multiple of 4. Moreover in each of these two cases exactly one such graph exists.*

Proposition 2.7. [40] *Let p be a prime and $G \cong \mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$. Then there is no tetravalent one-regular Cayley graph on G.*

Let X be a connected symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of $\mathrm{Aut}(X)$. For a normal subgroup N of G, the *quotient graph* X_N of X relative to the set of orbits of N is defined as the graph whose vertices are orbits of N on $V(X)$ with two orbits being adjacent in X_N if there is an edge between these two orbits in X. The following proposition is a 'reduction' theorem which is deduced from [22, Theorem 1.1].

Proposition 2.8. [22, Theorem 1.1] *Let X be a tetravalent connected symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of $\mathrm{Aut}(X)$. Then for each normal subgroup N of G one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N acts transitively on each of the two bipartition sets;*
(3) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} of X_N;*
(4) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-symmetric graph and X is a regular cover of X_N.*

To state the next result we need to introduce three families of tetravalent graphs that were first defined in [23]. First, let $C^{\pm 1}(p;4,2)$ be the graph with vertex set $\mathbb{Z}_p^2 \times \mathbb{Z}_4$, and adjacencies in $C^{\pm 1}(p;4,2)$ satisfying the following conditions: for $i, j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4$

$$(i,j,k) \sim \begin{cases} (i \pm 1, j, k+1) & \text{if } k \text{ is even} \\ (i, j \pm 1, k+1) & \text{if } k \text{ is odd} \end{cases}.$$

Second, for a prime $p \equiv \pm 1 (\mathrm{mod}\ 8)$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv 2 \pmod p$ the graph $\mathcal{NC}^0_{4p^2}$ is defined to have vertex set and edge set

$$\begin{aligned} V(\mathcal{NC}^0_{4p^2}) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x,y,z) \mid x,y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}^0_{4p^2}) &= \{(x,y,0)(x \pm 1, y, 1) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,1)(x, y \pm 1, 2) \mid x,y \in \mathbb{Z}_p\} \cup \\ &\quad \{(x,y,2)(x \mp 1, y \pm k, 3) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,3)(x \mp k, y \pm 1, 0) \mid x,y \in \mathbb{Z}_p\}. \end{aligned}$$

And third, for a prime p, $p \equiv 1 (\mathrm{mod}\ 8)$ or $p \equiv 3 (\mathrm{mod}\ 8)$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv -2 \pmod p$ the graph $\mathcal{NC}^1_{4p^2}$ is defined to have vertex set and edge set

$$\begin{aligned} V(\mathcal{NC}^1_{4p^2}) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x,y,z) \mid x,y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}^1_{4p^2}) &= \{(x,y,0)(x \pm 1, y, 1) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,1)(x, y \pm 1, 2) \mid x,y \in \mathbb{Z}_p\} \cup \\ &\quad \{(x,y,2)(x \pm 1, y \pm k, 3) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,3)(x \pm k, y \mp 1, 0) \mid x,y \in \mathbb{Z}_p\}. \end{aligned}$$

The graphs $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are extracted from [23, Lemma 8.4, Lemma 8.7]. We can now state the result of Gardiner and Praeger [23, Theorem 1.2] about connected tetravalent graphs admitting arc-transitive subgroups of automorphisms with normal elementary abelian p-groups N such that the corresponding quotient graph X_N is a cycle.

Proposition 2.9. [23, Theorem 1.2] *For an odd prime p let X be a connected, G-symmetric, tetravalent graph of order $4p^2$, let $N = \mathbb{Z}_p^2$ be a minimal normal subgroup of G with orbits of size p^2, and let K be the kernel of the action of G on $V(X_N)$. If $X_N = C_4$ and $K_v = \mathbb{Z}_2$ then X is isomorphic to one of the following graphs: $C^{\pm 1}(p; 4, 2)$, $\mathcal{N}C^0_{4p^2}$ and $\mathcal{N}C^1_{4p^2}$.*

In [23] it is proven that the three graphs in the above proposition all admit a one-regular subgroup of automorphisms. In the following two lemmas we improve this result by showing that $C^{\pm 1}(p; 4, 2)$ is not one-regular whereas $\mathcal{N}C^0_{4p^2}$ and $\mathcal{N}C^1_{4p^2}$ are.

Lemma 2.10. *Let p be a prime. Then $C^{\pm 1}(p; 4, 2)$ is not one-regular.*

Proof. First recall that the vertex set of $X = C^{\pm 1}(p, 4, 2)$ is equal to $V(X) = \{(i, j, k) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_p, k \in \mathbb{Z}_4\}$ and the edges are of the form

$$\begin{aligned} (i, j, 2l) &\sim (i \pm 1, j, 2l + 1), \text{ where } i, j \in \mathbb{Z}_p \text{ and } l \in \{0, 1\} \\ (i, j, 2l - 1) &\sim (i, j \pm 1, 2l), \text{ where } i, j \in \mathbb{Z}_p \text{ and } l \in \{0, 1\}. \end{aligned}$$

Then the reader can check that a permutation α of $V(X)$ defined by $(i, j, k)^\alpha = (-i, j, k)$ maps edges to edges, and hence α is an automorphism of X. Since α fixes the arc $(0, 0, 1)(0, 1, 2) \in A(X)$ it follows that X is not one-regular. $\square$

Lemma 2.11. *Let p be a prime. Then $\mathcal{N}C^0_{4p^2}$ and $\mathcal{N}C^1_{4p^2}$ are both one-regular graphs.*

Proof. Let $X \in \{\mathcal{N}C^0_{4p^2}, \mathcal{N}C^1_{4p^2}\}$ and let X^2 be the distance-2-graph of X, that is, $V(X^2) = V(X)$ with two vertices being adjacent in X^2 if and only if they are at distance 2 in X. Let

$$\Delta_i = \{(x, y, i) \mid x, y \in \mathbb{Z}_p\}, \quad i \in \mathbb{Z}_4.$$

Then for every $i \in \mathbb{Z}_4$ the subgraph $X^2[\Delta_i]$ of X^2 induced by the vertices in Δ_i is a 2-dimensional grid $C_p \times C_p$, whereas any edge uv in X^2 with endvertices $u \in \Delta_i$ and $v \in \Delta_j$, where $i \neq j$, is contained in an induced subgraph of X^2 isomorphic to the complete graph K_4. Moreover this induced subgraph isomorphic to K_4 containing the edge uv is unique. Take four vertices $u_1, u_2, u_3, u_4 \in \Delta_i$ such that the subgraph Y of X^2 induced on these four vertices is isomorphic to a 4-cycle C_4. Then Y^g for any $g \in \mathrm{Aut}(X^2)$ is an induced subgraph of X^2 isomorphic to C_4. Since there is no set of four vertices containing vertices from different sets Δ_i such that the induced subgraph of X^2 is isomorphic to C_4 it follows that Y^g is a subgraph of $X^2[\Delta_j]$ for some $j \in \mathbb{Z}_4$. This shows that the sets Δ_i, $i \in \mathbb{Z}_4$, are blocks of imprimitivity for $\mathrm{Aut}(X)$. Therefore every automorphism $g \in \mathrm{Aut}(X)$ that fixes the vertices $(0, 0, 0)$ and $(1, 0, 1)$, and thus the arc $(0, 0, 0), (1, 0, 1)$, also fixes the vertices $(2, 0, 0)$ and $(-1, 0, 1)$. Now looking at the action of g on X^2 we get that g fixes both Δ_0 and Δ_1 pointwise. Since all the vertices in Δ_1 are fixed by g and the induced bipartite subgraph $X[\Delta_1, \Delta_2]$ is a disjoint union of p $2p$-cycles it follows that also Δ_2 is fixed pointwise by g. Using the same argument for $X[\Delta_0, \Delta_3]$ one can see that g also fixes the vertices in Δ_3 and thus $g = 1$, which shows that X is one-regular. $\square$

To state the next result we need to introduce two additional families of tetravalent graphs that were first defined in [23]. The graph $C^{\pm 1}(p; 4p, 1)$ is defined to have the vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ and the edge set $\{(i, j)(i \pm 1, j + 1) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_{4p}\}$. The graph $C^{\pm\varepsilon}(p; 4p, 1)$ is a graph with vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ with adjacencies in $C^{\pm\varepsilon}(p; 4p, 1)$ satisfying the following conditions:

$$(i, j) \sim \begin{cases} (i \pm \varepsilon, j + 1) & \text{if } j \text{ is odd} \\ (i \pm 1, j + 1) & \text{if } j \text{ is even} \end{cases},$$

where $i \in \mathbb{Z}_p$, $j \in \mathbb{Z}_{4p}$ and ε is an element of order 4 in $\mathbb{Z}_p^*$.

Proposition 2.12. [23, Theorem 1.1] *Let p be an odd prime and let X be a connected, G-symmetric, tetravalent graph of order $4p^2$. Let $N = \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p and let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_{4p}$ and $K_v = \mathbb{Z}_2$ then X is isomorphic either to $C^{\pm 1}(p; 4p, 1)$ or to $C^{\pm\varepsilon}(p; 4p, 1)$.*

We end this subsection with a result on tetravalent arc-transitive graphs of order $4p$, where p is a prime. In order to state the result, first recall that the *lexicographic product* $X[Y]$ (sometimes also called the *wreath product*) of two graphs X and Y has vertex set $V(X) \times V(Y)$, and two vertices (a, u) and (b, v) are adjacent in $X[Y]$ if $ab \in E(X)$ or if $a = b$ and $uv \in E(Y)$. Second, following [44], for a prime p congruent to 1 modulo 4, an element w of order 4 in $\mathbb{Z}_p^*$ and the group $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, we use notation $\mathcal{CA}^0_{4p} = \mathrm{Cay}(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})$ and $\mathcal{CA}^1_{4p} = \mathrm{Cay}(G, \{a, a^{-1}, a^{w}b, a^{-w}b\})$. For the definition of the graph $C(2, p, 2)$ stated in the sixth row of Table 1 see Section 4. Finally, by [44, Example 3.7], $\mathfrak{g}_{28} = \mathrm{Cos}(G, T, TaT)$ is a coset graph of the group $G = \mathrm{PGL}(2, 7)$ with respect to a subgroup T isomorphic to A_4 and an involution a from the center of the normalizer of a Sylow 3-subgroup of T in G.

Proposition 2.13. [44, Theorem 4.1] *Let s be a positive integer and let p be a prime. Then a connected tetravalent graph of order $4p$ is s-arc-transitive if and only if it is isomorphic to one of the graphs listed in Table 1. Furthermore, all graphs listed in Table 1 are pairwise non-isomorphic.*

X	s	$\mathrm{Aut}(X)$	comments
$K_{4,4}$	3	$\mathbb{Z}_2 \ltimes (S_4 \times S_4)$	p=2
$C_{2p}[2K_1]$	1	$D_{4p} \ltimes \mathbb{Z}_2^{2p}$	$p > 2$
$\mathcal{CA}^0_{4p}$	1	$\mathbb{Z}_2^2 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2)$,	$p \equiv 1 (\mathrm{mod}\ 4)$
$\mathcal{CA}^1_{4p}$	1	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2)$,	$p \equiv 1 (\mathrm{mod}\ 4)$
$C(2, p, 2)$	1	$D_{2p} \ltimes \mathbb{Z}_2^{2p}$	$p > 2$
$\mathfrak{g}_{28}$	3	$\mathrm{PGL}(2, 7) \times \mathbb{Z}_2$	$p = 7$

Table 1: Tetravalent s-arc-transitive graphs of order $4p$.

3. Examples

In this section, we give examples of tetravalent one-regular graphs of order $4p^2$, where p is a prime. In this paper, the abbreviations $\mathcal{CA}$ and $\mathcal{CN}$ will mean a Cayley graph on abelian group and a Cayley graph on non-abelian group, respectively.

Example 3.1. Introduced by Wilson [39] the *bicycle wheels* are defined in the following way. Given natural numbers n, a, r and s, the graph $X = \mathcal{BW}_n(a, r, s)$ is defined to be the graph of order $3n$ with vertex set $V(X) = \{A_i, B_i, C_i \mid i \in \mathbb{Z}_n\}$ and edge set

$$E(X) = \{A_iB_i, B_iA_{i+1}, B_iC_i, C_iB_{i+a}, A_iA_{i+r}, C_iC_{i+s} \mid i \in \mathbb{Z}_n\}.$$

With the help of computer software package MAGMA [3] one can see that $\mathcal{BW}_{12}(5, 1, 5)$ is one-regular. In addition, it is a Cayley graph $\mathrm{Cay}(G_{36}, S)$ on the group $G_{36} = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1 = [a, b] = [a, c] = [b, c] = [c, d], d^{-1}ad = b, d^{-1}bd = ab\rangle$ with respect to the generating set $S = \{ad, (ad)^{-1}, bdc, (bdc)^{-1}\}$, and $\mathrm{Aut}(\mathcal{CA}^2_{36}) \cong G_{36} \rtimes \mathbb{Z}_2^2$.

Remark: The automorphism group of the graph $\mathcal{BW}_{12}(5, 1, 5)$ has a non-normal Sylow 3-subgroup. Since, by Theorem 5.1, the automorphism groups of the graphs $\mathcal{CA}^i_{4p^2}$, $i \in \{0, 1, 2\}$, given in Examples 3.3 and 3.4 and Lemma 3.6, all have normal Sylow p-subgroups, the graph $\mathcal{BW}_{12}(5, 1, 5)$ is not isomorphic to any of these graphs.

Example 3.2. Given natural numbers k and m, and a 2×2 matrix M over $\mathbb{Z}_n$ the 2-*dimensional generalized power spidergraph* $\mathcal{GPS}2(k,n,M)$ is defined to be the graph with vertex set $\mathbb{Z}_k\times\mathbb{Z}_n\times\mathbb{Z}_n$, and edge set $\{(i,x)(i+1,x+a_i),(i,x)(i+1,x+b_i)\mid i\in\mathbb{Z}_k, x\in\mathbb{Z}_n\times\mathbb{Z}_n\}$ where $a_i=(1,0)M^i$ and $b_i=(-1,0)M^i$ (see [39]). With the use of MAGMA [3] one can see that $\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$ is a one-regular graph. In addition, it is not a Cayley graph and the stabilizer of a vertex in the automorphism group is isomorphic to $\mathbb{Z}_4$.

Example 3.3. Let $p\equiv 1\pmod 4$ be a prime and w an element of order 4 in $\mathbb{Z}_p^*$ with $1\leq w\leq p-1$. Let $G^0_{4p^2}=\langle a\rangle\times\langle b\rangle\cong\mathbb{Z}_{2p^2}\times\mathbb{Z}_2$. Then, by [40, Proposition 3.3(iv)], the Cayley graph $\mathcal{CA}^0_{4p^2}=\mathrm{Cay}(G^0_{4p^2},\{a,a^{-1},a^wb,a^{-w}b\})$ is a tetravalent one-regular graph. Furthermore, $\mathrm{Aut}(\mathcal{CA}^0_{4p^2})\cong(\mathbb{Z}_{2p^2}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2^2$.

Example 3.4. Let p be an odd prime and $G^1_{4p^2}=\langle a,b\,|\,a^{4p}=b^p=1, ab=ba\rangle\cong\mathbb{Z}_{4p}\times\mathbb{Z}_p$. Then, by [40, Proposition 3.3], the Cayley graph $\mathcal{CA}^1_{4p^2}=\mathrm{Cay}(G^1_{4p^2},\{ab,a^{-1}b,ab^{-1},a^{-1}b^{-1}\})$ is a tetravalent one-regular graph. Furthermore, $\mathrm{Aut}(\mathcal{CA}^1_{4p^2})\cong(\mathbb{Z}_{4p}\times\mathbb{Z}_p)\rtimes\mathbb{Z}_2^2$. The graph $\mathcal{DW}(12,3)$ of order 36 given in [39] is the smallest example of such graphs.

For an odd prime p, the tetravalent graph $C^{\pm1}(p;4p,1)$ is defined in the paragraph preceding Proposition 2.12. In the following lemma we prove that $C^{\pm1}(p;4p,1)$ is isomorphic to $\mathcal{CA}^1_{4p^2}$, and thus it is one-regular in view of Example 3.4.

Lemma 3.5. *Let p be an odd prime, let $G^1_{4p^2}=\langle a,b\mid a^{4p}=b^p=1, ab=ba\rangle\cong\mathbb{Z}_{4p}\times\mathbb{Z}_p$, and let $S=\{ab,a^{-1}b,ab^{-1},a^{-1}b^{-1}\}$. Then $C^{\pm1}(p;4p,1)\cong\mathrm{Cay}(G^1_{4p^2},S)=\mathcal{CA}^1_{4p^2}$.*

Proof. Recall that $C^{\pm1}(p;4p,1)$ has vertex set $\mathbb{Z}_p\times\mathbb{Z}_{4p}$ and edge set $\{(i,j)(i\pm1,j+1)\mid i\in\mathbb{Z}_p, j\in\mathbb{Z}_{4p}\}$. The map defined by $(i,j)\mapsto a^jb^i$ is an isomorphism from $C^{\pm1}(p;4p,1)$ to the Cayley graph $\mathcal{CA}^1_{4p^2}$. We leave the details to the reader. □

Let $p\equiv 1\pmod 4$ be a prime and let $\varepsilon\in\mathbb{Z}_p$ be such that $\varepsilon^2\equiv -1\pmod p$. The following lemma shows that $C^{\pm\varepsilon}(p;4p,1)$ is a Cayley graph.

Lemma 3.6. *Let $p\equiv 1\pmod 4$ be a prime, let $\varepsilon\in\mathbb{Z}_p$ be such that $\varepsilon^2\equiv-1\pmod p$, let $G^2_{4p^2}=\langle a,b\mid a^{4p}=b^p=1, a^{-1}ba=b^\varepsilon\rangle$, and let $S=\{ab,a^{-1}b^\varepsilon,ab^{-1},a^{-1}b^{-\varepsilon}\}$. Then $\mathcal{CN}^2_{4p^2}=\mathrm{Cay}(G^2_{4p^2},S)$ is a symmetric graph isomorphic to $C^{\pm\varepsilon}(p;4p,1)$.*

Proof. Recall that the graph $C^{\pm\varepsilon}(p;4p,1)$ has vertex set $\mathbb{Z}_p\times\mathbb{Z}_{4p}$ with adjacencies defined as follows:

$$(i,j)\sim\begin{cases}(i\pm\varepsilon,j+1) & \text{if } j \text{ is odd}\\ (i\pm1,j+1) & \text{if } j \text{ is even}\end{cases}$$

where $i\in\mathbb{Z}_p$ and $j\in\mathbb{Z}_{4p}$.

Let $G=G^2_{4p^2}$ and $X=Cay(G;S)$. Then the map defined by $(i,j)\mapsto a^jb^i$ is an isomorphism from $C^{\pm\varepsilon}(p;4p,1)$ to X. Since, by [23], the graph $C^{\pm\varepsilon}(p;4p,1)$ is symmetric, the lemma holds. □

4. Analysis of tetravalent one-regular graphs of order $4p^2$

Let p be an odd prime. Then define $C(2,p,2)$ to be a graph with $V(C(2,p,2))=\mathbb{Z}_4\times\mathbb{Z}_p$ and adjacencies in $C(2,p,2)$ satisfying the following conditions:

$$\begin{aligned}
(0,i)\sim(0,j) &\iff j-i=\pm1,\\
(0,i)\sim(1,j) &\iff j-i=-1,\\
(0,i)\sim(2,j) &\iff j-i=1,\\
(1,i)\sim(2,j) &\iff j-i=\pm1,\\
(1,i)\sim(3,j) &\iff j-i=-1,\\
(2,i)\sim(3,j) &\iff j-i=1,\\
(3,i)\sim(3,j) &\iff j-i=\pm1.
\end{aligned}$$

Let $X = C(2, p, 2)$ and let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$. Observe that for each $j \in \mathbb{Z}_p$, $j \neq i$, the subgraph $X[B_i, B_j]$ induced on the union $B_i \cup B_j$ is not an independent set of vertices if and only if $j = i \pm 1$. Moreover, for each such j we have that $X[B_j, B_{j+1}] \cong 2C_4$, see also Figure 1. The following lemma shows that there is no one-regular $\mathbb{Z}_p$-cover of $C(2, p, 2)$.

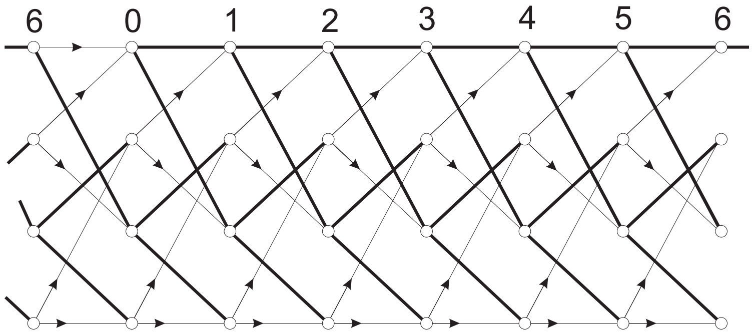

Figure 1: A spanning tree in the base graph $C(2, p, 2)$ for $p = 7$.

Lemma 4.1. *Let Y be a tetravalent one-regular graph of order $4p^2$, $p > 3$ a prime, such that there exists a normal subgroup H of $\mathrm{Aut}(Y)$ of order p. Then Y is not a regular $\mathbb{Z}_p$-cover of the graph $C(2, p, 2)$.*

Proof. Let $\mathcal{K} = \{1, \tau_1, \tau_2, \tau_3\}$ be the Klein 4-group acting on $\mathbb{Z}_4$ so that $\tau_1 = (0\,1)(2\,3)$, $\tau_2 = (0\,2)(1\,3)$ and $\tau_3 = (0\,3)(1\,2)$. Let $X = C(2, p, 2)$, let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$, and let K be the kernel of the action of $\mathrm{Aut}(X)$ on $\mathcal{B}$. We shall be sloppy and shall identify restrictions of elements of K to sets B_i by elements of $\mathcal{K}$. For instance, when we say that the restriction γ_i of $\gamma \in K$ to B_i is, for example, τ_1, we mean that $\gamma_i = ((0, i)(1, i))((2, i)(3, i))$. Now, the structure of X indicated in Figure 1 implies that the restrictions γ_i must satisfy the following conditions:

$$\gamma_i \in \{1, \tau_1\} \iff \gamma_{i+1} \in \{1, \tau_2\} \quad \forall i \in \mathbb{Z}_p. \tag{1}$$

Let the vertices of X be labeled in the following way: $a_i = (0, i)$, $b_i = (1, i)$, $c_i = (2, i)$ and $d_i = (3, i)$. Let $E = \langle \gamma_i \mid i \in \mathbb{Z}_p \rangle$. It is well known, see for instance [33, 44], that $\mathrm{Aut(X)} = E \rtimes \langle \rho, \tau \rangle \cong \mathbb{Z}_2^p \rtimes D_{2p}$ where

$$\rho = (a_0\, a_1\, \ldots\, a_{p-1})(b_0\, b_1\, \ldots\, b_{p-1})(c_0\, c_1\, \ldots\, c_{p-1})(d_0\, d_1\, \ldots\, d_{p-1})$$

and

$$\tau = (a_0)(b_0\, c_0)(d_0) \prod_{i=1}^{p-1} (a_i\, a_{-i})(b_i\, c_{-i})(c_i\, b_{-i})(d_i\, d_{-i}).$$

Now let Y be a tetravalent one-regular graph of order $4p^2$. Assume that $\mathrm{Aut}(Y)$ contains a normal subgroup H isomorphic to $\mathbb{Z}_p$ such that the corresponding quotient graph Y_H is isomorphic to $X = C(2, p, 2)$. Then, since the orbits of H form an $\mathrm{Aut}(Y)$-invariant partition, the whole automorphism group $\mathrm{Aut}(Y)$ of Y projects to a subgroup of $\mathrm{Aut}(X)$. On the other hand, the graph Y can be viewed as an H-covering graph (that is, a $\mathbb{Z}_p$-covering) of X, and it can therefore be derived from X through a suitable voltage assignment ζ. To find this voltage assignment fix the spanning tree T of X as indicated on Figure 1.

Let G be the largest subgroup of $\mathrm{Aut}(X)$ which lifts with respect to the natural projection $X \times_\zeta \mathbb{Z}_p \cong Y \to Y_H \cong X$, where ζ is as given in Figure 1. Clearly, since Y is arc-transitive, we may assume that $\rho, \tau \in G$. Let F denote the largest subgroup of E which lifts. Then $G = F \rtimes \langle \rho, \tau \rangle$ and thus $|G| = 2p|F|$. We will show that $|F| > 8$. This will then imply that the lift $\bar{G}$ of G is of order $|\bar{G}| = 2p^2|F| > 16p^2$, and consequently that Y is not one-regular.

Since $\rho, \tau \in G$, we have that

$$\text{if } \phi \in F \text{ then } \phi^\rho, \phi^\tau \in F. \tag{2}$$

It is convenient to view elements γ in E as vectors in $\mathbb{Z}_4^p$. Namely, we write $\gamma = (e_0, \ldots, e_{p-1})$ where $e_i = s$ if and only if $\gamma_i = \tau_s$ (where $e_i = 0$ means that $\gamma_i = \tau_0 = id$). Note that in this context (2) can be interpreted as follows: F is invariant under the "cyclic shift"

$$\phi = (f_0, f_1, \ldots, f_{p-1}) \mapsto (f_{p-1}, f_0, \ldots, f_{p-2}),$$

and under the "reflection around the first entry"

$$\phi = (f_0, f_1, \ldots, f_{p-1}) \mapsto (f'_0, f'_{p-1}, f'_{p-2}, \ldots, f'_2, f'_1),$$

where

$$f'_i = \begin{cases} 0 & , \quad \text{if } f_i = 0 \\ 1 & , \quad \text{if } f_i = 2 \\ 2 & , \quad \text{if } f_i = 1 \\ 3 & , \quad \text{if } f_i = 3 \end{cases}$$

Now choose $\phi \in F$. By (1) the first two components of ϕ can be one of the following pairs: $\phi = (0, 0, \ldots)$, $\phi = (0, 2, \ldots)$, $\phi = (1, 0, \ldots)$, $\phi = (1, 2, \ldots)$, $\phi = (2, 1, \ldots)$, $\phi = (2, 3, \ldots)$, $\phi = (3, 1, \ldots)$, or $\phi = (3, 3, \ldots)$. Since the lift of G acts arc-transitively on Y the group G must be of order $|G| = 2p|F| \geq 16p$ and thus $|F| \neq 1$.

Suppose first that there exist $\psi \in F$ such that $\psi \notin \{id, (3, 3, \ldots, 3)\}$. Since ρ is of prime order, the conjugacy class of ψ under $\langle \rho \rangle$ is of size p. But then, by (2), we have that $|F| > 8$, which implies that $\bar{G}$ is not acting one-regularly on Y.

Suppose now that $(3, 3, \ldots, 3)$ belongs to F. Then, since $\langle (3, 3, \ldots, 3) \rangle \leq F$ is of order 2 and $|G| = 2p|F| = 16p$, we have that there must also exist a non-identity automorphism $\psi \in F$ which is different from $(3, 3, \ldots, 3)$. But then, as above, the conjugacy class of ψ is of size p, and consequently $|F| > 8$. This shows that $\bar{G}$ is not acting one-regularly on Y, and the proof is completed. □

By the following lemma there are only two normal one-regular Cayley graphs on the group $G = \langle a, b, c, g \mid a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a \rangle$.

Lemma 4.2. *Let p be a prime and $G = \langle a, b, c, g \mid a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a \rangle$. Then a tetravalent normal Cayley graph X of order $4p^2$ on G is one-regular if and only if it is either isomorphic to*

$$\mathcal{CN}^3_{4p^2} = \mathrm{Cay}(G, \{ag, bcg, b^{-1}g, a^{-1}cg\}) \text{ or to } \mathcal{CN}^4_{4p^2} = \mathrm{Cay}(G, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\}).$$

Moreover, $\mathrm{Aut}(\mathcal{CN}^3_{4p^2}) \cong G \rtimes \mathbb{Z}_2^2$ *and* $\mathrm{Aut}(\mathcal{CN}^4_{4p^2}) \cong G \rtimes \mathbb{Z}_4$.

Proof. Let X be a tetravalent one-regular normal Cayley graph $\mathrm{Cay}(G, S)$ on the group G with respect to the generating set S. Since X is one-regular and normal, the stabilizer $A_1 = \mathrm{Aut}(G, S)$ of the vertex $1 \in G$ is transitive on S, and either $\mathrm{Aut}(G, S) \cong \mathbb{Z}_2^2$ or $\mathrm{Aut}(G, S) \cong \mathbb{Z}_4$. This implies that elements in S are all of the same order.

Observe that G contains elements of order 2, p and $2p$. In particular, elements of the form c, a^ib^jg and a^ib^jcg, where $p \mid i + j$, are of order 2; elements of the form a^ib^j are of order p; and elements of the form a^ib^jc, a^mb^ng and a^mb^ncg, where $p \nmid m + n$, are of order $2p$. In the following, we will show that up to isomorphism, there are only two generating sets of size 4 such that the corresponding Cayley graphs are normal and one-regular.

First, observe that neither four involutions nor two elements of order p can generate G. Moreover, G cannot be generated by the following pairs of elements of order $2p$: $a^{i_1}b^{j_1}c$ and $a^{i_2}b^{j_2}c$, $a^{m_1}b^{n_1}g$ and $a^{m_2}b^{n_2}g$, $a^{m_1}b^{n_1}cg$ and $a^{m_2}b^{n_2}cg$, where $m_i + n_i \neq 0\,(1 \leq i \leq 2)$. Second, $Z(G) = \langle ab, c \rangle = \langle ab \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_2$, and thus $\langle c \rangle$ char G. Also, since $\mathrm{Aut}(G, S)$ is transitive on S, we have that $S \neq \{a^ib^jc, a^mb^ng, (a^ib^jc)^{-1}, (a^mb^ng)^{-1}\}$ and $S \neq \{a^ib^jc, a^mb^ncg, (a^ib^jc)^{-1}, (a^mb^ncg)^{-1}\}$, where $m + n \neq 0$. Now suppose that G is generated by

$$S_0 = \{a^ib^jg, a^{m'}b^{n'}cg, (a^ib^jg)^{-1}, (a^{m'}b^{n'}cg)^{-1}\},$$

where $p \nmid i + j$ and $p \nmid m' + n'$.

Case 1. $\mathrm{Aut}(G, S_0) = \langle\alpha\rangle \times \langle\beta\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where α and β are such that $a^\alpha = a^{i_1}b^{j_1}$, $b^\alpha = a^{j_1}b^{i_1}$, $c^\alpha = c$, $g^\alpha = a^x b^{-x} cg$, $a^\beta = a^{i_2}b^{j_2}$, $b^\beta = a^{j_2}b^{i_2}$, $c^\beta = c$ and $g^\beta = a^y b^{-y} g$.

Subcase 1.1. Let $i = j$.

Since $ab \in Z(G)$, G can be generated by S_0 if and only if $m' \neq n'$. Now take an automorphism σ of G such that

$$a^\sigma = a^i,\ b^\sigma = b^i,\ c^\sigma = c,\ g^\sigma = g.$$

Then $(abg)^\sigma = a^i b^i g$, and hence

$$S = S_0{}^{\sigma^{-1}} = \{abg,\ a^m b^n cg,\ (abg)^{-1},\ (a^m b^n cg)^{-1}\} = \{abg,\ a^m b^n cg,\ a^{-1}b^{-1}g, a^{-n}b^{-m}cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$. Moreover, it can be easily seen that $m \neq n$.

Suppose first that $(abg)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = abg$, $(a^{-1}b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = a^{-1}b^{-1}g$. It follows that either $m + n = 2$ or $m + n = -2$. If $m + n = 2$ then, since $m \neq n$, we have that $m \neq 1$ and

$$a^\alpha = b,\ b^\alpha = a,\ c^\alpha = c,\ g^\alpha = a^{m-1}b^{1-m}cg.$$

If $m + n = -2$, then since $m \neq n$, we have $n \neq -1$ and

$$a^\alpha = a^{-1},\ b^\alpha = b^{-1},\ c^\alpha = c,\ g^\alpha = a^{-1-n}b^{1+n}cg.$$

Suppose now that $(abg)^\beta = a^{-1}b^{-1}g$. Then $(a^{-1}b^{-1}g)^\beta = abg$, $(a^m b^n cg)^\beta = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\beta = a^m b^n cg$. By a similar argument as above, one can get that

$$a^\beta = b^{-1},\ b^\beta = a^{-1},\ c^\beta = c,\ g^\beta = g.$$

Consequently, either $S_0 = S_1 = \{abg,\ a^m b^{2-m} cg,\ a^{-1}b^{-1}g,\ a^{m-2}b^{-m}cg\}$, where $m \neq 1$, or

$$S_0 = S_2 = \{abg,\ a^{-2-n}b^n cg,\ a^{-1}b^{-1}g,\ a^{-n}b^{n+2}cg\},$$

where $n \neq -1$. In addition, replacing $-n$ with m, it can be seen that $S_2 = S_1$. Moreover, it can be easily seen that G can indeed be generated by S_1. Namely, since $(abg)^p = g$ we have $g, ab \in \langle S_1\rangle$. Then, since $a^m b^{2-m} cg \in \langle S_1\rangle$, we get that $a^m b^{2-m} c \in \langle S_1\rangle$. Further, since $(a^m b^{2-m} c)^p = c$, also $c, a^m b^{2-m} \in \langle S_1\rangle$. Now, since $a^m b^{2-m} = a^m b^m b^{2-2m}$, $m \neq 1$, and $ab \in \langle S_1\rangle$, we get that $b^{2-2m} \in \langle S_1\rangle$. Finally, the fact that $b^g = a$ implies that $G = \langle S_1\rangle$.

Subcase 1.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^\sigma = a^i b^j$, $b^\sigma = a^j b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(ag)^\sigma = a^i b^j g$ and

$$S = S_0{}^{\sigma^{-1}} = \{ag,\ a^m b^n cg,\ (ag)^{-1},\ (a^m b^n cg)^{-1}\} = \{ag,\ a^m b^n cg,\ b^{-1}g,\ a^{-n}b^{-m}cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = ag$, $(b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = b^{-1}g$. In addition, either $m + n = 1$ or $m + n = -1$. If $m + n = 1$ then, since $\{ag,\ acg,\ b^{-1}g,\ b^{-1}cg\}$ cannot generate G, we have that $m \neq 1$. Thus α is mapping according to the rule: $a^\alpha = b$, $b^\alpha = a$, $c^\alpha = c$, and $g^\alpha = a^m b^{-m} cg$. If on the other hand $m + n = -1$ then, since $\{ag,\ b^{-1}cg,\ b^{-1}g,\ acg\}$ cannot generate G, we have that $n \neq -1$, and hence α is mapping according to the rule: $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$, $c^\alpha = c$, and $g^\alpha = a^{-n}b^n cg$.

Suppose now that $(ag)^\beta = b^{-1}g$. Then we have that $(b^{-1}g)^\beta = ag$, $(a^m b^n cg)^\beta = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\beta = a^m b^n cg$. Whenever $m + n = 1$ or $m + n = -1$, we can get that β is mapping according to the rule: $a^\beta = b^{-1}$, $b^\beta = a^{-1}$, $c^\beta = c$, and $g^\beta = g$. Thus, we can conclude that either $S_0 = S_3 = \{ag,\ a^m b^{1-m} cg,\ b^{-1}g,\ a^{m-1}b^{-m}cg\}$, where $m \neq 1$, or $S_0 = S_4 = \{ag,\ a^{-n-1}b^n cg,\ b^{-1}g, a^{-n}b^{n+1}cg\}$, where $n \neq -1$. Moreover, replacing $-n$ with m, it can be easily seen that $S_4 = S_3$. Also, since $(ag)^2 = ab$ and $aga^m b^{1-m} cg = a^{2-m} b^m c$, we get that $c, a^{2-m} b^m \in \langle S_3 \rangle$. Further, the facts that $a^{2-m} b^m = a^{2-2m} a^m b^m$, $m \neq 1$ and $ab \in \langle S_3 \rangle$ combined together imply that $a^{2-2m} \in \langle S_3 \rangle$. Since $ag \in \langle S_3 \rangle$, it follows that $g \in \langle S_3 \rangle$. Finally, since $a^g = b$, G is indeed generated by S_3.

Now considering the automorphism γ of G defined by $a^\gamma = a^{\frac{1}{2}}$, $b^\gamma = b^{\frac{1}{2}}$, $c^\gamma = c$, and $g^\gamma = a^{\frac{1}{2}} b^{-\frac{1}{2}} g$ we get that ${S_1}^\gamma = \{ag,\ a^{\frac{m+1}{2}} b^{1-\frac{m+1}{2}} cg,\ b^{-1} g,\ a^{\frac{m+1}{2}-1} b^{-\frac{m+1}{2} cg}\}$, where $m \neq 1$. Thus we only need to consider the generating set $S_3 = \{ag,\ a^m b^{1-m} cg,\ b^{-1} g,\ a^{m-1} b^{-m} cg\}$, where $m \neq 1$.

Case 2. $\mathrm{Aut}(G, S_0) = \langle \alpha \rangle \cong \mathbb{Z}_4$, where α is such that $a^\alpha = a^{i_1} b^{j_1}$, $b^\alpha = a^{j_1} b^{i_1}$, $c^\alpha = c$, and $g^\alpha = a^x b^{-x} cg$.

Subcase 2.1. Let $i = j$.

Since $ab \in Z(G)$, G can be generated by S_0 (where $p \nmid i$ and $p \nmid m' + n'$) if and only if $m' \neq n'$. Now take an automorphism σ of G such that $a^\sigma = a^i$, $b^\sigma = b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(abg)^\sigma = a^i b^i g$, and consequently

$$S = {S_0}^{\sigma^{-1}} = \{abg,\ a^m b^n cg,\ (abg)^{-1},\ (a^m b^n cg)^{-1}\} = \{abg,\ a^m b^n cg,\ a^{-1} b^{-1} g, a^{-n} b^{-m} cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$, and $m \neq n$.

Suppose first that $(abg)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = a^{-1} b^{-1} g$, $(a^{-1} b^{-1} g)^\alpha = a^{-n} b^{-m} cg$, $(a^{-n} b^{-m} cg)^\alpha = abg$. Hence either $m + n = \omega$ or $m + n = -\omega$, where $\omega^2 = -4$. If $m + n = \omega$ then since $m \neq n$, we have that $m \neq \frac{\omega}{2}$. It follows that $a^\alpha = a^i b^{\frac{\omega}{2}-i}$, $b^\alpha = a^{\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{m-\frac{\omega}{2}} b^{\frac{\omega}{2}-m} cg$, where $i = \frac{(m+1)\omega+2-2m}{2(2m-\omega)}$. If on the other hand $m + n = -\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^\alpha = a^i b^{-\frac{\omega}{2}-i}$, $b^\alpha = a^{-\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{-\frac{\omega}{2}-n} b^{\frac{\omega}{2}+n} cg$, where $i = \frac{2-2n-(n+1)\omega}{2(2n+\omega)}$.

Suppose now that $(abg)^\alpha = a^{-n} b^{-m} cg$. Then $(a^{-n} b^{-m} cg)^\alpha = a^{-1} b^{-1} g$, $(a^{-1} b^{-1} g)^\alpha = a^m b^n cg$, and $(a^m b^n cg)^\alpha = abg$. Hence, either $m + n = \omega$ or $m + n = -\omega$, where $\omega^2 = -4$. If $m + n = \omega$ then, since $m \neq n$, we have that $m \neq \frac{\omega}{2}$, and thus $a^\alpha = a^i b^{-\frac{\omega}{2}-i}$, $b^\alpha = a^{-\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{m-\frac{\omega}{2}} b^{\frac{\omega}{2}-m} cg$, where $i = \frac{(1-m)\omega-2m-2}{2(2m-\omega)}$. If however $m + n = -\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^\alpha = a^i b^{\frac{\omega}{2}-i}$, $b^\alpha = a^{\frac{\omega}{2}-i} b^i$, $c^\alpha = c$, and $g^\alpha = a^{-\frac{\omega}{2}-n} b^{\frac{\omega}{2}+n} cg$, where $i = \frac{(n-1)\omega-2n-2}{2(2n+\omega)}$.

We can conclude that either $S_0 = S_5 = \{abg,\ a^m b^{\omega-m} cg,\ a^{-1} b^{-1} g,\ a^{m-\omega} b^{-m} cg\}$, where $m \neq \frac{\omega}{2}$, or $S_0 = S_6 = \{abg,\ a^{-\omega-n} b^n cg,\ a^{-1} b^{-1} g,\ a^{-n} b^{n+\omega} cg\}$, where $n \neq -\frac{\omega}{2}$. Moreover, replacing $-n$ with m, it can be easily seen that $S_5 = S_6$. Also, the group G is indeed generated by S_5. Namely, since $(abg)^p = g$ we have that $g, ab \in \langle S_5 \rangle$. Further, since $a^m b^{\omega-m} cg \in \langle S_5 \rangle$, also $a^m b^{\omega-m} c \in \langle S_5 \rangle$, and the fact that $(a^m b^{\omega-m} c)^p = c$ implies that $c, a^m b^{\omega-m} \in \langle S_5 \rangle$. Finally, since $a^m b^{\omega-m} = a^m b^m b^{\omega-2m}$, $m \neq \frac{\omega}{2}$, and $ab \in \langle S_5 \rangle$, it follows that $b^{\omega-2m} \in \langle S_5 \rangle$. Now this fact and $b^g = a$ combined together imply that $G = \langle S_5 \rangle$.

Subcase 2.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^\sigma = a^i b^j$, $b^\sigma = a^j b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(ag)^\sigma = a^i b^j g$, and consequently

$$S = {S_0}^{\sigma^{-1}} = \{ag,\ a^m b^n cg,\ (ag)^{-1},\ (a^m b^n cg)^{-1}\} = \{ag,\ a^m b^n cg,\ b^{-1} g,\ a^{-n} b^{-m} cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = b^{-1} g$, $(b^{-1} g)^\alpha = a^{-n} b^{-m} cg$, and $(a^{-n} b^{-m} cg)^\alpha = ag$. Also, either $m + n = \varepsilon$ or $m + n = -\varepsilon$, where $\varepsilon^2 = -1$. If $m + n = \varepsilon$ then, since $\{ag,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg,\ b^{-1} g,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg\}$ cannot generate G (namely, for $\varphi \in \mathrm{Aut}(G)$ such that $a^\varphi = a^2$, $b^\varphi = b^2$, $c^\varphi = c$, and $g^\varphi = a^{-1} bg$ we have $\{ag,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg,\ b^{-1} g,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg\}^\varphi = \{abg,\ a^\varepsilon b^\varepsilon cg,\ a^{-1} b^{-1} g,\ a^{-\varepsilon} b^{-\varepsilon} cg\}$), we have that $m \neq \frac{\varepsilon+1}{2}$. It follows that

$$a^\alpha = a^i b^{\varepsilon-i},\ b^\alpha = a^{\varepsilon-i} b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{m-i} b^{i-m} cg,$$

where $i = \frac{m\varepsilon-m+1}{2m-\varepsilon-1}$. If on the other hand $m + n = -\varepsilon$ then, since G cannot be generated by

$$\{ag,\ a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}} cg, b^{-1} g,\ a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}} cg\},$$

we have that $n \neq -\frac{\varepsilon+1}{2}$, and so

$$a^\alpha = a^i b^{-\varepsilon-i},\ b^\alpha = a^{-\varepsilon-i}b^i,\ c^\alpha = c,\ \text{and}\ g^\alpha = a^{-\varepsilon-i-n}b^{\varepsilon+i+n}cg,$$

where $i = -\frac{(n+1)\varepsilon+n}{2n+\varepsilon+1}$.

Suppose now that $(ag)^\alpha = a^{-n}b^{-m}cg$. Then $(a^{-n}b^{-m}cg)^\alpha = b^{-1}g$, $(b^{-1}g)^\alpha = a^m b^n cg$, and $(a^m b^n cg)^\alpha = ag$. Also, either $m+n=\varepsilon$ or $m+n=-\varepsilon$, where $\varepsilon^2=-1$. If $m+n=\varepsilon$ then, since $\{ag,\ a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cg,\ b^{-1}g,\ a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cg\}$ cannot generate G, we have that $m \neq \frac{\varepsilon+1}{2}$, and thus

$$a^\alpha = a^i b^{-\varepsilon-i},\ b^\alpha = a^{-\varepsilon-i}b^i,\ c^\alpha = c,\ \text{and}\ g^\alpha = a^{m-\varepsilon-i}b^{\varepsilon+i-m}cg,$$

where $i = \frac{\varepsilon(1-m)-m}{2m-\varepsilon-1}$. If however $m+n=-\varepsilon$ then, since $\{ag,\ a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cg,\ b^{-1}g,\ a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cg\}$ cannot generate G, we have that $n \neq -\frac{\varepsilon+1}{2}$, and consequently

$$a^\alpha = a^i b^{\varepsilon-i},\ b^\alpha = a^{\varepsilon-i}b^i,\ c^\alpha = c,\ \text{and}\ g^\alpha = a^{-i-n}b^{i+n}cg,$$

where $i = \frac{n(\varepsilon-1)-1}{2n+\varepsilon+1}$.

We can conclude that either $S_0 = S_7 = \{ag,\ a^m b^{\varepsilon-m}cg,\ b^{-1}g,\ a^{m-\varepsilon}b^{-m}cg\}$, where $m \neq \frac{\varepsilon+1}{2}$, or $S_0 = S_8 = \{ag,\ a^{-n-\varepsilon}b^n cg,\ b^{-1}g, a^{-n}b^{n+\varepsilon}cg\}$, where $n \neq -\frac{\varepsilon+1}{2}$. Further, replacing $-n$ with m, one can see that $S_8 = S_7$. That G is indeed generated by S_7 can be seen in the following way. Since $(ag)^2 = ab$ and $aga^m b^{\varepsilon-m}cg = a^{\varepsilon+1-m}b^m c$, we have that $c, a^{\varepsilon+1-m}b^m \in \langle S_7\rangle$. Then, since $a^{\varepsilon+1-m}b^m = a^{\varepsilon+1-2m}a^m b^m$, $m \neq \frac{\varepsilon+1}{2}$, and $ab \in \langle S_7\rangle$, we get that $a^{\varepsilon+1-2m} \in \langle S_7\rangle$. Finally, since $ag \in \langle S_7\rangle$, it follows that also $g \in \langle S_7\rangle$. Now the fact that $a^g = b$ implies that $G = \langle S_7\rangle$.

Now considering the automorphism γ of G defined by

$$a^\gamma = a^{\frac{1}{2}},\ b^\gamma = b^{\frac{1}{2}},\ c^\gamma = c,\ \text{and}\ g^\gamma = a^{\frac{1}{2}}b^{-\frac{1}{2}}g,$$

gives that ${S_5}^\gamma = \{ag,\ a^{\frac{m+1}{2}}b^{\frac{\omega}{2}-\frac{m+1}{2}}cg,\ b^{-1}g,\ a^{\frac{m+1}{2}-\frac{\omega}{2}}b^{-\frac{m+1}{2}}cg\}$, where $m \neq \frac{\omega}{2}$. So we only need to consider the generating set $S_7 = \{ag,\ a^m b^{\varepsilon-m}cg,\ b^{-1}g,\ a^{m-\varepsilon}b^{-m}cg\}$, where $m \neq \frac{\varepsilon+1}{2}$ and $\varepsilon^2 = -1$. Observe also, that this implies that $p \equiv 1 \pmod 4$.

We have proved that when $\mathrm{Aut}(G, S_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there always exists an automorphism σ of G such that ${S_0}^\sigma = S = \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}$. Moreover, $\mathrm{Aut}(G, S) = \langle\alpha,\ \beta\rangle$, where

$$a^\alpha = b,\ b^\alpha = a,\ c^\alpha = c,\ g^\alpha = cg\, a^\beta = b^{-1},\ b^\beta = a^{-1},\ c^\beta = c,\ \text{and}\ g^\beta = g.$$

One the other hand when $\mathrm{Aut}(G, S_0) \cong \mathbb{Z}_4$ there always exists an automorphism δ of G such that ${S_0}^\delta = S = \{ag,\ b^\varepsilon cg,\ b^{-1}g,\ a^{-\varepsilon}cg\}$. Moreover, in this case $\mathrm{Aut}(G, S) = \langle\rho\rangle$, where

$$a^\rho = a^{\frac{\varepsilon-1}{2}}b^{\frac{\varepsilon+1}{2}},\ b^\rho = a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}},\ c^\rho = c,\ \text{and}\ g^\rho = a^{\frac{1-\varepsilon}{2}}b^{\frac{\varepsilon-1}{2}}cg.$$

Observe also that the following hold:

(1) If $\varepsilon^2 = -1$ then $\{ag,\ b^\varepsilon cg,\ b^{-1}g,\ a^{-\varepsilon}cg\}^\tau = \{ag,\ b^{-\varepsilon}cg,\ b^{-1}g,\ a^\varepsilon cg\}$, where τ is an automorphism of G mapping according to the rule $a^\tau = b^{-\varepsilon}$, $b^\tau = a^{-\varepsilon}$, $c^\tau = c$, and $g^\tau = cg$.
(2) Since $agbcg = a^2c$, $(a^2c)^2 = a^4$, $(a^2c)^p = c$, $a^g = b$ and p is an odd prime, we can conclude that $\langle\{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}\rangle = \langle ag,\ bcg\rangle = \langle a,\ b,\ c,\ g\rangle = G$.
(3) Let $\varepsilon^2 = -1$. Then $agb^\varepsilon cg = a^{1+\varepsilon}c$, $(a^{1+\varepsilon}c)^2 = a^{2(1+\varepsilon)}$, and $(a^{1+\varepsilon}c)^p = c$. Since p is an odd prime and $a^g = b$, we can conclude that $\langle\{ag,\ b^\varepsilon cg,\ b^{-1}g,\ a^{-\varepsilon}cg\}\rangle = \langle ag,\ b^\varepsilon cg\rangle = \langle a,\ b,\ c,\ g\rangle = G$.

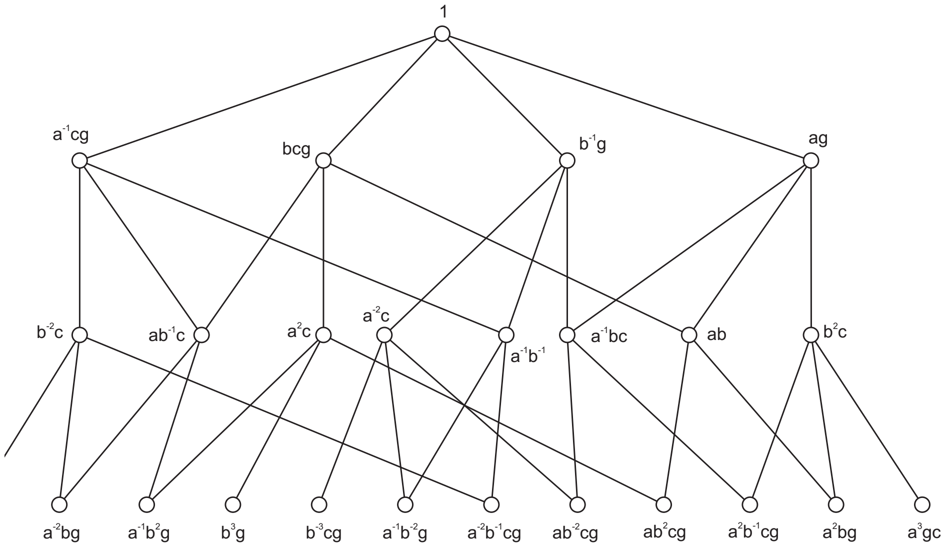

Figure 2: A local structure of the graph $CN^3_{4p^2}$.

To finish the proof, it is sufficient to prove that the graphs

$$Cay(G, \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}) \text{ and } Cay(G, \{ag,\ b^{\varepsilon}cg,\ b^{-1}g,\ a^{-\varepsilon}cg\})$$

are normal Cayley graphs.

First, let $X = Cay(G, \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\})$, let $A = \mathrm{Aut}(X)$ and let A_1^* be the subgroup of the stabilizer A_1 fixing the set $S = \{ag,\ bcg,\ b^{-1}g,\ a^{-1}cg\}$ pointwise. Then, since the 2-arc $(1, ag, a^{-1}bc)$ lies on a 6-cycle but the 2-arc $(1, ag, ab)$ does not, one can see that A_1^* fixes every vertex at distance 2 from 1 in X (see also Figure 2). By connectivity of X and transitivity of A on $V(X)$, A_1^* fixes every vertex in X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $\mathrm{Aut}(G, S) = \mathbb{Z}_2^2 \leq A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_2^2, D_8, A_4, S_4\}$. If $A_1 \in \{A_4, S_4\}$ then there exists a permutation δ in A_1 of order 3. We can, without loss of generality, assume that δ fixes ag, and cyclically permutates the other three neighbors of 1. But, however, considering the images of the vertices at distance 2 from 1, one can see that this is impossible (see Figure 2). If $A_1 = D_8$ then we may, without loss of generality, assume that there exists an involution $\gamma \in A_1$ such that $\gamma \notin \mathrm{Aut}(G, S)$, $(ag)^\gamma = ag$, $(b^{-1}g)^\gamma = b^{-1}g$, $(bcg)^\gamma = a^{-1}cg$ and $(a^{-1}cg)^\gamma = bcg$. However, ab is a common neighbor of ag and bcg in X, but there is no common neighbor of ag and $a^{-1}cg$, and thus this case cannot occur. It follows that $A_1 = \mathrm{Aut}(G, S) = \mathbb{Z}_2^2$, and so X is a normal one-regular Cayley graph as claimed.

Now let $X = Cay(G, \{ag,\ b^{\varepsilon}cg,\ b^{-1}g,\ a^{-\varepsilon}cg\})$, let $A = \mathrm{Aut}(X)$ and let A_1^* be the subgroup of the stabilizer A_1 fixing S pointwise. Then considering 6-cycles passing through the vertex 1 one can see that A_1^* fixes all the vertices at distance 2 from 1 in X (see also Figure 3). Then, connectivity and vertex-transitivity of X combined together imply that A_1^* fixes every vertex of X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $\mathrm{Aut}(G, S) \cong \mathbb{Z}_4 \lesssim A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_4, D_8, S_4\}$. If $A_1 \in \{D_8, S_4\}$ then, without loss of generality, we may assume that there exists an involution $\zeta \in A_1$ such that $\zeta \notin \mathrm{Aut}(G, S)$, $(ag)^\zeta = ag$, $(b^{-1}g)^\zeta = b^{-1}g$, $(b^{\varepsilon}cg)^\zeta = a^{-\varepsilon}cg$, and $(a^{-\varepsilon}cg)^\zeta = (b^{\varepsilon}cg)$. Since there is no 6-cycle passing through $b^{-1}g, 1, ag$ and ab, it follows that ζ fixes ab. On the other hand, since ζ normalizes a Sylow p-subgroup P of G ($P \trianglelefteq A$, see Theorem 5.1), we have that $(xy)^\zeta = 1^{R(xy)\zeta} = 1^{\zeta^{-1}(R(x)R(y))\zeta} = 1^{R(x)^\zeta R(y)^\zeta} = R(x)^\zeta R(y)^\zeta = 1^{R(x)^\zeta} 1^{R(y)^\zeta} = x^\zeta y^\zeta$, for every $x, y \in \langle a, b\rangle$. In other words, ζ induces an automorphism on $\langle a, b\rangle$. Thus, ζ fixes $\langle ab\rangle$ pointwise, and, in particular, ζ fixes both $a^{\varepsilon}b^{\varepsilon}$ and $a^{-\varepsilon}b^{-\varepsilon}$, a contradiction. This means that $A_1 = \mathrm{Aut}(G, S) = \mathbb{Z}_4$, and thus X is a normal one-regular Cayley graph as claimed.

□

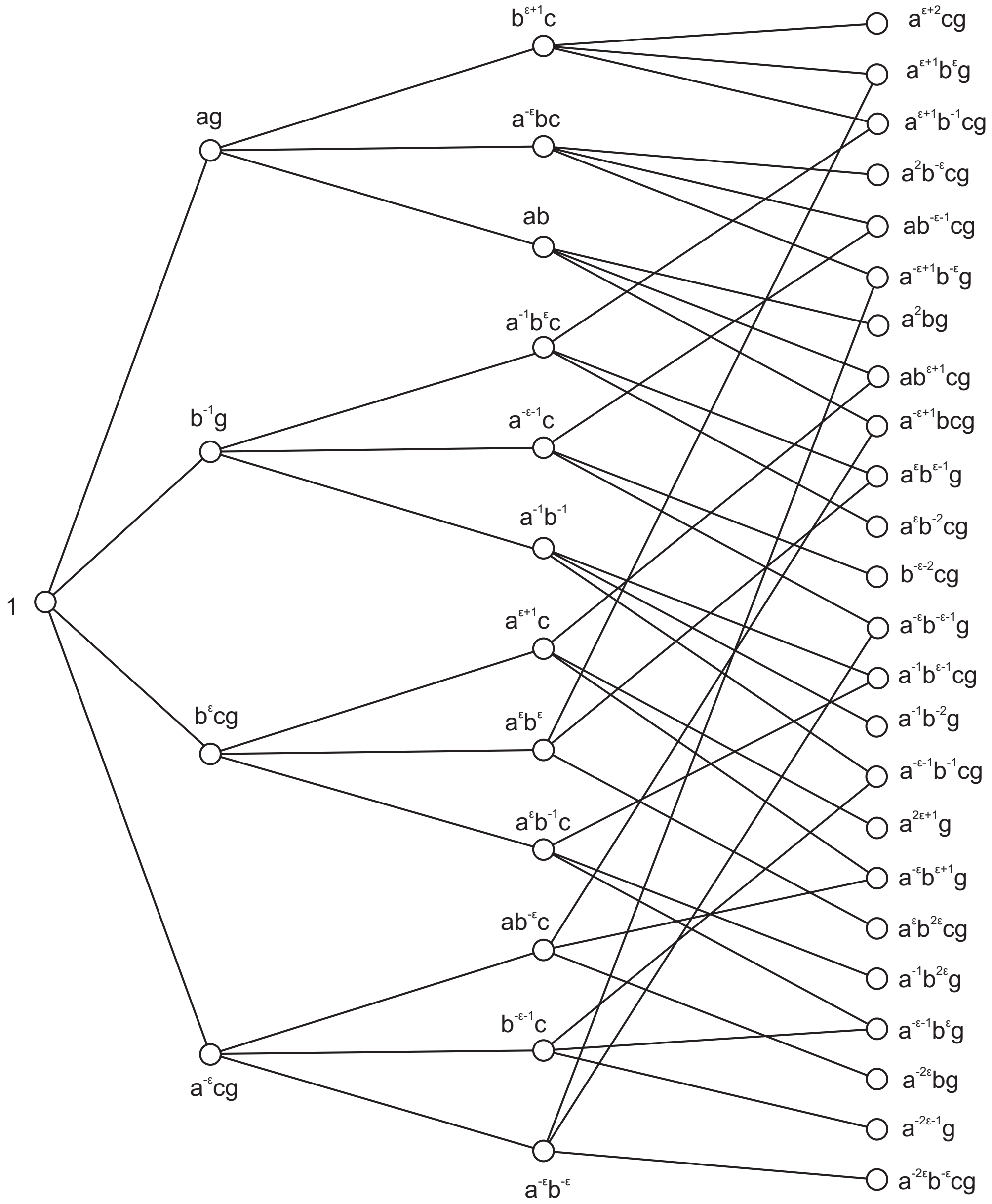

Figure 3: A local structure of the graph $\mathcal{CN}^{4}_{4p^2}$.

Lemma 4.3. $\mathcal{CA}^{1}_{4p^2} \cong \mathcal{CN}^{3}_{4p^2}$.

Proof. Let $G^{1}_{4p^2} = \langle a, b \mid a^{4p} = b^{p} = 1, ab = ba\rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_{p}$ and let $G^{3}_{4p^2} = \langle a, b, c, g \mid a^{p} = b^{p} = c^{2} = g^{2} = [a,b] = [c,g] = [a,c] = [b,c] = 1, a^{g} = b, b^{g} = a\rangle$. Then the automorphism group of $\mathcal{CN}^{3}_{4p^2} = \mathrm{Cay}(G^{3}_{4p^2}, \{ag, bcg, b^{-1}g, a^{-1}cg\})$, is equal to $\mathrm{Aut}(\mathcal{CN}^{3}_{4p^2}) = R(G^{3}_{4p^2}) \rtimes A_1 = R(G^{3}_{4p^2}) \rtimes \langle \alpha, \beta\rangle \cong G^{3}_{4p^2} \rtimes \mathbb{Z}_2^2$, where $a^{\alpha} = b$, $b^{\alpha} = a$, $c^{\alpha} = c$, $g^{\alpha} = cg$, $a^{\beta} = b^{-1}$, $b^{\beta} = a^{-1}$, $c^{\beta} = c$, $g^{\beta} = g$.

Let $H = \langle R(ag)\alpha, R(b)\rangle$. Then it is easy to see that $H = \langle R(ag)\alpha\rangle \times \langle R(b)\rangle \cong G^{1}_{4p^2}$. Since $H_1 \leq A_1 = \langle \alpha, \beta\rangle \cong \mathbb{Z}_2^2$ and subgroups of order 4 in H are cyclic, we have that $H_1 < A_1$. Moreover, since $(R(ag)\alpha)^{2p}$ is a unique element of order 2 in H and $1^{(R(ag)\alpha)^{2p}} \neq 1$, we have that $H_1 \notin \{\langle\alpha\rangle, \langle\beta\rangle, \langle\alpha\beta\rangle\}$. Thus $H_1 = 1$, that is, H is a regular subgroup of $\mathrm{Aut}(\mathcal{CN}^3_{4p^2})$. Now Proposition 2.6 and Example 3.4 combined together imply that $\mathcal{CA}^1_{4p^2} \cong \mathcal{CN}^3_{4p^2}$. □

Lemma 4.4. $\mathcal{CN}^2_{4p^2} \cong \mathcal{CN}^4_{4p^2}$.

Proof. Let $G^2_{4p^2} = \langle a, b \,|\, a^{4p} = b^p = 1, a^{-1}ba = b^{\varepsilon}, \varepsilon^2 \equiv -1 (\mathrm{mod}\ p)\rangle$, and let $G^3_{4p^2} = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a,b] = [c,g] = [a,c] = [b,c] = 1, a^g = b, b^g = a\rangle$. Let 4^{-1} be the inverse of 4 in $\mathbb{Z}_p$ and let $r = 4^{-1}(\varepsilon - 1)$. Observe that $8r(\varepsilon + 1) + 4 \equiv 0 \pmod{4p}$ and that $4r \neq (\varepsilon - 1)$ in $\mathbb{Z}_{4p}$.

Now define a map α from the vertex set of $\mathcal{CN}^4_{4p^2} = \mathrm{Cay}(G^3_{4p^2}, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\})$ to the vertex set of $\mathcal{CN}^2_{4p^2} = \mathrm{Cay}(G^2_{4p^2}, \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\})$ in the following way:

$$\begin{array}{rcl} a^ib^j & \mapsto & a^{4r(i-j)}b^{i+j} \\ a^ib^jc & \mapsto & a^{4r(i-j+\varepsilon+1)+2}b^{i+j} \\ a^ib^jg & \mapsto & a^{4r(j-i+1)+1}b^{i+j} \\ a^ib^jgc & \mapsto & a^{4r(j-i-\varepsilon)-1}b^{i+j} \end{array}$$

where c and g are involutions in $G^3_{4p^2}$. Then

$$\begin{array}{rcl} (a^ib^j, ag\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^{j+1}b^ig)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-j-1+1)+1}b^{i+j+1}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)+1}b^{i+j+1}) = (a^{4r(i-j)}b^{i+j}, ab\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, b^{\varepsilon}cg\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^jb^{i+\varepsilon}gc)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i+\varepsilon-j-\varepsilon)-1}b^{i+j+\varepsilon}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)-1}b^{i+j+\varepsilon}) = (a^{4r(i-j)}b^{i+j}, a^{-1}b^{\varepsilon}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, b^{-1}g\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^jb^{i-1}g)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-1-j+1)+1}b^{i-1+j}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)+1}b^{i-1+j}) = (a^{4r(i-j)}b^{i+j}, ab^{-1}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, a^{-\varepsilon}cg\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^{j-\varepsilon}b^igc)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-j+\varepsilon-\varepsilon)-1}b^{i+j-\varepsilon}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)-1}b^{i+j-\varepsilon}) = (a^{4r(i-j)}b^{i+j}, a^{-1}b^{-\varepsilon}\cdot a^{4r(i-j)}b^{i+j}). \end{array}$$

Similarly, it can be checked that for any edge $(u, s\cdot u)$, we have that $(u, s\cdot u)^{\alpha} = (v, \bar{s}\cdot v)$, where

$$\begin{array}{rcl} u & \in & \{a^ib^jc, a^ib^jg, a^ib^jgc\}, \\ v & \in & \{a^{4r(i-j+\varepsilon+1)+2}b^{i+j}, a^{4r(j-i+1)+1}b^{i+j}, a^{4r(j-i-\varepsilon)-1}b^{i+j}\}, \\ s & \in & \{ag,\ b^{\varepsilon}cg,\ b^{-1}g,\ a^{-\varepsilon}cg\}, \text{ and} \\ \bar{s} & \in & \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\}. \end{array}$$

From this it follows that α is an isomorphism from $\mathcal{CN}^2_{4p^2}$ to $\mathcal{CN}^4_{4p^2}$. The details are omitted. □

Lemma 4.5. *The graphs $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$, $\mathcal{CA}^i_{4p^2}$, $i \in \{0,1\}$, $\mathcal{CN}^2_{4p^2}$, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$, are pairwise non-isomorphic.*

Proof. First, by the remark subsequent to Example 3.1, the graph $\mathcal{BW}_{12}(5,1,5)$ is not isomorphic to any of the other graphs listed in the lemma. Next, Example 3.2 shows that $\mathcal{GPS}2(4,3,(0\,1):(1\,2))$ is not isomorphic to any of the other graphs listed in the lemma. Then, since the automorphism group of $\mathcal{CA}^0_{4p^2}$ has a cyclic Sylow p-subgroup, $\mathcal{CA}^0_{4p^2}$ is not isomorphic to $\mathcal{CA}^1_{4p^2}$ and $\mathcal{CN}^2_{4p^2}$. Also, Example 3.4 and Lemmas 4.3 and 4.4 combined together show that $\mathcal{CA}^1_{4p^2}$ and $\mathcal{CN}^2_{4p^2}$ are not isomorphic. Namely, the stabilizer of a vertex in $\mathcal{CA}^1_{4p^2}$ is isomorphic to $\mathbb{Z}_2^2$ whereas the stabilizer of a vertex in $\mathcal{CN}^2_{4p^2}$ is isomorphic to $\mathbb{Z}_4$. Finally, since the automorphism groups of both $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ have a minimal normal Sylow p-subgroup and the automorphism groups of $\mathcal{CA}^1_{4p^2}$, $\mathcal{CN}^2_{4p^2}$, do not have a minimal normal Sylow p-subgroups, we have that none of $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathcal{CA}^1_{4p^2}, \mathcal{CN}^2_{4p^2}$. Moreover, since the automorphism groups of both $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ have an elementary abelian Sylow p-subgroup and the automorphism group of $\mathcal{CA}^0_{4p^2}$ has a cyclic Sylow p-subgroup, which follows that none of $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathcal{CA}^0_{4p^2}$. The result now follows from the fact that the stabilizer of a vertex in $\mathcal{NC}^0_{4p^2}$ is isomorphic to $\mathbb{Z}_2^2$ whereas the stabilizer of a vertex in $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathbb{Z}_4$ (see [23, Lemmas 8.4 and 8.7] and Lemma 2.11). □

5. The classification

X	$\lvert V(X)\rvert$	$\mathrm{Aut}(X)$	*References*
$\mathcal{BW}_{12}(5,1,5)$	36	$G_{36} \rtimes \mathbb{Z}_2^2$	Example 3.1
$\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$	36	$\lvert\mathrm{Aut}(X)\rvert = 144$	Example 3.2
$\mathcal{NC}^0_{4p^2}$	$4p^2, p > 7$, $p \equiv \pm 1 \pmod 8$	given in [23, Lemma 8.4]	Lemma 2.11
$\mathcal{NC}^1_{4p^2}$	$4p^2, p > 7$, or $p \equiv 1$ or $3 \pmod 8$	given in [23, Lemma 8.7]	Lemma 2.11
$\mathcal{CA}^0_{4p^2}$	$4p^2, p \equiv 1 \pmod 4$	$(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	Example 3.3
$\mathcal{CA}^1_{4p^2}$	$4p^2, p > 2$	$(\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$	Example 3.4
$\mathcal{CN}^2_{4p^2}$	$4p^2, p \equiv 1 \pmod 4$	$G^3_{4p^2} \rtimes \mathbb{Z}_4$	Lemmas 4.2 and 3.6

Table 2: Tetravalent one-regular graphs of order $4p^2$.

We are now ready to state the main theorem of this paper.

Theorem 5.1. *Let p be a prime. Then a tetravalent graph X of order $4p^2$ is one-regular if and only if it is isomorphic to one of the graphs listed in Table 2. Furthermore, all the graphs listed in Table 2 are pairwise non-isomorphic.*

Proof. Let X be a tetravalent one-regular graph of order $4p^2$. Let $A = \mathrm{Aut}(X)$ and let A_v be the stabilizer of $v \in V(X)$ in A. By [39], there is no tetravalent one-regular graph of order 16, and $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS}2[4,3,(0\ 1):(1\ 2)]$ and $\mathcal{CA}^1_{36}$ are the only tetravalent one-regular graphs of order 36 (see also Examples 3.1, 3.2 and 3.4). Thus, we may assume that $p > 3$. Since X is one-regular we have that $|A| = 16p^2$, and thus A is a solvable group. Let P be a Sylow p-subgroup of A.

Claim: P is normal in A.

Since $|A| = 16p^2$ Sylow's theorems imply that the number of Sylow p-subgroups of A is equal to $|A : N_A(P)| = kp + 1$. In addition, this number divides 16. Hence, if $p > 7$ then we clearly have that P is normal in A as claimed. Now we will prove that P is normal in A also when $p \in \{5, 7\}$.

Let $N = O_2(A)$ be the largest normal 2-subgroup of A. Suppose first that $|N| = 16$ and consider the quotient graph X_N. Then $N \leq K$, where K is the kernel of A acting on $V(X_N)$, X_N is a symmetric graph of valency 2 or 4, and, by Proposition 2.8, A/K acts arc-transitively on X_N. But then $2 \mid |A/K|$, which is clearly impossible since $|A| = 16p^2$. Therefore $|N| \mid 8$. Now we distinguish three different cases depending on the order of N. Let T be a minimal normal subgroup of A.

Case 1. $|N| = 1$.

Then either $|T| = p^2$ or $|T| = p$. In the former case we have that $T = P$ and thus $P \trianglelefteq A$ as claimed. We may therefore assume that $|T| = p$. Let X_T be the quotient graph of X relative to the orbits of T, and let K be the kernel of A acting on $V(X_T)$. Then $T \leq K$ and A/K acts arc-transitively on X_T. If A/T is abelian then, since A/K is a quotient group of the group A/T, also A/K is abelian. But since A/K is vertex-transitive on X_T, Proposition 2.1 implies that it is regular on X_T, contradicting arc-transitivity of A/K on X_T. Thus A/T is a non-abelian group. Let $C = C_A(T)$. Then $T \leq C$ and, by Proposition 2.2, A/C is isomorphic to a subgroup of $\mathrm{Aut}(T) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and consequently $T < C$. Let L/T be a minimal normal subgroup of A/T contained in C/T. Then $L/T \cong \mathbb{Z}_p$, and therefore $P = L \trianglelefteq A$.

Case 2. $|N| = 2$.

Then $|T| \in \{p^2, p, 2\}$. If $|T| = p^2$ then $P \trianglelefteq A$ as claimed. Suppose now that $|T| = 2$, and let $C = C_A(T)$. Then $T \leq C$ and, moreover, by Proposition 2.2, $|A/C| = 1$ which implies that $T < C$. Let L/T be a minimal normal subgroup of C/T. Then either $|L/T| = p^2$ or $|L/T| = p$. In the former case it follows that $|L| = 2p^2$, and consequently P char $L \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed. In the later case we have $L = \mathbb{Z}_2 \times \mathbb{Z}_p$. Suppose first that A/L is abelian and consider the quotient graph X_L of X relative to the orbits of L. Let K be the kernel of A acting on $V(X_L)$. Then $L \leq K$, A/K is a quotient group of A/L, and as such also abelian. But since A/K is vertex-transitive on X_L, Proposition 2.1 implies that A/K is regular on X_L, which is impossible since A/K acts arc-transitively on X_L. Thus, A/L is a non-abelian group. Let $C = C_A(L)$. Then $L \leq C$ and, by Proposition 2.2, $A/C \lesssim \mathrm{Aut}(L) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and so $L < C$. Let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$ and thus $M \trianglelefteq A$ and $|M| = 2p^2$. In addition, since P char $M \trianglelefteq A$, we have that $P \trianglelefteq A$ as claimed.

Assume now that $|T| = p$. Then an argument similar to the one used above shows that A/T is a non-abelian group. Let $C = C_A(T)$. Then, by Proposition 2.2, we have that $A/C \lesssim \mathrm{Aut}(T) \cong \mathbb{Z}_{p-1}$. Thus A/C is abelian, which implies that $T < C$. Let L/T be a minimal normal subgroup A/T contained in C/T. Then either $L/T \cong \mathbb{Z}_p$ or $L/T \cong \mathbb{Z}_2$. If $L/T \cong \mathbb{Z}_p$, then clearly $L = P \trianglelefteq A$. If however $L/T \cong \mathbb{Z}_2$, then $L \cong \mathbb{Z}_{2p}$ and, by Proposition 2.2, $A/C \lesssim \mathrm{Aut}(L) \cong \mathbb{Z}_{p-1}$ where $C = C_A(L)$. Hence A/C is abelian, and consequently $L < C$. Now let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$, and so $|M| = 2p^2$. But then P char $M \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed.

Case 3. $|N| \in \{4, 8\}$.

Then either $|A/N| = 2p^2$ or $|A/N| = 4p^2$. Clearly PN/N is a Sylow p-subgroup of A/N and by Sylow's theorems, $PN/N \trianglelefteq A/N$. Moreover, $PN \trianglelefteq A$. If $|N| = 4$ then for $p \in \{5, 7\}$ we have that P is characteristic in PN, and hence normal in A. Also, if $|N| = 8$ and $p = 5$ then one can easily see that P is characteristic in PN and hence normal in A. Therefore we can now assume that $|N| = 8$ and $p = 7$. Then N is isomorphic to one of the following groups: D_8, Q_8 (the quaternion group), $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2^3$. Let $C = C_A(N)$. By Proposition 2.2, we have that $A/C \lesssim \mathrm{Aut}(N)$. If $N \not\cong \mathbb{Z}_2^3$ then $7 \nmid |\mathrm{Aut}(N)|$ and hence $7^2 \mid |C|$, which implies that $P \leq C$. It follows that P is characteristic in PN and hence normal in A. If however $N \cong \mathbb{Z}_2^3$ then $N \leq C$ and $\mathrm{Aut}(N) \cong \mathrm{PSL}(2,7)$. Observe that $|A/N| = 98$ and $A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PSL}(2,7)$. But $\mathrm{Aut}(N) = \mathrm{PSL}(2,7)$ has no subgroup of order 98 since $|\mathrm{PSL}(2,7)| = 168$, implying that $A/N \neq A/C$, and therefore $N < C$. Note also that $|C| > 8$, but $16 \nmid |C|$. Namely, if $16 \mid |C|$, the fact that A/K acts arc-transitively on X_C, where K is the kernel of A acting on $V(X_C)$, implies that $2 \mid |A/K|$. But this is impossible since $C \leq K$. Therefore $7 \mid |C|$. If $7^2 \nmid |C|$ then $|C| = 8 \cdot 7 = 56$. But then A/C is a group of order $|A/C| = 2 \cdot 7 = 14$ isomorphic to a subgroup of $\mathrm{Aut}(N) \cong \mathrm{PSL}(2,7)$, which by Proposition 2.3 is impossible. Therefore $7^2 \mid |C|$, and consequently $P \leq C_A(N)$. It follows that P is characteristic in PN, and thus normal in A. This proves that A always has a normal Sylow p-subgroup as claimed.

Assume first that P is cyclic. Let X_P be the quotient graph of X relative to the orbits of P and let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, the orbits of P are of length p^2. Thus $|V(X_P)| = 4$, $P \leq K$ and A/K acts arc-transitively on X_P. By Proposition 2.8, we have that $X_P \cong C_4$ and hence $A/K \cong D_8$, forcing $|K| = 2p^2$. Since A/K is a quotient group of A/P, it follows that A/P is a non-abelian group. Moreover, $|K| = 2p^2$ and thus K is not semiregular on $V(X)$. Then $K_v \cong \mathbb{Z}_2$ where $v \in V(X)$. By Proposition 2.2, $A/C \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since A/P is not abelian, we have that P is a proper subgroup of C. If $C \cap K \neq P$ then $C \cap K = K$ ($|K| = 2p^2$). Since K_v is a Sylow 2-subgroup of K, K_v is characteristic in K and so normal in A, implying that $K_v = 1$, a contradiction. Thus, $C \cap K = P$ and $1 \neq C/P = C/(C \cap K) \cong CK/K \trianglelefteq A/K \cong D_8$. If $C/P \cong \mathbb{Z}_2$ then C/P is in the center of A/P and since $(A/P)/(C/P) \cong A/C$ is cyclic, A/P is abelian, a contradiction. It follows that $|C/P| \in \{4, 8\}$, and hence C/P has a characteristic subgroup of order 4, say H/P. Thus, $|H| = 4p^2$ and $H/P \trianglelefteq A/P$, implying that $H \trianglelefteq A$. In addition, since $H \leq C = C_A(P)$, we have that H is abelian. Clearly, $|H_v| \in \{1, 2, 4\}$. First, suppose that $|H_v| = 4$. Then H_v is a Sylow 2-subgroup of H, implying that H_v is characteristic in H. The normality of H in A implies that $H_v \trianglelefteq A$, forcing $H_v = 1$, a contradiction. Second, suppose that $|H_v| = 2$, and let Q be a Sylow 2-subgroup of H. Then $Q \trianglelefteq A$ and $Q_v = H_v$. Consider the quotient graph X_Q of X relative to the orbits of Q. Since $|Q| = 4$ and $Q_v \cong \mathbb{Z}_2$, Proposition 2.8 implies that $X_Q \cong C_{2p^2}$ and hence $X \cong C_{2p^2}[2K_1]$, contradicting one-regularity of X. Thus, we have that $H_v = 1$, and since $|H| = 4p^2$, H is regular on $V(X)$. It follows that X is a Cayley graph on an abelian group with a cyclic Sylow p-subgroup P. By elementary group theory, we know that up to isomorphism $\mathbb{Z}_{4p^2}$ and $\mathbb{Z}_{2p^2} \times \mathbb{Z}_2$, where $p > 3$, are the only abelian groups with a cyclic Sylow p-subgroup. However, by Xu [41, Theorems 3], there is no tetravalent one-regular Cayley graph on $\mathbb{Z}_{4p^2}$, and so $H \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$. Proposition 2.6 and Example 3.3 combined together now imply that $X \cong \mathcal{CA}^0_{4p^2}$.

Now assume that P is elementary-abelian. Suppose first that P is a minimal normal subgroup of A, and consider the quotient graph X_P of X relative to the orbits of P. Let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, we have that the orbits of P are of length p^2, and thus $|V(X_P)| = 4$. By Proposition 2.8, $X_P \cong C_4$, and hence $A/K \cong D_8$, forcing $|K| = 2p^2$ and thus $K_v = \mathbb{Z}_2$. Proposition 2.9 now implies that X is isomorphic to $C^{\pm 1}(p, 4, 2)$, $\mathcal{NC}^0_{4p^2}$ or $\mathcal{NC}^1_{4p^2}$. However, by Lemma 2.10, $C^{\pm 1}(p, 4, 2)$ is not one-regular whereas, by Lemma 2.11, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ both are one-regular. Conditions on the prime p written in Table 2 follow from the definition of these graphs (see page 288).

Suppose now that P is not a minimal normal subgroup of A. Then a minimal normal subgroup N of A is isomorphic to $\mathbb{Z}_p$. Let X_N be the quotient graph of X relative to the orbits of N and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and A/K is transitive on $V(X_N)$. Moreover, we have that $|V(X_N)| = 4p$. By Proposition 2.8, X_N is a cycle of length $4p$, or N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-arc-transitive graph and X is a regular cover of X_N. If $X_N \cong C_{4p}$, and hence $A/K \cong D_{8p}$, then $|K| = 2p$ and thus $K_v = \mathbb{Z}_2$. Applying Proposition 2.12 we get that X is either isomorphic to $C^{\pm 1}(p; 4p, 1)$ or to $C^{\pm \varepsilon}(p; 4p, 1)$. By Lemmas 3.5 and 3.6 and Example 3.4, these two graphs are both one-regular and they are, respectively, isomorphic to $\mathcal{CA}^0_{4p^2}$ and $\mathcal{CA}^1_{4p^2}$. If, however, X_N is a tetravalent connected G/N-symmetric graph, then, by Proposition 2.8, X is a covering graph of a symmetric graph of order $4p$. By Proposition 2.13, there are six tetravalent symmetric graphs of order $4p$: $K_{4,4}$, $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$, $\mathcal{CA}^1_{4p}$, $C(2, p, 2)$ and $\mathfrak{g}_{28}$. But, since there is no tetravalent one-regular graph of order 16, the automorphism group of $\mathfrak{g}_{28}$ does not admit a one-regular subgroup, and since, by Lemma 4.1, there is no one-regular $\mathbb{Z}_p$-cover of $C(2, p, 2)$, we only need to consider the covering graphs of $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$ and $\mathcal{CA}^1_{4p}$. Observe that in each of these three graphs a one-regular subgroup of automorphisms contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$. Let H be a one-regular subgroup of automorphisms of X_N. Since X is one-regular graph, A is the lift of H. Since H contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$ also A contains a normal regular subgroup. Therefore X is a normal Cayley graph of order $4p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_{2p} \times \mathbb{Z}_2 \trianglelefteq H$, there exists a normal subgroup G of A such that $G/\mathbb{Z}_p \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$. The classification of groups of order $4p^2$, given in [5, 6], and a detail analysis of all these groups give that G is either isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$ or to $G = \langle a, b, c, g \mid a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_{2p}) \rtimes \mathbb{Z}_2$. However, by Proposition 2.7, there is no tetravalent one-regular graph on $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$, whereas for the latter group, Lemmas 4.2, 4.3 and 4.4, combined together imply that X is either isomorphic to $\mathcal{CA}^1_{4p^2}$ or to $\mathcal{CN}^2_{4p^2}$. Since, by Lemma 4.3, graphs listed in Table 2 are pairwise non-isomorphic the proof is completed. □

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