Filomat 28:3 (2014), 435–439
DOI 10.2298/FIL1403435B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost Increasing Sequences and Their New Applications II

Hüseyin Bor[a]

[a]*P. O. Box 121, TR-06502 Bahçelievler, Ankara, Turkey*

Abstract. In this paper, we generalize a known theorem dealing with $\mid C,\alpha\mid_k$ summability factors to the $\mid C,\alpha,\beta;\delta\mid_k$ summability factors of infinite series. This theorem also includes some known and new results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence sequence if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta>-1$, of the sequence (na_n), that is (see [5])

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{1}$$

where

$$A_n^{\alpha+\beta}=O(n^{\alpha+\beta}),\quad A_0^{\alpha+\beta}=1\quad and\quad A_{-n}^{\alpha+\beta}=0\quad for\quad n>0. \tag{2}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha,\beta\,;\delta\mid_k$, $k\geq 1$ and $\delta\geq 0$, if (see [3])

$$\sum_{n=1}^{\infty}n^{\delta k-1}\mid t_n^{\alpha,\beta}\mid^k<\infty. \tag{3}$$

If we take $\delta=0$, then $\mid C,\alpha,\beta\,;\delta\mid_k$ summability reduces to $\mid C,\alpha,\beta\mid_k$ summability (see [6]). If we set $\beta=0$ and $\delta=0$, then $\mid C,\alpha,\beta\ \delta\mid_k$ summability reduces to $\mid C,\alpha\mid_k$ summability (see [7]).
Also, if we take $\beta=0$, then we get $\mid C,\alpha;\delta\mid_k$ summability (see [8]).

2. Known result

The following theorems are known dealing with an application of almost increasing sequences.
Theorem 2.1[11] Let (φ_n) be a positive sequence and (X_n) be an almost increasing

2010 *Mathematics Subject Classification*. 40D15, 40F05; 26D15, 40G05
Keywords. Almost increasing sequences; Cesàro mean; absolute summability; infinite series; Hölder inequality; Minkowski inequality.
Received: 08 May 2013; Accepted: 15 May 2013
Communicated by Dragan S. Djordjević
Email address: hbor33@gmail.com (Hüseyin Bor)

sequence. If the conditions

$$\sum_{n=1}^{\infty} n\left|\Delta^2\lambda_n\right| X_n < \infty, \tag{4}$$

$$|\lambda_n| X_n = O(1) \; as \; n \to \infty, \tag{5}$$

$$\varphi_n = O(1) \; as \; n \to \infty, \tag{6}$$

$$n\Delta\varphi_n = O(1) \; as \; n \to \infty, \tag{7}$$

$$\sum_{v=1}^{n} \frac{|t_v|^k}{vX_v^{k-1}} = O(X_n) \; as \; n \to \infty, \tag{8}$$

are satisfied, then the series $\sum a_n\lambda_n\varphi_n$ is summable $|C,1|_k$, $\mathbf{k} \geq 1$.
Theorem 2.2 [4] Let (φ_n) be a positive sequence and let (X_n) be an almost increasing sequence.
If the conditions (4), (5), (6) and (7) are satisfied and the sequence (w_n^α) defined by (see [10])

$$w_n^\alpha = \begin{cases} \left|t_n^\alpha\right|, & \alpha = 1, \\ \max_{1\leq v\leq n}\left|t_v^\alpha\right|, & 0 < \alpha < 1, \end{cases} \tag{9}$$

satisfies the condition

$$\sum_{v=1}^{n} \frac{(w_v^\alpha)^k}{vX_v^{k-1}} = O(X_n) \; as \; n \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n\varphi_n$ is summable $\mid C,\alpha \mid_k$, $0 < \alpha \leq 1$, $(\alpha-1)\mathbf{k} > -1$ and $\mathbf{k} \geq 1$.
Remark 2.3 It should be noted that if we take $\alpha = 1$, then we get Theorem 2.1. In this case, condition (10) reduces to condition (8) and the condition '$(\alpha-1)\mathbf{k} > -1$' is trivial.

3. The main result

The aim of this paper is to generalize Theorem 2. 2 in the following form ;
Theorem 3.1 Let (φ_n) be a positive sequence and let (X_n) be an almost increasing sequence. If the conditions (4), (5), (6) and (7) are satisfied and the sequence $(w_n^{\alpha,\beta})$ defined by

$$(w_n^{\alpha,\beta}) = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1\leq v\leq n}\left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1 \end{cases} \tag{11}$$

satisfies the condition

$$\sum_{v=1}^{n} v^{\delta k}\frac{(w_v^{\alpha,\beta})^k}{vX_v^{k-1}} = O(X_n), \; as \; n \to \infty, \tag{12}$$

then the series $\sum a_n\lambda_n\varphi_n$ is summable $\mid C,\alpha,\beta;\delta \mid_k$, $0 < \alpha \leq 1$, $\delta \geq 0$, $(\alpha+\beta-\delta-1)k > -1$, and $k \geq 1$.

We need the following lemmas for the proof of our theorem.
Lemma 3.2 [2]) If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p\right|. \tag{13}$$

Lemma 3.3 [9]) Under the conditions (4) and (5), we have that

$$nX_n\,|\Delta\lambda_n| = O(1) \; as \; n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} X_n\,|\Delta\lambda_n| < \infty. \tag{15}$$

4. Proof of Theorem 3.1 Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean, with $0 < \alpha \leq 1$ and $\beta > -1$, of the sequence $(na_n\lambda_n\varphi_n)$. Then, by (1), we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v \varphi_n. \tag{16}$$

Thus, applying Abel's transformation first and then using Lemma 3.2, we have that

$$\begin{aligned} T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta(\lambda_v\varphi_n)\sum_{p=1}^{v} A_{n-p}^{\alpha-1}A_p^{\beta} p a_p + \frac{\lambda_n\varphi_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v, \\ &= \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}(\lambda_v\Delta\varphi_v + \varphi_{v+1}\Delta\lambda_v)\sum_{p=1}^{v} A_{n-p}^{\alpha-1}A_p^{\beta} p a_p + \frac{\lambda_n\varphi_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v. \end{aligned}$$

$$\begin{aligned} |T_n^{\alpha+\beta}| &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\lambda_v\Delta\varphi_v|\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1}A_p^{\beta} p\, a_p\right| + \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\varphi_{v+1}\Delta\lambda_v|\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1}A_p^{\beta} p\, a_p\right| \\ &\quad + \frac{|\lambda_n\varphi_n|}{A_n^{\alpha+\beta}}\left|\sum_{v=1}^{v} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v\right| \\ &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1} A_v^{\alpha+\beta} w_v^{\alpha,\beta}|\lambda_v||\Delta\varphi_v| + \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1} A_v^{\alpha+\beta} w_v^{\alpha,\beta}|\varphi_{v+1}||\Delta\lambda_v| + |\lambda_n||\varphi_n| w_n^{\alpha,\beta} \\ &= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} + T_{n,3}^{\alpha,\beta}. \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1}|T_{n,r}^{\alpha,\beta}|^k < \infty, \; for \; r = 1, 2, 3.$$

When $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1}|T_{n,1}^{\alpha,\beta}|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1}(A_n^{\alpha+\beta})^{-k}\left\{\sum_{v=1}^{n-1} A_v^{\alpha+\beta} w_v^{\alpha,\beta}|\Delta\varphi_v||\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta-\delta)k}}\sum_{v=1}^{n-1}(v^{\alpha+\beta})^k(w_v^{\alpha,\beta})^k|\Delta\varphi_v|^k|\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2+(\alpha+\beta-\delta-1)k}}\sum_{v=1}^{n-1} v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\lambda_v|^k\frac{1}{v^k} \\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k v^{-k}|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2+(\alpha+\beta-\delta-1)k}} \\
&= O(1)\sum_{v=1}^{m} v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k v^{-k}|\lambda_v|^k\int_v^\infty\frac{dx}{x^{2+(\alpha+\beta-\delta-1)k}} \\
&= O(1)\sum_{v=1}^{m}(w_v^{\alpha,\beta})^k|\lambda_v||\lambda_v|^{k-1}v^{\delta k}\frac{1}{v} \\
&= O(1)\sum_{v=1}^{m} v^{\delta k}\frac{(w_v^{\alpha,\beta})^k|\lambda_v|}{v\, X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v} r^{\delta k}\frac{(w_r^{\alpha,\beta})^k}{r\, X_r^{k-1}} + O(1)|\lambda_m|\sum_{v=1}^{m} v^{\delta k}\frac{(w_v^{\alpha,\beta})^k}{v\, X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m = O(1), \ m\to\infty
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Again, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1}|T_{n,2}^{\alpha,\beta}|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1}(A_n^{\alpha+\beta})^{-k}\left\{\sum_{v=1}^{n-1} A_v^{\alpha+\beta} w_v^{\alpha,\beta}|\varphi_{v+1}||\Delta\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta-\delta)k}}\left\{\sum_{v=1}^{n-1} v^{\alpha+\beta}(w_v^{\alpha,\beta})|\Delta\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta-\delta)k}}\sum_{v=1}^{n-1}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}}\left\{\sum_{v=1}^{n-1}X_v|\Delta\lambda_v|\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+(\alpha+\beta-\delta)k}}\sum_{v=1}^{n-1}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{1}{n^{1+(\alpha+\beta-\delta)k}}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha,\beta})^k|\Delta\lambda_v|}{X_v^{k-1}}\int_v^{\infty}\frac{dx}{x^{1+(\alpha+\beta-\delta)k}}\\
&= O(1)\sum_{v=1}^{m} v|\Delta\lambda_v|v^{\delta k}\frac{(w_v^{\alpha,\beta})^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v} r^{\delta k}\frac{(w_r^{\alpha,\beta})^k}{rX_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m} v^{\delta k}\frac{(w_v^{\alpha,\beta})^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1} v|\Delta^2\lambda_v|X_v + O(1)\sum_{v=1}^{m-1} X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m\\
&= O(1),\ as\ m\to\infty,
\end{aligned}
$$

by hypotheses of Theorem 3.1 and Lemma 3.3. Finally, as in $T_{n,1}^{\alpha,\beta}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1}|T_{n,3}^{\alpha,\beta}|^k &= \sum_{n=1}^{m} n^{\delta k-1}|\lambda_n\ \varphi_n\ w_n^{\alpha,\beta}|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k}\frac{(w_n^{\alpha,\beta})^k|\lambda_n|}{nX_n^{k-1}} = O(1),\ as\ m\to\infty.
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. This completes the proof of Theorem 3.1. It should be noted that, if we take β=0, δ=0 and α=1, then we get Theorem 2. 1. If we set δ=0, then we get a result concerning the $|\ C,\alpha,\beta\ |_k$ summability factors of infinite series. Also, if we take β=0 and δ=0 , then we obtain Theorem 2. 2. Finally, if we take $k=1$, δ=0 and $\beta=0$, then we get a new result dealing with the $|\ C,\alpha\ |$ summability factors of infinite series.

References

[1] N. K. Bari and S. B. Stečkin, Best approximation and differential properties of two conjugate functions, Trudy. Moskov. Mat. Obš č. 5 (1956) 483–522 (Russian)

[2] H. Bor, On a new application of power increasing sequences, Proc. Est. Acad. Sci. 57 (2008) 205–209

[3] H. Bor, An application of almost increasing sequences, Appl. Math. Lett. 24 (2011) 289–301

[4] H. Bor, Almost increasing sequences and their new applications, J. Inequal. Appl. 2013. 2013: 207

[5] D. Borwein, Theorems on some methods of summability, Quart. J. Math., Oxford, Ser. (2) 9 (1958) 310–316

[6] G. Das, A Tauberian theorem for absolute summability, Proc. Camb. Phil. Soc. 67 (1970) 321–326

[7] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc. 7 (1957) 113–141

[8] T.M. Flett, Some more theorems concerning the absolute summability of Fourier series and power series, Proc. London Math. Soc.7 (1958) 357–387

[9] S. M. Mazhar, Absolute summability factors of infinite series, Kyungpook Math. J. 39 (1999) 67–73

[10] T. Pati, The summability factors of infinite series, Duke Math. J. 21 (1954) 271–284

[11] W. T. Sulaiman, On a new application of almost increasing sequences, Bull. Math. Anal. Appl. 4(3) (2012) 29–33