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# On Generalized Absolute Cesàro Summablity Factors

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**Abstract.** In this paper, we prove a known theorem dealing with $\varphi- \mid C, \alpha, \beta \mid_k$ summability factors under more weaker conditions. This theorem also includes several new results.

## 1. Introduction

A positive sequence $(b_n)$ is said to be an almost increasing sequence if there exists a positive increasing sequence $(c_n)$ and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [2]). A sequence $(d_n)$ is said to be $\delta$-quasi-monotone, if $d_n \to 0$, $d_n > 0$ ultimately and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and $\delta$=$(\delta_n)$ is a sequence of positive numbers (see [3]. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^{\eta}(\log n)^{\gamma}, \gamma \geq 0, 0 < \eta < 1\}$ (see [14]). If we set $\gamma$=0, then we get a quasi-$\eta$-power increasing sequence (see [13]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the $n$th Cesàro mean of order $(\alpha, \beta)$, with $\alpha + \beta > -1$, of the sequence $(na_n)$, that is (see [9])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad and \quad A_{-n}^{\alpha+\beta} = 0 \quad for \quad n > 0. \tag{2}$$

Let $(\varphi_n)$ be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, if ( see [5])

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\beta} \mid^k < \infty. \tag{3}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - \mid C, \alpha, \beta \mid_k$ summability is the same as $\mid C, \alpha, \beta \mid_k$ summability (see [10]). Also, if we take $\varphi_n = n^{\sigma+1-\frac{1}{k}}$), then $\varphi - \mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha, \beta; \sigma \mid_k$ summability (see

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[6]. If we take $\beta = 0$, then we have $\varphi - \mid C, \alpha \mid_k$ summability (see [1]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [11]. Finally, if we take $\varphi_n = n^{\sigma+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C, \alpha; \sigma \mid_k$ summability (see [12]).

## 2. The known results

The following theorems are known dealing with $\varphi - \mid C, \alpha, \beta \mid_k$ summability factors of infinite series.

**Theorem 2.1 [7]** Let $(X_n)$ be an almost increasing sequence such that $\mid \Delta X_n \mid = O(\frac{X_n}{n})$ and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ such that it is $\delta$-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum B_nX_n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and if the sequence $(\theta_n^{\alpha,\beta})$ defined by

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1 \leq v \leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1 \end{cases} \tag{4}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k} (\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k = O(X_m) \quad \text{as} \quad m \to \infty, \tag{5}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\beta > -1$ and $(\alpha + \beta)k + \epsilon > 1$.

**Theorem 2.2 [8]** Let $(X_n)$ be a quasi-f-power increasing sequence and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ such that it is $\delta$-quasi-monotone with $\Delta B_n \leq \delta_n$, $\sum n\delta_nX_n < \infty$, $\sum B_nX_n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and the condition (5) is satisfied, then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\beta > -1$ and $(\alpha + \beta)k + \epsilon > 1$.

## 3. The main result

The aim of this paper is to prove Theorem 2.2 under more weaker conditions. Now, we shall prove the following theorem.

**Theorem 3.1** Let $(X_n)$ be a quasi-f-power increasing sequence and let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers $(B_n)$ such that it is $\delta$-quasi-monotone with $\Delta B_n \leq \delta_n$, $\sum n\delta_nX_n < \infty$, $\sum B_nX_n$ is convergent and $\mid \Delta\lambda_n \mid \leq \mid B_n \mid$ for all n. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{n^k X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty \tag{6}$$

satisfies, then the series $\sum a_n\lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\beta > -1$ and $(\alpha + \beta - 1)k + \epsilon > 0$.

**Remark 3.2** It should be noted that condition (6) is the same as condition (5) when k=1. When $k > 1$ condition (6) is weaker than condition (5), but the converse is not true. As in [15] we can show that if (5) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{n^k X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{n^k} = O(X_m).$$

If (6) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{n^k} = \sum_{n=1}^{m} X_n^{k-1} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{nX_n^{k-1}} = O(X_m^{k-1}) \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\beta})^k}{nX_n^{k-1}} = O(X_m^k) \neq O(X_m).$$

We need the following lemmas for the proof of our theorem.
**Lemma 3.3 [4]** If $0 < \alpha \le 1, \beta > -1$ and $1 \le v \le n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \le \max_{1 \le m \le v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{7}$$

**Lemma 3.4 [8]** Under the conditions regarding $(\lambda_n)$ and $(X_n)$ of Theorem 3.1, we have

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{8}$$

**Lemma 3.5 [8]** Let $(X_n)$ be a quasi-f-power increasing sequence. If $(B_n)$ is a $\delta$-quasi-monotone sequence with $\Delta B_n \le \delta_n$ and $\sum n\delta_n X_n < \infty$ , then

$$\sum_{n=1}^{\infty} nX_n \mid \Delta B_n \mid < \infty, \tag{9}$$

$$nB_n X_n = O(1) \quad as \quad n \to \infty. \tag{10}$$

**4. Proof of Theorem 3.1** Let $(T_n^{\alpha,\beta})$ be the $n$th $(C, \alpha, \beta)$ mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 3.3, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$\begin{aligned} \mid T_n^{\alpha,\beta} \mid &\le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} \\ &= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}. \end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad \text{for} \quad r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$
\begin{aligned}
\sum_{n=2}^{m+2} n^{-k} \mid \varphi_n T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\beta})^{-k} \mid \varphi_n \mid^k \{\sum_{v=1}^{n-1} A_v^{\alpha+\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid\}^k \\
&\leq \sum_{n=2}^{m+1} n^{-k} n^{-(\alpha+\beta)k} \mid \varphi_n \mid^k \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k \left\{\sum_{v=1}^{n-1} 1\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k \sum_{n=v+1}^{m+1} \frac{n^{-k} \mid \varphi_n \mid^k}{n^{1+(\alpha+\beta-1)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} \mid \varphi_n \mid^k}{n^{1+(\alpha+\beta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k \mid B_v \mid^k v^{\epsilon-k} \mid \varphi_v \mid^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\theta_v^{\alpha,\beta})^k v^{\epsilon-k} \mid \varphi_v \mid^k \mid B_v \mid^k \int_v^{\infty} \frac{dx}{x^{1+(\alpha+\beta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} \mid B_v \mid\mid B_v \mid^{k-1} (\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k \\
&= O(1) \sum_{v=1}^{m} \mid B_v \mid \frac{1}{v^{k-1} {X_v}^{k-1}} (\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v \mid B_v \mid) \sum_{r=1}^{v} \frac{(\theta_r^{\alpha,\beta} \mid \varphi_r \mid)^k}{r^k {X_r}^{k-1}} + O(1) m \mid B_m \mid \sum_{v=1}^{m} \frac{(\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k}{v^k {X_v}^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v \mid B_v \mid) \mid X_v + O(1) m \mid B_m \mid X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1) \Delta \mid B_v \mid - \mid B_v \mid\mid X_v + O(1) m \mid B_m \mid X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta B_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid B_v \mid X_v + O(1) m \mid B_m \mid X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.5. Finally, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\beta} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid \mid \lambda_n \mid^{k-1} n^{-k} (\theta_n^{\alpha,\beta} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \frac{1}{{X_n}^{k-1}} n^{-k} (\theta_n^{\alpha,\beta} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \frac{(\theta_v^{\alpha,\beta} \mid \varphi_v \mid)^k}{v^k {X_v}^{k-1}} + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \frac{(\theta_n^{\alpha,\beta} \mid \varphi_n \mid)^k}{n^k {X_n}^{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_m + O(1) \mid \lambda_m \mid X_m \\
&= O(1)\sum_{n=1}^{m-1} \mid B_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.4. This completes the proof of Theorem 3.1. If we take $(X_n)$ as an almost increasing sequence such that $\mid \Delta X_n \mid = O(\frac{X_n}{n})$, then we get Theorem 2.1. In this case the condition " $\Delta B_n \leq \delta_n$ " is not need. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result for $\mid C, \alpha, \beta \mid_k$ summability. If we take $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\sigma+1-\frac{1}{k}}$, then we get a new result for $\mid C, \alpha; \sigma \mid_k$ summability. Also, if we take $\beta = 0$, then we get another new result dealing with $\varphi- \mid C, \alpha \mid_k$ summability factors. Finally if we take $\gamma$=0, then we get a new result dealing with quasi-$\eta$-power increasing sequences.

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