

The Best Bounds for Toader Mean in Terms of the Centroidal and Arithmetic Means

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Abstract. In the paper, the authors discover the best constants $\alpha_1, \alpha_2, \beta_1,$ and β_2 for the double inequalities

$$\alpha_1 \bar{C}(a, b) + (1 - \alpha_1)A(a, b) < T(a, b) < \beta_1 \bar{C}(a, b) + (1 - \beta_1)A(a, b)$$

and

$$\frac{\alpha_2}{A(a, b)} + \frac{1 - \alpha_2}{\bar{C}(a, b)} < \frac{1}{T(a, b)} < \frac{\beta_2}{A(a, b)} + \frac{1 - \beta_2}{\bar{C}(a, b)}$$

to be valid for all $a, b > 0$ with $a \neq b$, where

$$\bar{C}(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)}, \quad A(a, b) = \frac{a + b}{2}, \quad \text{and} \quad T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta$$

are respectively the centroidal, arithmetic, and Toader means of two positive numbers a and b . As an application of the above inequalities, the authors also find some new bounds for the complete elliptic integral of the second kind.

1. Introduction

In [24], Toader introduced a mean

$$T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta = \begin{cases} \frac{2a}{\pi} \mathcal{E}\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right), & a > b, \\ \frac{2b}{\pi} \mathcal{E}\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right), & a < b, \\ a, & a = b, \end{cases}$$

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where

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \theta} \, d\theta$$

for $r \in [0, 1]$ is the complete elliptic integral of the second kind. The quantities

$$\bar{C}(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)}, \quad A(a, b) = \frac{a + b}{2}, \quad S(a, b) = \sqrt{\frac{a^2 + b^2}{2}}$$

are called in the literature the centroidal, arithmetic, and quadratic means of two positive real numbers a and b with $a \neq b$. For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the p -th power mean $M_p(a, b)$ is defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

It is well known that

$$M_{-1}(a, b) < A(a, b) = M_1(a, b) < \bar{C}(a, b) < S(a, b) = M_2(a, b)$$

for all $a, b > 0$ with $a \neq b$.

In [25], Vuorinen conjectured that

$$M_{3/2}(a, b) < T(a, b) \tag{1.2}$$

for all $a, b > 0$ with $a \neq b$. This conjecture was verified by Qiu and Shen [23] and by Barnard, Pearce, and Richards [7]. In [1], Alzer and Qiu presented that

$$T(a, b) < M_{(\ln 2)/\ln(\pi/2)}(a, b) \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$, which gives a best possible upper bound for Toader mean in terms of the power mean. From (1.2) and (1.3), one concludes that

$$A(a, b) < T(a, b) < S(a, b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$. In [12], the authors demonstrated that the double inequality

$$\alpha S(a, b) + (1 - \alpha)A(a, b) < T(a, b) < \beta S(a, b) + (1 - \beta)A(a, b) \tag{1.5}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{2}$ and $\beta \geq \frac{4-\pi}{(\sqrt{2}-1)\pi}$.

Motivated by the double inequality (1.5), we naturally ask a question: What are the best constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1)$ such that the double inequalities

$$\alpha_1 \bar{C}(a, b) + (1 - \alpha_1)A(a, b) < T(a, b) < \beta_1 \bar{C}(a, b) + (1 - \beta_1)A(a, b) \tag{1.6}$$

and

$$\frac{\alpha_2}{A(a, b)} + \frac{1 - \alpha_2}{\bar{C}(a, b)} < \frac{1}{T(a, b)} < \frac{\beta_2}{A(a, b)} + \frac{1 - \beta_2}{\bar{C}(a, b)} \tag{1.7}$$

hold for all $a, b > 0$ with $a \neq b$?

The main aim of this paper is to affirmatively answer the above question.

Theorem 1.1. *The double inequality (1.6) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq \frac{3}{4}$ and $\beta_1 \geq \frac{12}{\pi} - 3$.*

Theorem 1.2. *The double inequality (1.7) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq \pi - 3$ and $\beta_1 \geq \frac{1}{4}$.*

As an immediate applications of Theorem 1.1, we will derive a new bounds in terms of elementary functions for the complete elliptic integral of the second kind.

Theorem 1.3. *For $r \in (0, 1)$ and $r' = \sqrt{1 - r^2}$, we have*

$$\frac{\pi}{2} \left[\frac{1 + r' + (r')^2}{2(1 + r')} + \frac{1 + r'}{8} \right] < \mathcal{E}(r) < \frac{\pi}{2} \left[\left(\frac{8}{\pi} - 2 \right) \frac{1 + r' + (r')^2}{1 + r'} + \left(2 - \frac{6}{\pi} \right) (1 + r') \right]. \tag{1.8}$$

In Section 4 we will compare the above main results with some well-known ones.

Remark 1.1. Some estimates for the three kinds of complete elliptic integrals were established in [1–3, 5, 7, 10, 13, 19, 20, 22, 25–27]. and there is a short review and survey in [21, pp. 40–46] for these estimates.

2. Lemmas

For proving our main results, we need the following lemmas.

For $0 < r < 1$, denote $r' = \sqrt{1 - r^2}$. It is known that Legendre’s complete elliptic integrals of the first and second kind are defined respectively by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 \theta}} d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \frac{\pi}{2}, \\ \mathcal{K}(1) = \infty \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \theta} d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \frac{\pi}{2}, \\ \mathcal{E}(1) = 1. \end{cases}$$

See [9, 11]. For $0 < r < 1$, the following formulas were presented in [3, Appendix E, pp. 474–475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - (r')^2\mathcal{K}}{r(r')^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r}, \quad \frac{d(\mathcal{E} - (r')^2\mathcal{K})}{dr} = r\mathcal{K},$$

$$\frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{(r')^2}, \quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E} - (r')^2\mathcal{K}}{1+r}.$$

Lemma 2.1 ([3, Theorem 3.21]). *The function $\frac{\mathcal{E} - (r')^2\mathcal{K}}{r^2}$ is strictly increasing from $(0, 1)$ onto $(\frac{\pi}{4}, 1)$.*

Lemma 2.2. *The function $5\mathcal{E}(r) - 3(r')^2\mathcal{K}(r)$ is positive and strictly increasing on $(0, 1)$.*

Proof. Let $f(r) = 5\mathcal{E}(r) - 3(r')^2\mathcal{K}(r)$ for $r \in (0, 1)$ and $r' = \sqrt{1 - r^2}$. A simple computation leads to

$$f'(r) = \frac{2\mathcal{E}(r) - 2\mathcal{K}(r) + 3r^2\mathcal{K}(r)}{r} \triangleq \frac{g(r)}{r}.$$

A direct differentiation yields $g'(r) = \frac{r(\mathcal{E}(r) + 3(r')^2\mathcal{K}(r))}{(r')^2} > 0$ for all $r \in (0, 1)$, that is, the function $g(r)$ is strictly increasing on $(0, 1)$. Hence, it is derived that $g(r) > g(0) = 0$, that $f'(r) > 0$, that $f(r)$ is increasing on $(0, 1)$, and that $f(x) > f(0) = \pi > 0$. \square

Lemma 2.3 ([3, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) \neq 0$ on (a, b) . If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing respectively) on (a, b) , so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

Remark 2.1. Lemma 2.3 and its variants have been extensively applied in, for example, [6, 15, 18, 22] and many references listed in [21], and have been generalized in, for example, [14, Lemma 2.2] and [15] and closely related references therein. For more information, please read the first sentence after [6, p. 582, Lemma 2.1], the references [4, 8, 16, 17] and [18, Remark 2.2]

3. Proofs of Main Results

Now we are in a position to prove our main results.

Proof. [Proof of Theorem 1.1] Without loss of generality, we assume that $a > b$. Let $t = \frac{b}{a} \in (0, 1)$ and $r = \frac{1-t}{1+t}$. Then

$$\frac{T(a, b) - A(a, b)}{\overline{C}(a, b) - A(a, b)} = \frac{\frac{\pi}{2} \mathcal{E}'(t) - \frac{1+t}{2}}{\frac{2}{3} \frac{1+t+t^2}{1+t} - \frac{1+t}{2}} = \frac{\frac{2}{\pi} \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) - \frac{1}{1+r}}{\frac{1}{3} \frac{r^2}{1+r}} = 3 \frac{\frac{2}{\pi} [2\mathcal{E} - (r')^2 \mathcal{K}] - 1}{r^2}.$$

Let $f_1(r) = \frac{2}{\pi} [2\mathcal{E} - (r')^2 \mathcal{K}] - 1$, $f_2(r) = r^2$, and $f(r) = 3 \frac{f_1(r)}{f_2(r)} = 3 \frac{\frac{2}{\pi} [2\mathcal{E} - (r')^2 \mathcal{K}] - 1}{r^2}$. Simple computations lead to

$$f_1(0) = f_2(0) = 0, \quad f_1'(r) = \frac{2}{\pi} \frac{\mathcal{E} - (r')^2 \mathcal{K}}{r}, \quad f_2'(r) = 2r, \quad \frac{f_1'(r)}{f_2'(r)} = \frac{1}{\pi} \frac{\mathcal{E} - (r')^2 \mathcal{K}}{r^2}.$$

Combining this with Lemmas 2.1 and 2.3 reveals that the function $f(r)$ is strictly increasing on $(0, 1)$. Further making use of L'Hôpital's rule gives $\lim_{r \rightarrow 0^+} f(r) = \frac{3}{4}$ and $\lim_{r \rightarrow 1^-} f(r) = \frac{12}{\pi} - 3$. Theorem 1.1 is thus proved. \square

Proof. [Proof of Theorem 1.2] Without loss of generality, we assume that $a > b$. Let $t = \frac{b}{a} \in (0, 1)$ and $r = \frac{1-t}{1+t}$. Then

$$\frac{1/T(a, b) - 1/\overline{C}(a, b)}{1/A(a, b) - 1/\overline{C}(a, b)} = \frac{1 - T(a, b)/\overline{C}(a, b)}{T(a, b) \left[\frac{1}{A(a, b)} - \frac{1}{\overline{C}(a, b)} \right]} = \frac{1 - \frac{2}{\pi} \mathcal{E}'(t) / \frac{2}{3} \frac{1+t+t^2}{1+t}}{\frac{2}{\pi} \mathcal{E}'(t) \left[\frac{2}{1+t} - \frac{3(1+t)}{2(1+t+t^2)} \right]} = \frac{3 + r^2 - \frac{6}{\pi} [2\mathcal{E} - (r')^2 \mathcal{K}]}{\frac{2}{\pi} r^2 [2\mathcal{E} - (r')^2 \mathcal{K}]}.$$

Let $f_1(r) = 3 + r^2 - \frac{6}{\pi} [2\mathcal{E} - (r')^2 \mathcal{K}]$, $f_2(r) = \frac{2}{\pi} r^2 [2\mathcal{E} - (r')^2 \mathcal{K}]$, and $f(r) = \frac{f_1(r)}{f_2(r)} = \frac{3+r^2 - \frac{6}{\pi} [2\mathcal{E} - (r')^2 \mathcal{K}]}{\frac{2}{\pi} r^2 [2\mathcal{E} - (r')^2 \mathcal{K}]}$. Simple computations lead to

$$f_1(0) = f_2(0) = 0, \quad f_1'(r) = 2r - \frac{6}{\pi} \frac{\mathcal{E} - (r')^2 \mathcal{K}}{r}, \quad f_2'(r) = \frac{2}{\pi} [5\mathcal{E} - 3(r')^2] \mathcal{K} r, \quad \text{and} \quad \frac{f_1'(r)}{f_2'(r)} = \frac{2 - \frac{6}{\pi} \frac{[\mathcal{E} - (r')^2 \mathcal{K}]}{r^2}}{\frac{2}{\pi} [5\mathcal{E} - 3(r')^2 \mathcal{K}]}.$$

Combining this with Lemmas 2.1, 2.2, and 2.3 yields that the function $f(r)$ is strictly decreasing on $(0, 1)$. Making use of L'Hôpital's rule shows that $\lim_{r \rightarrow 0^+} f(r) = \frac{1}{4}$ and $\lim_{r \rightarrow 1^-} f(r) = \pi - 3$. Thus, Theorem 1.2 is proved. \square

Proof. [Proof of Theorem 1.3] Without loss of generality, assume that $a > b$. Substituting $r' = \frac{b}{a}$, $\alpha_1 = \frac{3}{4}$, and $\beta_1 = \frac{12}{\pi} - 3$ into Theorem 1.1 produces Theorem 1.3. \square

4. Comparisons with some Known Results

In [12], it was obtained that

$$\frac{\pi}{2} \left[\frac{1}{2} \sqrt{\frac{1+(r')^2}{2}} + \frac{1+r'}{4} \right] < \mathcal{E}(r) < \frac{\pi}{2} \left[\frac{4-\pi}{(\sqrt{2}-1)\pi} \sqrt{\frac{1+(r')^2}{2}} + \frac{(\sqrt{2}\pi-4)(1+r')}{2(\sqrt{2}-1)\pi} \right] \tag{4.1}$$

for all $r \in (0, 1)$ and $r' = \sqrt{1-r^2}$. Guo and Qi proved in [13] that

$$\frac{\pi}{2} - \frac{1}{2} \ln \frac{(1+r)^{1-r}}{(1-r)^{1+r}} < \mathcal{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r} \ln \frac{1+r}{1-r} \tag{4.2}$$

for all $r \in (0, 1)$. It was pointed out in [12] that the bounds in (4.1) for $\mathcal{E}(r)$ are better than those in (4.2) for some $r \in (0, 1)$. Very recently, Yin and Qi obtained in [27] that

$$\frac{\pi}{2} \frac{\sqrt{6 + 2\sqrt{1 - r^2} - 3r^2}}{2\sqrt{2}} \leq \mathcal{E}(r) \leq \frac{\pi}{2} \frac{\sqrt{10 - 2\sqrt{1 - r^2} - 5r^2}}{2\sqrt{2}}. \tag{4.3}$$

Let

$$g(x) = \frac{1 + x + x^2}{2(1 + x)} + \frac{1 + x}{8} - \left(\frac{1}{2} \sqrt{\frac{1 + x^2}{2}} + \frac{1 + x}{4} \right)$$

for $x \in (0, 1)$. Then a simplification leads to

$$g(x) = \frac{3x^2 + 2x + 3 - 2(1 + x)\sqrt{2(1 + x^2)}}{8(1 + x)}.$$

Since

$$(3x^2 + 2x + 3)^2 - [2(1 + x)\sqrt{2(1 + x^2)}]^2 = (1 - x)^4 > 0,$$

the lower bound in (1.8) for $\mathcal{E}(r)$ is better than the one in (4.1).

Since

$$\frac{1 + x + x^2}{2(1 + x)} + \frac{1 + x}{8} > \frac{\sqrt{6 + 2x - 3(1 - x^2)}}{2\sqrt{2}}$$

is equivalent to

$$(5x^2 + 6x + 5)^2 > 8(x + 1)^2(3x^2 + 2x + 3)$$

and $(x - 1)^4 > 0$, the lower bound in (1.8) for $\mathcal{E}(r)$ is better than the one in (4.3).

Let

$$J(r) = \frac{\pi}{2} \left[\left(\frac{8}{\pi} - 2 \right) \frac{1 + r' + (r')^2}{1 + r'} + \left(2 - \frac{6}{\pi} \right) (1 + r') \right], \quad Q(r) = \frac{\pi - 1}{2} + \frac{1 - r^2}{4r} \ln \frac{1 + r}{1 - r'}$$

$$D(r) = \frac{\pi}{2} \left[\frac{4 - \pi}{(\sqrt{2} - 1)\pi} \sqrt{\frac{1 + (r')^2}{2}} + \frac{(\sqrt{2}\pi - 4)(1 + r')}{2(\sqrt{2} - 1)\pi} \right], \quad Y(r) = \frac{\pi}{2} \frac{\sqrt{10 - 2\sqrt{1 - r^2} - 5r^2}}{2\sqrt{2}}.$$

The values of these functions at points 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 can be numerical computed and listed in Table 1. This implies that the upper bound in (1.8) for $\mathcal{E}(r)$ are better than those in (4.1), (4.2), and (4.3) for some $r \in (0, 1)$.

In conclusion, the double inequality (1.8) is better than some known results in [12, 13, 27] somewhere.

References

- [1] H. Alzer and S.-L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, *J. Comput. Appl. Math.* 172 (2004) 289–312; Available online at <http://dx.doi.org/10.1016/j.cam.2004.02.009>.
- [2] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen, Generalized elliptic integrals and modular equations, *Pacific J. Math.* 192 (2000) 1–37; Available online at <http://dx.doi.org/10.2140/pjm.2000.192.1>.
- [3] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.
- [4] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, Functional inequalities for hypergeometric functions and complete elliptic integrals, *SIAM J. Math. Anal.* 23 (1992) 512–524; Available online at <http://dx.doi.org/10.1137/0523025>.

Table 1: Values of $J(r)$, $D(r)$, $Q(r)$, and $Y(r)$

r	$J(r)$	$D(r)$	$Q(r)$	$Y(r)$
0.1	1.566862174...	1.566862736...	1.567456298...	1.566866887...
0.2	1.554972309...	1.554981471...	1.557354457...	1.555049510...
0.3	1.534853276...	1.534901499...	1.540234393...	1.535259718...
0.4	1.506007907...	1.506169094...	1.515627704...	1.507368120...
0.5	1.467637170...	1.468061483...	1.482775936...	1.471228040...
0.6	1.418485626...	1.419455645...	1.440474824...	1.426746617...
0.7	1.356514851...	1.358548915...	1.386741519...	1.374078083...
0.8	1.278097245...	1.282149209...	1.317984092...	1.314222496...
0.9	1.175305090...	1.183095913...	1.226197273...	1.251499407...

[5] S. András and Á. Baricz, Bounds for complete elliptic integral of the first kind, *Expo. Math.* 28 (2010) 357–364; Available online at <http://dx.doi.org/10.1016/j.exmath.2009.12.005>.

[6] A. Baricz, Bounds for modified Bessel functions of the first and second kinds, *Proc. Edinburgh Math. Soc.* 53 (2010) 575–599; Available online at <http://dx.doi.org/10.1017/S0013091508001016>.

[7] R. W. Barnard, K. Pearce, and K. C. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, *SIAM J. Math. Anal.* 31 (2000) 693–699; Available online at <http://dx.doi.org/10.1137/S0036141098341575>.

[8] M. Biernacki and J. Krzyż, On the monotony of certain functionals in the theory of analytic functions, *Annales Univ. Mariae Curie-Skłodowska A* 9 (1955) 135–147.

[9] F. Bowman, *Introduction to Elliptic Functions with Applications*, Dover Publications, New York, 1961.

[10] P. Bracken, An arithmetic-geometric mean inequality, *Expo. Math.* 19 (2001) 273–279; Available online at [http://dx.doi.org/10.1016/S0723-0869\(01\)80006-2](http://dx.doi.org/10.1016/S0723-0869(01)80006-2).

[11] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag, New York, 1971.

[12] Y.-M. Chu, M.-K. Wang, and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, *Proc. Indian Acad. Sci. Math. Sci.* 122 (2012) 41–51; Available online at <http://dx.doi.org/10.1007/s12044-012-0062-y>.

[13] B.-N. Guo and F. Qi, Some bounds for the complete elliptic integrals of the first and second kind, *Math. Inequal. Appl.* 14 (2011) 323–334; Available online at <http://dx.doi.org/10.7153/mia-14-26>.

[14] S. Koumandos and H. L. Pedersen, On the asymptotic expansion of the logarithm of Barnes triple Gamma function, *Math. Scand.* 105 (2009) 287–306.

[15] I. Pinelis, L'Hospital rules for monotonicity and the Wilker-Anglesio inequality, *Amer. Math. Monthly* 111 (2004) 905–909; Available online at <http://dx.doi.org/10.2307/4145099>.

[16] S. Ponnusamy and M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, *Mathematika* 44 (1997) 278–301.

[17] F. Qi, Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions, *Math. Inequal. Appl.* 18 (2015), in press; Available online at <http://arxiv.org/abs/1302.6731>.

[18] F. Qi and C. Berg, Complete monotonicity of a difference between the exponential and trigamma functions and properties related to a modified Bessel function, *Mediterr. J. Math.* 10 (2013) 1683–1694; Available online at <http://dx.doi.org/10.1007/s00009-013-0272-2>.

[19] F. Qi, L.-H. Cui, and S.-L. Xu, Some inequalities constructed by Tchebysheff's integral inequality, *Math. Inequal. Appl.* 2 (1999) 517–528; Available online at <http://dx.doi.org/10.7153/mia-02-42>.

[20] F. Qi and Z. Huang, Inequalities for complete elliptic integrals, *Tamkang J. Math.* 29 (1998) 165–169; Available online at <http://dx.doi.org/10.5556/j.tkjm.29.1998.165-169>.

[21] F. Qi, D.-W. Niu, and B.-N. Guo, Refinements, generalizations, and applications of Jordan's inequality and related problems, *J. Inequal. Appl.* 2009 Article ID 271923 52 pages; Available online at <http://dx.doi.org/10.1155/2009/271923>.

[22] F. Qi and A. Sofo, An alternative and united proof of a double inequality for bounding the arithmetic-geometric mean, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* 71 (2009) 69–76.

[23] S.-L. Qiu and J.-M. Shen, On two problems concerning means, *J. Hangzhou Insitute Electronic Engineering* 17 (1997) (3) 1–7. (Chinese)

[24] Gh. Toader, Some mean values related to the arithmetic-geometric mean, *J. Math. Anal. Appl.* 218 (1998) 358–368; <http://dx.doi.org/10.1006/jmaa.1997.5766>.

[25] M. Vuorinen, Hypergeometric functions in geometric function theory, in: *Special Functions and Differential Equations, Proceedings of a Workshop held at The Institute of Mathematical Sciences, Madras, India, January 13-24, 1997*, Allied Publ., New Delhi, 1998, 119–126.

[26] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, *J. Math. Anal. Appl.* 388 (2012) 1141–1146; Available online at <http://dx.doi.org/10.1016/j.jmaa.2011.10.063>.

[27] L. Yin and F. Qi, Some inequalities for complete elliptic integrals, available online at <http://arxiv.org/abs/1301.4385>.