Filomat 28:4 (2014), 839–847
DOI 10.2298/FIL1404839W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Second Order Parallel Tensors on Almost Kenmotsu Manifolds Satisfying the Nullity Distributions

Yaning Wang[a], Ximin Liu[b]

[a]*College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, P. R. China*
[b]*School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, Liaoning, P. R. China*

Abstract. In this paper, we prove that if there exists a second order symmetric parallel tensor on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution, then either M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, or the second order parallel tensor is a constant multiple of the associated metric tensor of M^{2n+1}. Furthermore, some properties of an almost Kenmotsu manifold admitting a second order parallel tensor with ξ belonging to the (k, μ)-nullity distribution are also obtained.

1. Introduction

In 1923, L. P. Eisenhart [12] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, H. Levy [16] proved that a second order parallel symmetric non-degenerated tensor in a space form is proportional to the metric tensor. Since then, many authors investigated the Eisenhart problem of finding symmetric and skew symmetric parallel tensors on various spaces and obtained fruitful results. For instance, by giving a global approach based on the Ricci identity, R. Sharma firstly investigated Eisenhart problem on complex space forms in [21]. In addition to space forms, R. Sharma considered the Eisenhart problem on contact geometry in [22] and [24], for example for K-contact manifolds in [23]. Note that the Eisenhart problem have also been studied in [17] on P-Sasakian manifolds with a coefficient k, in [8] on P-Sasakian manifolds, in [7] on α-Sasakian manifold and in [6] on $N(k)$-quasi Einstein manifolds, respectively.

However, the results of Eisenhart problem on Kenmotsu manifolds are lack, until recently De-Mondal [9] and Calin-Crasmareanu [5] have obtained some theorems of Eisenhart problem on 3-dimensional normal almost contact metric manifolds and on f-Kenmotsu manifolds respectively. Thus, motivated by the related results mentioned above, the object of this paper is to start the study of the Eisenhart problem on a type of almost Kenmotsu manifolds. In fact, let M^{2n+1} be an almost Kenmotsu manifold admitting a second order symmetric parallel covariant tensor with ξ belonging to the $(k, \mu)'$-nullity distribution, by

2010 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25, 53D10, 53D15
Keywords. Almost Kenmotsu manifold, Second order parallel tensor, Nullity distribution.
Received: 29 April 2013; Accepted: 25 July 2013
Communicated by Ljubica Velimirović
This work is supported by Natural Science Foundation of China (No. 10931005) and Natural Science Foundation of Guangdong Province of China (No. S2011010000471)
Email addresses: wyn051@163.com (Yaning Wang), ximinliu@dlut.edu.cn (Ximin Liu)

using the classification theorems of almost Kenmotsu manifolds proved by Dileo-Pastore in [11] we prove that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$, provided that the second order symmetric parallel tensor is not a constant multiple of the associated metric tensor of M^{2n+1}. Moreover, we also obtain some results concerning the existence of second order parallel covariant tensors on almost Kenmotsu manifolds such that ξ belongs to the (k,μ)-nullity distribution.

This paper is organized in the following way. In Section 2, we provide some basic formulas and properties of almost Kenmotsu manifolds. Section 3 is devoted to investigating the Eisenhart problem on an almost Kenmotsu manifold M^{2n+1} whose characteristic vector field ξ belongs to the (k,μ)-nullity distribution. Finally, in Section 4, we study the existence of symmetric and skew symmetric parallel covariant tensors on an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distributions respectively. Some classification theorems are obtained in this paper.

2. Almost Kenmotsu Manifolds

Firstly, we shall recall some basic notions and properties of almost Kenmotsu manifolds (see [10, 11]). An almost contact structure (see [3]) on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\phi^2=-\mathrm{Id}+\eta\otimes\xi,\quad \eta(\xi)=1, \tag{1}$$

which implies that $\phi(\xi)=0$, $\eta\circ\phi=0$ and $\mathrm{rank}(\phi)=2n$. A Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ,ξ,η) if

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y) \tag{2}$$

for any vector fields X,Y. An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. The fundamental 2-form Φ is defined by $\Phi(X,Y)=g(X,\phi Y)$ for any vector fields X and Y on M^{2n+1}. An almost Kenmotsu manifold is defined as an almost contact metric manifold together with $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$. The normality of an almost contact structure is expressed by the vanishing of the tensor $N_\phi=[\phi,\phi]+2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. According to Janssens-Vanhecke [14], a normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

Now let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold. We denote by $l=R(\cdot,\xi)\xi$ and $h=\frac{1}{2}\mathcal{L}_\xi\phi$ on M^{2n+1}, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. Thus, the two $(1,1)$-type tensor fields l and h are symmetric and satisfy

$$h\xi=0,\;\; l\xi=0,\;\; \mathrm{tr}h=0,\;\; \mathrm{tr}(h\phi)=0,\;\; h\phi+\phi h=0. \tag{3}$$

We also have the following formulas presented in [10, 11]

$$\nabla_X\xi=-\phi^2X-\phi hX\;(\Rightarrow\nabla_\xi\xi=0), \tag{4}$$

$$\phi l\phi-l=2(h^2-\phi^2), \tag{5}$$

$$\mathrm{tr}(l)=S(\xi,\xi)=g(Q\xi,\xi)=-2n-\mathrm{tr}h^2, \tag{6}$$

$$R(X,Y)\xi=\eta(X)(Y-\phi hY)-\eta(Y)(X-\phi hX)+(\nabla_Y\phi h)X-(\nabla_X\phi h)Y, \tag{7}$$

$$\nabla_\xi h=-\phi-2h-\phi h^2-\phi l \tag{8}$$

for any $X,Y\in\Gamma(TM)$, where S, Q, ∇ and $\Gamma(TM)$ denote the Ricci curvature tensor, the Ricci operator, the Levi-Civita connection of g and the Lie algebra of all vector fields on M^{2n+1}, respectively.

Finally, we recall the definitions of the nullity distributions. Blair-Koufogiorgos-Papantoniou [4] introduced and studied a generalized notion of the k-nullity distribution (see [13, 25]), namely, the (k,μ)-nullity distribution on contact metric manifolds $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined by

$$N_p(k,\mu)=\left\{Z\in T_pM:R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]+\mu[g(Y,Z)hX-g(X,Z)hY]\right\}, \tag{9}$$

where $\mathcal{L}$ denotes the Lie differentiation and $k, \mu \in R$.

Recently, Dileo-Pastore [11] introduced another generalized notion of the k-nullity distribution named the $(k,\mu)'$-nullity distribution on almost Kenmotsu manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined by

$$N_p(k,\mu) = \left\{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\right\}, \tag{10}$$

where $h' = h \circ \phi$, $\mathcal{L}$ denotes the Lie differentiation and $k, \mu \in R$.

Here, it is worth to point out that almost Kenmotsu pseudo-metric manifolds satisfying the (k,μ) or $(k,\mu)'$-nullity distributions were studied by the present authors in [26]. For some results on the k-nullity distributions, the generalized $(k,\mu)'$ and (k,μ)-nullity distributions on almost Kenmotsu manifolds, we refer the reader to Pastore-Saltarelli [19, 20]. Some classification theorems of almost Kenmotsu manifolds with ξ belonging to the nullity distributions are also obtained by the present authors in [27].

3. ξ Belongs to the (k,μ)-Nullity Distribution

A covariant tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g. Let α be a $(0,2)$-type symmetric tensor field on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ such that $\nabla\alpha = 0$, then it follows that

$$\alpha(R(W,X)Y,Z) + \alpha(Y,R(W,X)Z) = 0 \tag{11}$$

for arbitrary vector fields $X, Y, Z, W \in \Gamma(TM)$.

Let M^{2n+1} be an almost Kenmotsu manifold for which ξ belongs to the (k,μ)-nullity distribution, that is

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{12}$$

Thus, it follows from (12) that

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX]. \tag{13}$$

Theorem 3.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution. If M^{2n+1} admits a second order symmetric parallel tensor, then the second order parallel tensor is a constant multiple of the associated metric tensor of M^{2n+1}.*

Proof. Substituting $Y = Z = W = \xi$ in (11) gives that $\alpha(R(\xi,X)\xi,\xi) + \alpha(\xi,R(\xi,X)\xi) = 0$, then it follows from the symmetry of α that

$$\alpha(R(\xi,X)\xi,\xi) = 0 \tag{14}$$

for the arbitrary vector field $X \in \Gamma(TM)$. Replacing Y by ξ in (13) gives that $R(\xi,X)\xi = k[\eta(X)\xi - X] - \mu hX$, by substituting this equation into (14) we obtain

$$k[g(X,\xi)\alpha(\xi,\xi) - \alpha(X,\xi)] - \mu\alpha(hX,\xi) = 0 \tag{15}$$

for any $X \in \Gamma(TM)$. It follows from Dileo-Pastore [11] that an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution satisfies $h = 0$ and $k = -1$. Then from (15) it is easy to see

$$\alpha(X,\xi) = g(X,\xi)\alpha(\xi,\xi) \tag{16}$$

for any $X \in \Gamma(TM)$. Noticing that α is parallel, then, by differentiating (16) along the arbitrary vector field Y on M^{2n+1} we obtain

$$\alpha(\nabla_Y X,\xi) + \alpha(X,\nabla_Y\xi) = g(\nabla_Y X,\xi)\alpha(\xi,\xi) + g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi) \tag{17}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, replacing X by $\nabla_Y X$ in (16) yields that

$$\alpha(\nabla_Y X,\xi) = g(\nabla_Y X,\xi)\alpha(\xi,\xi) \tag{18}$$

for any $X, Y \in \Gamma(TM)$. Thus, it follows from (17) and (18) that

$$\alpha(X, \nabla_Y \xi) = g(X, \nabla_Y \xi)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\nabla_Y \xi, \xi). \tag{19}$$

Taking into account (4) and the fact that $h = 0$, then we get $\nabla_Y \xi = Y - \eta(Y)\xi$ for any $Y \in \Gamma(TM)$, substituting this equation into (19) gives that

$$\alpha(X, Y) = g(X, Y)\alpha(\xi, \xi) + \eta(Y)\alpha(X, \xi) - \eta(X)\eta(Y)\alpha(\xi, \xi) + 2\eta(X)\alpha(Y, \xi) - 2\eta(X)\eta(Y)\alpha(\xi, \xi). \tag{20}$$

Finally, using (16) in (20) yields the following relation

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y) \tag{21}$$

for any $X, Y \in \Gamma(TM)$. This completes the proof. □

Corollary 3.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold for which ξ belongs to the (k, μ)-nullity distribution. If M^{2n+1} is Ricci symmetric, that is, $\nabla S = 0$, then the Ricci curvature tensor is given by*

$$S(X, Y) = -2ng(X, Y) \tag{22}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Noticing that $h = 0$ in this context, then it follows from (7) that

$$R(X, \xi)\xi = \eta(X)\xi - X \tag{23}$$

for any vector field $X \in \Gamma(TM)$. Hence, taking the inner product with X on both sides of (23) gives that $R(X, \xi, X, \xi) = (\eta(X))^2 - g(X, X)$ for any $X \in \Gamma(TM)$, contracting X in the above equation gives that $S(\xi, \xi) = -2n$. By applying Theorem 3.1 we complete the proof. □

We remark that the above corollary was proved in [2] on Kenmotsu manifolds.

Theorem 3.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold for which ξ belongs to the (k, μ)-nullity distribution. Then there exists no non-zero second order skew symmetric parallel tensor on M^{2n+1}.*

Proof. Let α be a non-zero second order skew symmetric parallel tensor (that is, α is a parallel 2-form) on M^{2n+1}, then (11) holds. Substituting $Y = W = \xi$ into (11) and using (13) we obtain

$$\begin{aligned}
&\alpha(R(\xi, X)\xi, Z) + \alpha(\xi, R(\xi, X)Z) \\
&= \alpha(k\eta(X)\xi - kX - \mu hX, Z) + \alpha(\xi, k[g(X, Z)\xi - \eta(Z)X]) + \alpha(\xi, \mu[g(hX, Z)\xi - \eta(Z)hX]) \\
&= -\eta(X)\alpha(\xi, Z) + \alpha(X, Z) + \eta(Z)\alpha(\xi, X) - g(X, Z)\alpha(\xi, \xi) \\
&= 0
\end{aligned} \tag{24}$$

for any $X, Z \in \Gamma(TM)$. The second equality in (24) follows because we used the conclusion of Dileo-Pastore [11] that an almost Kenmotsu manifold with ξ belonging to the (k, μ)-nullity distribution satisfies $h = 0$ and $k = -1$. Moreover, in view of the skew symmetry of α we see that $\alpha(X, X) = 0$ for any $X \in \Gamma(TM)$, thus it follows from (24) that

$$\alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X) \tag{25}$$

for any $X, Z \in \Gamma(TM)$. We denote by A the dual $(1, 1)$-type tensor which is metrically equivalent to α, that is, $\alpha(X, Y) = g(AX, Y)$. Thus, (25) is equivalent to the following relation

$$AX = \eta(X)A\xi - g(A\xi, X)\xi. \tag{26}$$

Taking the covariant differentiation along the arbitrary vector field Y on (26) gives that

$$\nabla_Y AX = Y(\eta(X))A\xi + \eta(X)\nabla_Y A\xi - Y(g(A\xi, X))\xi - g(A\xi, X)\nabla_Y \xi \tag{27}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, replacing X by $\nabla_Y X$ in (26) implies that

$$A\nabla_Y X = \eta(\nabla_Y X)A\xi - g(A\xi, \nabla_Y X)\xi \tag{28}$$

for any $X, Y \in \Gamma(TM)$. Taking into account the assumption that the 2-form α is parallel and hence A is parallel, then it follows from (27) and (28) that

$$\begin{aligned} & g(\nabla_Y\xi, X)A\xi + \eta(X)\nabla_Y A\xi - g(\nabla_Y A\xi, X)\xi - g(A\xi, X)\nabla_Y\xi \\ &= g(X, Y)A\xi - 2g(X, \xi)g(Y, \xi)A\xi + g(X, \xi)AY - g(AY, X)\xi + 2g(Y, \xi)g(A\xi, X)\xi - g(A\xi, X)Y \\ &= 0 \end{aligned} \tag{29}$$

for any $X, Y \in \Gamma(TM)$. Substituting $X = \xi$ into (29) gives

$$AY + g(AY, \xi)\xi = g(Y, \xi)A\xi \tag{30}$$

for any $Y \in \Gamma(TM)$. Taking the inner product with ξ on both sides of (30) we obtain

$$g(AY, \xi) = \alpha(Y, \xi) = 0 \tag{31}$$

for any $Y \in \Gamma(TM)$. Since that α is parallel, then taking the covariant differentiation on (31) along the arbitrary vector field X and using $\nabla_Y\xi = Y - \eta(Y)\xi$ in the resulting equation we obtain the following relation

$$\alpha(Y, X) = 0 \tag{32}$$

for any $X, Y \in \Gamma(TM)$. This completes the proof. □

Remark 3.4. We observe from relation (7), Theorem 4.1 of [11] and Section 3 of [20] that on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ the following three conditions are equivalent:
(1) the tensor field h vanishes;
(2) the characteristic vector field ξ belongs to the (k, μ)-nullity distribution;
(3) the Reeb foliation is conformal.
Therefore, under one of the above conditions the conclusions of Theorem 3.1 and 3.3 still hold.

4. ξ Belongs to the $(k, \mu)'$-Nullity Distribution

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold for which ξ belongs to the $(k, \mu)'$-nullity distribution, that is

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y] \tag{33}$$

for any $X, Y \in \Gamma(TM)$. Clearly, it follows from (33) that

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X] \tag{34}$$

for any $X, Y \in \Gamma(TM)$.

It is easy to see that $h'X = \lambda X$ implies $h'\phi X = -\lambda\phi X$ for any $X \in \mathcal{D}$ and $\lambda \neq 0$, where $\mathcal{D}$ is the contact distribution defined by $\mathcal{D} = \ker(\eta) = \mathrm{Im}(\phi)$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the eigenvalue $\lambda \neq 0$ and $-\lambda$ of h', respectively. Before presenting our main theorems, we need the following result due to G. Dileo and A. M. Pastore [11].

Lemma 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and* $\mathrm{Spec}(h') = \{0, \lambda, -\lambda\}$, *with 0 as simple eigenvalue and*

$\lambda = \sqrt{-k-1}$. *The distributions* $[\xi] \oplus [\lambda]'$ *and* $[-\lambda]'$ *are integrable with totally geodesic leaves and integrable with totally umbilical leaves, respectively. Furthermore, the sectional curvatures are given as follows:*

(*a*) $K(X,\xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and* $K(X,\xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$;

(*b*) $K(X,Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$; $K(X,Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and* $K(X,Y) = -(k+2)$ *if* $X \in [\lambda]', Y \in [-\lambda]'$;

(*c*) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k-2n)$.

By applying the above result, we may present our main results as follows.

Theorem 4.2. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold such that* ξ *belongs to the* $(k,\mu)'$*-nullity distribution and* $h' \neq 0$. *If* M^{2n+1} *admits a second order symmetric parallel tensor, then either* M^{2n+1} *is locally isometric to the Riemannian product of an* $(n+1)$*-dimensional manifold of constant sectional curvature* -4 *and a flat* n*-dimensional manifold, or the second order parallel tensor is a constant multiple of the associated metric tensor* g *of* M^{2n+1}.

Proof. Replacing Y by ξ in (34) gives $R(\xi,X)\xi = k[\eta(X)\xi - X] - \mu h'X$ for any $X \in \Gamma(TM)$, by substituting this equation into (14) we obtain

$$k[g(X,\xi)\alpha(\xi,\xi) - \alpha(X,\xi)] - \mu\alpha(h'X,\xi) = 0$$

for any $X \in \Gamma(TM)$. Noticing that Dileo-Pastore [11] proved that an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ satisfies $\mu = -2$ and $h'^2X = (k+1)\phi^2X$ for any $X \in \Gamma(TM)$, hence we have $k < -1$ and obtain

$$k[g(X,\xi)\alpha(\xi,\xi) - \alpha(X,\xi)] + 2\alpha(h'X,\xi) = 0 \tag{35}$$

for any $X \in \Gamma(TM)$. Replacing X by $h'X$ in (35) gives

$$k\alpha(h'X,\xi) + 2(k+1)\alpha(X,\xi) - 2(k+1)\eta(X)\alpha(\xi,\xi) = 0 \tag{36}$$

for any $X \in \Gamma(TM)$. Substituting (36) into (35) implies that

$$(k+2)^2[\alpha(X,\xi) - g(X,\xi)\alpha(\xi,\xi)] = 0 \tag{37}$$

for any $X \in \Gamma(TM)$. Now we separate our discussion into two cases as following:

case 1: $k \neq -2$. It follows from (37) that

$$\alpha(X,\xi) = g(X,\xi)\alpha(\xi,\xi) \tag{38}$$

for any $X \in \Gamma(TM)$. Noticing that α is parallel, then by differentiating (38) along the arbitrary vector field Y on M^{2n+1} we obtain

$$\alpha(\nabla_Y X,\xi) + \alpha(X,\nabla_Y\xi) = g(\nabla_Y X,\xi)\alpha(\xi,\xi) + g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi) \tag{39}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, replacing X by $\nabla_Y X$ in (38) yields that

$$\alpha(\nabla_Y X,\xi) = g(\nabla_Y X,\xi)\alpha(\xi,\xi) \tag{40}$$

for any $X, Y \in \Gamma(TM)$. Thus, it follows from (39) and (40) that

$$\alpha(X,\nabla_Y\xi) = g(X,\nabla_Y\xi)\alpha(\xi,\xi) + 2g(X,\xi)\alpha(\nabla_Y\xi,\xi). \tag{41}$$

From (4) we see that $\nabla_Y\xi = Y - \eta(Y)\xi - \phi hY = Y - \eta(Y)\xi + h'Y$ for any $Y \in \Gamma(TM)$, substituting this equation into (41) gives that

$$\begin{aligned}\alpha(X,Y) =& [g(X,Y) - \eta(X)\eta(Y) + g(X,h'Y)]\alpha(\xi,\xi) + 2\eta(X)\alpha(Y,\xi)\\ &+ 2\eta(X)\alpha(h'Y,\xi) - \alpha(X,h'Y) + \eta(Y)\alpha(X,\xi) - 2\eta(X)\eta(Y)\alpha(\xi,\xi).\end{aligned} \tag{42}$$

Using (38) and the fact that $h'\xi = 0$ in (42) we have the following equation

$$\alpha(X,Y) + \alpha(X,h'Y) = g(X,Y)\alpha(\xi,\xi) + g(X,h'Y)\alpha(\xi,\xi) \tag{43}$$

for any $X,Y \in \Gamma(TM)$. Substituting Y by $h'Y$ in (43) and using (38) we have

$$\alpha(X,h'Y) - (k+1)\alpha(X,Y) = g(X,h'Y)\alpha(\xi,\xi) - (k+1)g(X,Y)\alpha(\xi,\xi) \tag{44}$$

for any $X,Y \in \Gamma(TM)$. Finally, subtracting (44) from (43) and noticing the assumption $k \neq -2$ we obtain

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y) \tag{45}$$

for any $X,Y \in \Gamma(TM)$, this means that α is a constant multiple of the associated metric tensor g of M^{2n+1}.

case 2: $k = -2$. Noticing that the relation $h'^2X = (k+1)\phi^2X$ for any $X \in \Gamma(TM)$ holds in this context, then we see that the nonzero eigenvalue of h' is either 1 or -1 with the same multiplication n. Without losing the generality we now choose $\lambda = 1$, in view of $\mu = -2$ and then it follows from Lemma 4.1 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Also from Lemma 4.1 we see that $K(X,Y) = -4$ for any $X,Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X,Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [11] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves given by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field of the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Since that $\lambda = 1$, then we know that two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we conclude that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof. □

Corollary 4.3. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold for which ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If M^{2n+1} is Ricci symmetric, that is, $\nabla S = 0$, then M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. We may apply Theorem 4.2 and Corollary 3.2 and obtain that either M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, or the Ricci curvature tensor is given by $S(X,Y) = -2ng(X,Y)$ for any vector fields X,Y. However, the later case can not occur. In fact, the later case implies that $S(\xi,\xi) = -2n$, comparing this relation with (6) we have $\mathrm{tr}h^2 = 0$. Noticing that $h^2 = (k+1)\phi^2$ holds in this case, then we obtain $h = 0$, this is a contradiction. Thus we complete the proof. □

Theorem 4.4. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold for which ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If M^{2n+1} admits a second order skew symmetric parallel tensor, i.e., a 2-from, then M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Let α be a non-zero second order skew symmetric parallel tensor (that is, α is a parallel 2-form) on M^{2n+1}, then relation (11) follows in this context. From [11] we know that under the assumption of Theorem 4.4 it is easy to get $\mu = -2$. Then substituting $Y = W = \xi$ into (11) and using (34) we obtain

$$\begin{aligned}
&\alpha(R(\xi,X)\xi,Z) + \alpha(\xi,R(\xi,X)Z)\\
&= \alpha(k\eta(X)\xi - kX - \mu h'X,Z) + \alpha(\xi,k[g(X,Z)\xi - \eta(Z)X]) + \alpha(\xi,\mu[g(h'X,Z)\xi - \eta(Z)h'X])\\
&= -k\alpha(X,Z) + 2\alpha(h'X,Z) + k\eta(X)\alpha(\xi,Z) - k\eta(Z)\alpha(\xi,X)\\
&+ [kg(X,Z) + \mu g(h'X,Z)]\alpha(\xi,\xi) + 2\eta(Z)\alpha(\xi,h'X)\\
&= 0
\end{aligned} \tag{46}$$

for any $X,Z \in \Gamma(TM)$. Also, in view of the skew symmetry of α we see that $\alpha(X,X) = 0$ for any $X \in \Gamma(TM)$, thus it follows from (46) that

$$-k\alpha(X,Z) + 2\alpha(h'X,Z) + k\eta(X)\alpha(\xi,Z) - k\eta(Z)\alpha(\xi,X) + 2\eta(Z)\alpha(\xi,h'X) = 0. \tag{47}$$

for any $X,Z \in \Gamma(TM)$. Replacing X by $h'X$ in (47) and using $h'^2X = (k+1)\phi^2X$ for any $X \in \Gamma(TM)$ we obtain the following relation

$$-k\alpha(h'X,Z) - 2(k+1)\alpha(X,Z) + 2(k+1)\eta(X)\alpha(\xi,Z) - k\eta(Z)\alpha(\xi,h'X) - 2(k+1)\eta(Z)\alpha(\xi,X) = 0. \tag{48}$$

Consequently, it follows from relations (47) and (48) that

$$(k+2)^2[\alpha(X,Z)+\eta(Z)\alpha(\xi,X)-\eta(X)\alpha(\xi,Z)]=0 \tag{49}$$

for any $X,Z\in\Gamma(TM)$. Similarly, now we separate our proof into two cases as following:
case 1: $k\neq -2$. It follows from (49) that

$$\alpha(X,Z)=\eta(X)\alpha(\xi,Z)-\eta(Z)\alpha(\xi,X) \tag{50}$$

for $X,Z\in\Gamma(TM)$. We denote by A the dual $(1,1)$-type tensor which is metrically equivalent to α, that is, $\alpha(X,Y)=g(AX,Y)$. Thus, (50) is equivalent to the following equation

$$AX=\eta(X)A\xi-g(A\xi,X)\xi \tag{51}$$

for any $X\in\Gamma(TM)$. If $A\xi=0$, then from (51) we know that $AX=0$ for any $X\in\Gamma(TM)$ and hence $\alpha=0$. Now we assume that $A\xi\neq 0$, then taking the inner product with $A\xi$ on both sides of (51) and using the skew symmetry $\alpha(X,X)=0$ for any $X\in\Gamma(TM)$, we obtain $g(X,A^2\xi)=-\eta(X)g(A\xi,A\xi)$ for any $X\in\Gamma(TM)$, which means that

$$A^2\xi=-\|A\xi\|^2\xi, \tag{52}$$

where $\|A\xi\|^2=g(A\xi,A\xi)$. Differentiating (52) covariantly along $X\in\Gamma(TM)$ and using (4) in the resulting equation, it follows that

$$\begin{aligned}\nabla_X A^2\xi = A^2\nabla_X\xi &= A^2X-\eta(X)A^2\xi+A^2h'X\\ &=2g(X+h'X,A^2\xi)\xi-\|A\xi\|^2(X+h'X)+3\eta(X)\|A\xi\|^2\xi\end{aligned} \tag{53}$$

for any $X\in\Gamma(TM)$. Using (52) in (53) we have that

$$A^2X+A^2h'X+\|A\xi\|^2(X+h'X)=0 \tag{54}$$

for any $X\in\Gamma(TM)$. Replacing X by $h'X$ in (54) and using $h'^2X=(k+1)\phi^2X$ gives that

$$A^2h'X-(k+1)A^2X+(k+1)\eta(X)A^2\xi+\|A\xi\|^2[h'X-(k+1)X+(k+1)\eta(X)\xi]=0 \tag{55}$$

for any $X\in\Gamma(TM)$. Substituting (52) into (55) we obtain

$$A^2h'X-(k+1)A^2X+\|A\xi\|^2[h'X-(k+1)X]=0,$$

then subtracting the above equation from (54) implies that

$$(k+2)[A^2X+\|A\xi\|^2X]=0 \tag{56}$$

for any $X\in\Gamma(TM)$. Noticing the hypothesis $k\neq -2$, then it follows from (56) that

$$A^2X+\|A\xi\|^2X=0 \tag{57}$$

for any $X\in\Gamma(TM)$. The following proof is similar to that of Theorem 3.2 of [18]. Now if $\|A\xi\|\neq 0$, then $J=\frac{1}{\|A\xi\|}A$ is an almost complex structure on U, where U is a non-empty open subset of M^{2n+1} such that $k\neq -2$. In fact, (J,g) is a Kähler structure on U. The fundamental second order skew-symmetric parallel tensor is $g(JX,Y)=\kappa g(AX,Y)=\kappa\alpha(X,Y)$ for any $X,Y\in\Gamma(TM)$, where $\kappa=\frac{1}{\|A\xi\|}$ is a nonzero constant. However, relation (50) means that $\alpha(X,Y)=\eta(X)\alpha(\xi,Y)-\eta(Y)\alpha(\xi,X)$ and thus α is degenerate, which is a contradiction. Therefore we have $\|A\xi\|=0$ and hence it follows from (50)-(52) that $\alpha=0$ on U. Since that α is parallel on U, then $\alpha=0$ on M^{2n+1}.

case 2: $k=-2$. The proof for this case follows from case 2 of Theorem 4.2. Thus, we complete the proof. □

References

[1] C.L. Bejan, M. Crasmareanu, Ricci solitons in manifolds with quasi-constant curvature, Publicationes Mathematicae Debrecen 78(1) (2011), 235–243.
[2] T.Q. Binh, L. Tamássy, U.C. De, M. Tarafdar, Some remarks on almost Kenmotsu manifolds, Mathematica Pannonica 13(1) (2002), 31–39.
[3] D.E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, Volume 203, Birkhäuser, 2010.
[4] D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel Journal of Mathematics 91 (1995), 189-214.
[5] C. Călin, M. Crasmareanu, From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds, Bulletin of the Malaysian Mathematical Sciences Society, 33(3) (2010), 361-368.
[6] M. Crasmareanu, Parallel tensors and Ricci solitons in $N(k)$-quasi Einstein manifolds, Indian Journal of Pure and Applied Mathematics 43(4) (2012), 359-369.
[7] L. Das, Second order parallel tensor on α-Sasakian manifold, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 23(1) (2007), 65-69.
[8] U.C. De, Second order parallel tensor on P-Sasakian manifolds, Publicationes Mathematicae Debrecen 49 (1996), 33-37.
[9] U.C. De, A.K. Mondal, On 3-dimensional normal almost contact metric manifolds satisfying certain curvatures conditions, Communications of the Korean Mathematical Society 24(2) (2009), 265-275.
[10] G. Dileo, A.M. Pastore, Almost Kenmotsu manifolds and local symmetry, Bulletin of the Belgian Mathematical Society-Simon Stevin 14 (2007), 343-354.
[11] G. Dileo, A.M. Pastore, Almost Kenmotsu manifolds and nullity distributions, Journal of Geometry 93 (2009), 46-61.
[12] L.P. Eisenhart, Symmetric tensors of the second order whose first covariant derivatives are zero, Transactions of the American Mathematical Society 25(2) (1923), 297-306.
[13] A. Gray, Spaces of contancy of curvature operators, Proceedings of the American Mathematical Society 17 (1966), 897-902.
[14] D. Janssens, L. Vanhecke, Almost contact structures and curvature tensors, Kodai Mathematical Journal 4 (1981), 1-27.
[15] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tôhoku Mathematical Journal 24 (1972), 93-103.
[16] H. Levy, Symmetric tensors of the second order whose covariant derivatives vanish, Annals of Mathematics 27(2) (1925), 91-98.
[17] Z. Li, Second order parallel tensors on P-Sasakian manifolds with a coefficient k, Soochow Journal of Mathematics 23(1) (1997), 97-102.
[18] A.K. Mondal, U.C. De, C. Özgür, Second order parallel tensors on (k,μ)-contact metric manifolds, Analele Ştiinţifice ale Universităţii Ovidius Constanţa-Seria Matematică 18(1) (2010), 229-238.
[19] A.M. Pastore, V. Saltarelli, Generalized nullity distributions on almost Kenmotsu manifolds, International Electronic Journal of Geometry 4(2) (2011), 168-183.
[20] A.M. Pastore, V. Saltarelli, Almost Kenmotsu manifolds with conformal Reeb foliation, Bulletin of the Belgian Mathematical Society-Simon Stevin 21 (2012), 343-354.
[21] R. Sharma, Second order parallel tensor in real and complex space forms, International Journal of Mathematics and Mathematical Sciences 12 (1989), 787-790.
[22] R. Sharma, Second order parallel tensor on contact manifolds, Algebras, Groups and Geometries 7 (1990), 787-790.
[23] R. Sharma, Second order parallel tensor on contact manifolds II, C. R. Math. Rep. Acad. Sci. Canada 13(6) (1991), 259-264.
[24] R. Sharma, On the curvature of contact metric manifolds, Journal of Geometry 53 (1995), 179–190.
[25] S. Tanno, Some differential equations on Riemannian manifolds, Journal of the Mathematical Society of Japan 30 (1978), 509–531.
[26] Y.N. Wang, X.M. Liu, Almost Kenmotsu pseudo-metric manifolds, to appear in Analele Ştiinţifice ale Universităţii Al. I. Cuza din Iaşi-Serie Nouă-Matematică
[27] Y.N. Wang, X.M. Liu, On almost Kenmotsu manifolds satisfying some nullity distributions, submitted.