



## Measure of Noncompactness for Compact Matrix Operators on some BK Spaces

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**Abstract.** In this paper, we characterize the matrix classes  $(\ell_1, \ell_p^A)$  ( $1 \leq p < \infty$ ). We also obtain estimates for the norms of the bounded linear operators  $L_A$  defined by these matrix transformations and find conditions to obtain the corresponding subclasses of compact matrix operators by using the Hausdorff measure of noncompactness.

### 1. Preliminaries

We shall write  $\omega$  for the set of all complex sequences  $x = (x_k)_{k=0}^\infty$ . Let  $\phi, \ell_\infty, c$  and  $c_0$  denote the sets of all finite, bounded, convergent and null sequences, respectively. We write  $\ell_p = \{x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty\}$  for  $1 \leq p < \infty$ . By  $e$  and  $e^{(n)}$  ( $n \in \mathbb{N}$ ), we denote the sequences such that  $e_k = 1$  for  $k = 0, 1, \dots$ , and  $e_n^{(n)} = 1$  and  $e_k^{(n)} = 0$  ( $k \neq n$ ). For any sequence  $x = (x_k)_{k=0}^\infty$  let  $x^{[n]} = \sum_{k=0}^n x_k e^{(k)}$  be its  $n$ -section.

A sequence  $(b^{(n)})_{n=0}^\infty$  in a linear metric space  $X$  is called *Schauder basis* if for every  $x \in X$ , there is a unique sequence  $(\lambda_n)_{n=0}^\infty$  of scalars such that  $x = \sum_{n=0}^\infty \lambda_n b^{(n)}$ . A sequence space  $X$  with a linear topology is called a *K-space* if each of the maps  $p_i : X \rightarrow \mathbb{C}$  defined by  $p_i(x) = x_i$  is continuous for all  $i \in \mathbb{N}$ . A *K-space* is called an *FK-space* if  $X$  is a complete linear metric space; a *BK-space* is a normed *FK-space*. An *FK-space*  $X \supset \phi$  is said to have *AK* if every sequence  $x = (x_k)_{k=0}^\infty \in X$  has a unique representation  $x = \sum_{k=0}^\infty x_k e^{(k)}$ , that is,  $x^{[n]} \rightarrow x$  as  $n \rightarrow \infty$  (cf. [20]).

The classical sequence spaces  $c_0, c$  and  $\ell_p$  ( $1 \leq p < \infty$ ) all have Schauder bases but  $\ell_\infty$  has no Schauder basis; the spaces  $c_0$  and  $\ell_p$  ( $1 \leq p < \infty$ ) have *AK*.

Let  $(X, \|\cdot\|)$  be a normed space. Then the unit sphere and closed unit ball in  $X$  are denoted by  $S_X := \{x \in X : \|x\| = 1\}$  and  $\bar{B}_X := \{x \in X : \|x\| \leq 1\}$ . If  $X$  and  $Y$  are normed spaces then we write, as usual,  $\mathcal{B}(X, Y)$  for the space of all bounded linear operators  $L : X \rightarrow Y$  normed by  $\|L\| = \sup\{\|L(x)\| : x \in S_X\}$ ; if  $Y$  is a Banach space, so is  $\mathcal{B}(X, Y)$ .

Throughout this paper, the matrices are infinite matrices of complex numbers. If  $A$  is an infinite matrix with complex entries  $a_{nk}$  ( $n, k \in \mathbb{N}$ ), then we write  $A = (a_{nk})$  instead of  $A = (a_{nk})_{n,k=0}^\infty$ . Also, we write  $A_n$  for

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the sequence in the  $n^{\text{th}}$  row of  $A$ , that is,  $A_n = (a_{nk})_{k=0}^{\infty}$  for every  $n \in \mathbb{N}$ . In addition, if  $x = (x_k) \in \omega$ , then we define the  $A$ -transform of  $x$  as the sequence  $Ax = (A_n x)_{n=0}^{\infty}$ , where  $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$  ( $n = 0, 1, \dots$ ) provided the series on the right converges for each  $n$ .

An infinite matrix  $T = (t_{nk})$  is said to be a *triangle* if  $t_{nk} = 0$  for  $k > n$  and  $t_{nn} \neq 0$  ( $n = 0, 1, \dots$ ).

Let  $X$  and  $Y$  be subsets of  $\omega$  and  $A = (a_{nk})$  an infinite matrix. Then the set  $X_A = \{x \in \omega : Ax \in X\}$  is called the *matrix domain of  $A$  in  $X$* . We say that  $A$  defines a matrix mapping from  $X$  into  $Y$ , and we denote this by writing  $A : X \rightarrow Y$ , if  $Ax$  exists and is in  $Y$  for all  $x \in X$ . By  $(X, Y)$ , we denote the class of all infinite matrices that map  $X$  into  $Y$ . Thus  $A \in (X, Y)$  if and only if  $X \in Y_A$ , that is,  $A_n \in X^\beta$  for all  $n \in \mathbb{N}$  and  $Ax \in Y$  for all  $x \in X$ .

The following results are well known and give some relations between the classes  $(X, Y)$  and  $\mathcal{B}(X, Y)$ .

**Lemma 1.1.** *Let  $X \supset \phi$  and  $Y$  be BK-spaces.*

(a) *Then we have  $(X, Y) \subset \mathcal{B}(X, Y)$ , that is, every matrix  $A \in (X, Y)$  defines an operator  $L_A \in \mathcal{B}(X, Y)$  by  $L_A(x) = Ax$  for all  $x \in X$  ([20, Theorem 4.2.8].*

(b) *If  $X$  has AK, then  $\mathcal{B}(X, Y) \subset (X, Y)$ , that is, for every operator  $L \in \mathcal{B}(X, Y)$  there exists a matrix  $A \in (X, Y)$  such that  $L(x) = Ax$  for all  $x \in X$  ([8, Theorem 1.9]).*

In case of Lemma 1.1 (b), we say that  $L \in \mathcal{B}(X, Y)$  is represented by a matrix  $A \in (X, Y)$ .

## 2. $\lambda$ -Sequence Spaces

We consider some  $\lambda$ -sequence spaces which are the matrix domains of the matrices of weighted means matrix in  $\ell_p$  for  $1 < p < \infty$ .

Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to naught.

Let  $(r_k)_{k=0}^{\infty}$  be a sequence of nonnegative real numbers with  $r_0 > 0$  and  $R_n = \sum_{k=0}^n r_k$  for  $n = 0, 1, \dots$ . Then the triangle  $\bar{N}_r = (a_{nk})$  of weighted means is given by  $a_{nk} = r_k/R_n$  ( $0 \leq k \leq n; n = 0, 1, \dots$ ). If we write  $\lambda_n = R_n$  for  $n = 0, 1, \dots$ ,  $r_k = \Delta R_k = R_k - R_{k-1} = \Delta \lambda_k$  then  $(\lambda_n)$  is a nondecreasing sequence of positive reals and, defining the triangle  $\Lambda = (\lambda_{nk})$  by  $\lambda_{nk} = (\lambda_k - \lambda_{k-1})/\lambda_n$  ( $0 \leq k \leq n; n = 0, 1, \dots$ ), we obtain  $\bar{N}_r = \Lambda$ .

Conversely, let  $(\lambda_n)$  be a nondecreasing sequence of positive reals and the matrix  $\Lambda = (\lambda_{nk})$  be defined as above. If we write  $r_k = \Delta \lambda_k$ , then  $r_0 > 0$ ,  $r_k \geq 0$  for all  $k \geq 1$ ,  $R_n = \sum_{k=0}^n r_k = \lambda_n > 0$  and  $\Lambda = \bar{N}_r$ .

So, let  $\lambda = (\lambda_k)_{k=0}^{\infty}$  be a nondecreasing sequence of positive reals numbers. We say that a sequence  $x = (x_k) \in \omega$  is  $\lambda$ -convergent to the number  $\xi \in \mathbb{C}$ , called the  $\lambda$ -limit of  $x$ , if  $\Lambda_n x \rightarrow \xi$  as  $n \rightarrow \infty$ , where

$$\Lambda_n x = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \quad (n \in \mathbb{N}). \tag{1}$$

In particular, we say that  $x$  is a  $\lambda$ -null sequence if  $\Lambda_n x \rightarrow 0$  as  $n \rightarrow \infty$ . Furthermore, we say that  $x$  is  $\lambda$ -bounded if  $\sup_n |\Lambda_n x| < \infty$ . We also say that the associated series  $\sum_{k=0}^{\infty} x_k$  is  $p$ -absolutely convergent of type  $\lambda$  if  $\sum_{n=0}^{\infty} |\Lambda_n x|^p < \infty$ , where  $0 < p < \infty$ .

Recently, the sequence spaces  $\ell_{\infty}^{\lambda}$ ,  $c^{\lambda}$ ,  $c_0^{\lambda}$  and  $\ell_p^{\lambda}$  have been defined and studied by Mursaleen and Noman (cf. [16]) which are the sets of all  $\lambda$ -bounded,  $\lambda$ -convergent,  $\lambda$ -null sequences and  $\lambda(p)$ -absolutely convergent series, respectively, that is, the matrix domains of the triangle  $\Lambda$  in the spaces  $\ell_{\infty}$ ,  $c$  and  $c_0$ , respectively,

$$\ell_{\infty}^{\lambda} = (\ell_{\infty})_{\Lambda}, \quad c^{\lambda} = c_{\Lambda}, \quad c_0^{\lambda} = (c_0)_{\Lambda} \quad \text{and} \quad \ell_p^{\lambda} = (\ell_p)_{\Lambda}.$$

The following result is known.

**Lemma 2.1.** ([16]) *The spaces  $\ell_{\infty}^{\lambda}$ ,  $c^{\lambda}$  and  $c_0^{\lambda}$  are BK spaces with the same norm given by  $\|x\|_{\ell_{\infty}^{\lambda}} = \|\Lambda x\|_{\infty}$ , that is,*

$$\|x\|_{\ell_{\infty}^{\lambda}} = \sup_n |\Lambda_n x|.$$

The space  $\ell_p^\lambda$  ( $1 \leq p \leq \infty$ ) is a BK space with the norm  $\|x\|_{\ell_p^\lambda} = \|\Lambda x\|_p$ , that is,

$$\|x\|_{\ell_p^\lambda} = \left( \sum_n |\Lambda_n x|^p \right)^{1/p} \quad (1 \leq p < \infty).$$

### 3. The Hausdorff Measure of Noncompactness

The Hausdorff measure of noncompactness can be most effectively used to characterize compact operators between Banach spaces.

Here we recall some fundamental definitions and results, and give a short outline of how the Hausdorff measure of noncompactness can be applied in the characterization of compact matrix operators between BK spaces when the final space has a Schauder basis.

There are several measures of noncompactness in use. Here we only mention two of them. We refer the reader to the the monographs [2, 3] for further studies.

The first measure of noncompactness, the function  $\alpha$ , was defined and studied by Kuratowski [9] in 1930. Darbo [5] used this measure to generalize both the classical Schauder fixed point principle and (a special variant of) Banach’s contraction mapping principle for so called condensing operators. The Hausdorff or ball measure of noncompactness  $\chi$  was introduced by Goldenštejn, Gohberg and Markus [6] in 1957, and later studied by Goldenštejn and Markus [7].

Let  $X$  and  $Y$  be infinite dimensional Banach spaces. We recall that a linear operator  $L$  from  $X$  into  $Y$  is called *compact* if its domain is all of  $X$  and, for every bounded sequence  $(x_n)$  in  $X$ , the sequence  $(Lx_n)$  has a convergent subsequence. We denote the class of all compact operators in  $\mathcal{B}(X, Y)$  by  $C(X, Y)$ .

Let  $(X, d)$  be a metric space,  $x_0 \in X$  and  $r > 0$ . Then we write, as usual,  $B(x, r) = \{x \in X : d(x, x_0) < r\}$  for the open ball of radius  $r$  and center  $x_0$ . Let  $\mathcal{M}_X$  denote the class of all bounded subsets of  $X$ . If  $Q \in \mathcal{M}_X$ , then the *Hausdorff measure of noncompactness of the set*  $Q$ , denoted by  $\chi(Q)$ , is defined by

$$\chi(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{k=1}^n B(x_k, r_k), x_k \in X, r_k < \epsilon \ (k = 1, 2, \dots), n \in \mathbb{N} \right\}.$$

The function  $\chi : \mathcal{M}_X \rightarrow [0, \infty)$  is called the *Hausdorff measure of noncompactness*.

The basic properties of the Hausdorff measure of noncompactness can be found in [1–3, 12, 13].

Now we recall the definition of the Hausdorff measure of noncompactness operators between Banach spaces. Let  $X$  and  $Y$  be Banach spaces and  $\chi_1$  and  $\chi_2$  be the Hausdorff measures of noncompactness on  $X$  and  $Y$ , respectively. An operator  $L : X \rightarrow Y$  is said to be  $(\chi_1, \chi_2)$ -*bounded* if  $L(Q) \in \mathcal{M}_Y$  for all  $Q \in \mathcal{M}_X$  and there exist a constant  $C \geq 0$  such that  $\chi_2(L(Q)) \leq C\chi_1(Q)$  for all  $Q \in \mathcal{M}_X$ . If an operator  $L$  is  $(\chi_1, \chi_2)$ -bounded then the number

$$\|L\|_{(\chi_1, \chi_2)} = \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}_X\}$$

is called the  $(\chi_1, \chi_2)$ -*measure of noncompactness of*  $L$ . If  $\chi_1 = \chi_2 = \chi$ , then we write  $\|L\|_{(\chi_1, \chi_2)} = \|L\|_\chi$ .

Now we outline the applications of the Hausdorff measure of noncompactness to the characterization of compact operators between Banach spaces. Let  $X$  and  $Y$  be Banach spaces and  $L \in \mathcal{B}(X, Y)$ . Then the Hausdorff measure of noncompactness of  $L$  is given by ([13, Theorem 2.25])

$$\|L\|_\chi = \chi(L(S_X)) \tag{2}$$

and  $L$  is compact if and only if ([13, Corollary 2.26 (2.58)])

$$\|L\|_\chi = 0. \tag{3}$$

The identities in (2) and (3) reduce the characterization of compact operators  $L \in \mathcal{B}(X, Y)$  to the determination of the Hausdorff measure of noncompactness  $\chi(Q)$  of bounded sets  $Q$  in a Banach space  $X$ . If  $X$  has a Schauder basis, then there exist estimates or even identities for  $\chi(Q)$ .

**Theorem 3.1 (Goldenštejn, Gohberg, Markus).** ([6] or [13, Theorem 2.23])

Let  $X$  be a Banach space with a Schauder basis  $(b_k)_{k=0}^\infty$ ,  $Q \in \mathcal{M}_X$ ,  $\mathcal{P}_n : X \rightarrow X$  be the projectors onto the linear span of  $\{b_0, b_1, \dots, b_n\}$  and  $\mathcal{R}_n = I - \mathcal{P}_n$  for  $n = 0, 1, \dots$ , where  $I$  denotes the identity map on  $X$ . Then we have

$$\frac{1}{a} \cdot \limsup_{n \rightarrow \infty} \left( \sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \limsup_{n \rightarrow \infty} \left( \sup_{x \in Q} \|\mathcal{R}_n(x)\| \right),$$

where  $a = \limsup_{n \rightarrow \infty} \|\mathcal{R}_n\|$ .

In particular, the following result shows how to compute the Hausdorff measure of noncompactness in the spaces  $c_0$  and  $\ell_p$  ( $1 \leq p < \infty$ ) which are BK-spaces with AK.

**Theorem 3.2.** ([13, Theorem 2.15]) Let  $Q$  be a bounded subset of the normed space  $X$ , where  $X$  is  $\ell_p$  for  $1 \leq p < \infty$  or  $c_0$ . If  $\mathcal{P}_n : X \rightarrow X$  is the operator defined by  $\mathcal{P}_n(x) = x^{[n]}$  for all  $x = (x_k)_{k=0}^\infty \in X$  and  $\mathcal{R}_n = I - \mathcal{P}_n$  for  $n = 0, 1, \dots$ , then we have

$$\chi(Q) = \lim_{n \rightarrow \infty} \left( \sup_{x \in Q} \|\mathcal{R}_n(x)\| \right). \tag{4}$$

Since matrix mappings between BK spaces define bounded linear operators between these spaces which are Banach spaces, it is natural to use the above results and the Hausdorff measure of noncompactness to obtain necessary and sufficient conditions for matrix operators between BK spaces with a Schauder basis or AK to be compact operators. This technique has recently been used by several authors in many research papers (see for instance [4, 10, 15, 18]). In this paper, we characterize the matrix classes  $(\ell_1, \ell_p^\lambda)$  ( $1 \leq p < \infty$ ). We also obtain an identity for the norms of the bounded linear operators  $L_A$  defined by these matrix transformations and find conditions to obtain the corresponding subclasses of compact matrix operators by using the Hausdorff measure of noncompactness.

#### 4. Main Results

Here we characterize the classes  $\mathcal{B}(\ell_1, \ell_p^\lambda)$  for  $(1 \leq p < \infty)$  and compute the norm of operators in  $\mathcal{B}(\ell_1, \ell_p^\lambda)$ . We also apply the results of the previous section to determine the Hausdorff measure of noncompactness of operators in  $\mathcal{B}(\ell_1, \ell_p^\lambda)$  and to characterize the classes  $\mathcal{C}(\ell_1, \ell_p)$  for  $1 \leq p < \infty$ .

The following result is useful.

**Lemma 4.1.** ([13, Theorem 3.8]) Let  $T$  be a triangle and  $X$  and  $Y$  be arbitrary subsets of  $\omega$ .

- (a) Then we have  $A \in (X, Y_T)$  if and only if  $C = T \cdot A \in (X, Y)$ , where  $C$  denotes the matrix product of  $T$  and  $A$ .
- (b) If  $X$  and  $Y$  are B spaces and  $A \in (X, Y_T)$  then

$$\|L_A\| = \|L_C\|. \tag{5}$$

First we establish the characterizations of the classes  $\mathcal{B}(\ell_1, \ell_p^\lambda)$  for  $(1 \leq p < \infty)$  and an identity for the operator norm.

**Theorem 4.2.** Let  $1 \leq p < \infty$ .

- (a) We have  $L \in \mathcal{B}(\ell_1, \ell_p^\lambda)$  if and only if there exists an infinite matrix  $A \in (\ell_1, \ell_p^\lambda)$  such that

$$\|A\| = \sup_k \left( \sum_{n=0}^\infty \left| \frac{1}{\lambda_n} \sum_{j=0}^n (\lambda_j - \lambda_{j-1}) a_{jk} \right|^p \right)^{1/p} < \infty \tag{6}$$

and

$$L(x) = Ax \text{ for all } x \in \ell_1. \tag{7}$$

- (b) If  $L \in \mathcal{B}(\ell_1, \ell_p^\lambda)$  then

$$\|L\| = \|A\|. \tag{8}$$

*Proof.* Since  $\ell_1$  is a BK space with AK it follows from Lemma 1.1 that  $L \in \mathcal{B}(\ell_1, \ell_p^\lambda)$  for  $1 \leq p < \infty$  if and only if there exists an infinite matrix  $A \in (\ell_1, \ell_p^\lambda)$  such that (7) holds. Also we have by Lemma 4.1 (a) that  $A \in (\ell_1, \ell_p^\lambda)$  if and only if  $C = \Lambda \cdot A \in (\ell_1, \ell_p)$ , where the entries of the triangle  $C$  are given by

$$c_{nk} = \frac{1}{\lambda_n} \sum_{j=0}^n (\lambda_j - \lambda_{j-1}) a_{jk} \text{ for } 0 \leq k \leq n \text{ and } n = 0, 1, \dots$$

Furthermore, we have by [20, Example 8.4.1D] that  $C \in (\ell_1, \ell_p)$  if and only if

$$\|C\| = \sup_k \left( \sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p} < \infty.$$

This completes the proof of Part (a).

(b) If  $L \in \mathcal{B}(\ell_1, \ell_p^\lambda)$  then it follows from (5) that  $\|L\| = \|L_C\|$ , where  $L_C \in \mathcal{B}(\ell_1, \ell_p)$  is given by  $L_C(x) = Cx$  for all  $x \in \ell_1$ . It follows by Minkowski’s inequality that

$$\begin{aligned} \|L_C(x)\|_p &= \left( \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} c_{nk} x_k \right|^p \right)^{1/p} \leq \sum_{k=0}^{\infty} |x_k| \left( \sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p} \\ &\leq \|C\| \cdot \|x\| = \|A\| \cdot \|x\|, \end{aligned}$$

and so

$$\|L\| \leq \|A\|. \tag{9}$$

We also obtain for  $e^{(k)} \in S_{\ell_1}$  ( $k \in \mathbb{N}$ )

$$\|L_C(e^{(k)})\| = \left( \sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p},$$

and so  $\|L\| \geq \|A\|$ . This and (9) yield (8).  $\square$

Now we are going to establish a formula for the Hausdorff measure of noncompactness of operators in  $\mathcal{B}(\ell_1, \ell_p^\lambda)$ . We need the following result.

**Lemma 4.3.** ([14, Theorem 4.2]) *Let  $X$  be a linear metric space with a translation invariant metric,  $T$  be a triangle and  $\chi$ , and  $\chi_T$  denote the Hausdorff measures of noncompactness on  $\mathcal{M}_X$  and  $\mathcal{M}_{X_T}$ , respectively. Then  $\chi_T(Q) = \chi(TQ)$  for all  $Q \in \mathcal{M}_{X_T}$ .*

**Theorem 4.4.** *Let  $L \in \mathcal{B}(\ell_1, \ell_p^\lambda)$  ( $1 \leq p < \infty$ ) and  $A$  denote the matrix which represents  $L$ . Then we have*

$$\|L\|_{\chi_{\ell_p^\lambda}} = \lim_{m \rightarrow \infty} \left( \sup_k \sum_{n=m}^{\infty} \left| \frac{1}{\lambda_n} \sum_{j=0}^n (\lambda_j - \lambda_{j-1}) a_{jk} \right|^p \right)^{1/p}. \tag{10}$$

*Proof.* We write  $S = S_{\ell_1}$ , for short, and  $C^{[m]}$  ( $m \in \mathbb{N}$ ) for the matrix with the rows  $C_n^{[m]} = 0$  for  $0 \leq n \leq m$  and  $C_n^{[m]} = C_n$  for  $n \geq m + 1$ . It follows from (2), Lemma 4.3, (4), (8) and (6)

$$\begin{aligned} \|L\|_{\chi_{\ell_p^\lambda}} &= \chi_{\ell_p^\lambda}(L(S)) = \chi_{\ell_p}(L_C(S)) = \lim_{m \rightarrow \infty} \left( \sup_{x \in S} \|\mathcal{R}_m(Cx)\|_p \right) \\ &= \lim_{m \rightarrow \infty} \left( \sup_{x \in S} \|C^{[m]}x\|_p \right) = \lim_{m \rightarrow \infty} \|C^{[m]}\| \end{aligned}$$

$$= \lim_{m \rightarrow \infty} \left( \sup_k \sum_{n=m+1}^{\infty} \left| \frac{1}{\lambda_n} \sum_{j=0}^n (\lambda_j - \lambda_{j-1}) a_{jk} \right|^p \right)^{1/p}.$$

□

Finally, the characterization of  $C(\ell_1, \ell_p^\lambda)$  is an immediate consequence of Theorem 4.4 and (3).

**Corollary 4.5.** *Let  $L \in \mathcal{B}(\ell_1, \ell_p^\lambda)$  ( $1 \leq p < \infty$ ) and  $A$  denote the matrix which represents  $L$ . Then  $L$  is compact if and only if*

$$\lim_{m \rightarrow \infty} \left( \sup_k \sum_{n=m}^{\infty} \left| \frac{1}{\lambda_n} \sum_{j=0}^n (\lambda_j - \lambda_{j-1}) a_{jk} \right|^p \right) = 0.$$

**Remark 4.6.** *The characterizations of the classes  $C(\ell_1, \ell_p^\lambda)$  ( $1 \leq p < \infty$ ) could be obtained from the characterizations of the classes  $C(\ell_1, \ell_p)$  in ([19], p. 85). The Hausdorff measure of noncompactness is, however, not used in [19].*

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