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Stability of Delay Parabolic Difference Equations

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Abstract. In the present paper, the stability of difference schemes for the approximate solution of the initial value problem for delay differential equations with unbounded operators acting on delay terms in an arbitrary Banach space is studied. Theorems on stability of these difference schemes in fractional spaces are established. In practice, the stability estimates in Hölder norms for the solutions of difference schemes for the approximate solutions of the mixed problems for delay parabolic equations are obtained.

1. Introduction

Delay differential equations have been studied extensively in a series of works (see, for example, [1]-[6] and the references therein) and developed over the last three decades. In the literature mostly the sufficient condition

$$|b(t)| \leq \ Re\ a(t), t \geq 0 \tag{1.1}$$

was considered for the stability of the following test delay differential equation

$$\frac{dv(t)}{dt} + a(t)v(t) = b(t)v(t-\omega), t > 0 \tag{1.2}$$

with the initial condition

$$v(t) = g(t)(-\omega \leq t \leq 0). \tag{1.3}$$

It is known that delay differential equations can be solved by applying standard numerical methods for ordinary differential equations without the presence of delay. However, it is difficult to generalize any numerical method to obtain high order of accuracy algorithms, because high order methods may not lead to efficient results. It is well-known that even if $a(t), b(t)$ and $g(t)$ are arbitrary differentiable functions, $v(t)$ may not possess the higher order derivatives for a sufficiently large t. Therefore, we may have a non-smooth solution of delay differential equations for given smooth data. This is the main difficulty in the study of convergence of numerical methods for delay differential equations.

Delay partial differential equations arise from various applications, like in control theory, biology, medicine, climate models, and many others (see, for example, [7] and the references therein).

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The theory of delay partial differential equations has received less attention than of delay ordinary differential equations. A situation which occurs in delay partial differential equations when the delay term is an operator of lower order with respect to other operator terms is widely investigated (see, for example, [7]-[9] and the references therein). In the case where the delay term is an operator of the same order with respect to other operator terms is studied mainly if H is a Hilbert space (see, for example, [10] and the references therein). In fact there are very few papers which allow E to be a general Banach space (see, [11]-[14]) and in these works, authors look only for partial differential equations under regular data. Moreover, approximate solutions of the delay parabolic equations in the case where the delay term is a simple operator of the same order with respect to other operator terms were studied recently in papers [15]- [19].

It is known that various initial-boundary value problems for linear evolutionary delay partial differential equations can be reduced to an initial value problem of the form

$$\begin{cases} \frac{dv(t)}{dt} + Av(t) = B(t)v(t-\omega) + f(t), t \geq 0, \\ v(t) = g(t)(-\omega \leq t \leq 0) \end{cases} \tag{1.4}$$

in an arbitrary Banach space E with the unbounded linear operators A and $B(t)$ in E with dense domains $D(A) \subseteq D(B(t))$. Let A be a strongly positive operator, i.e. $-A$ is the generator of the analytic semigroup $\exp\{-tA\}(t \geq 0)$ of the linear bounded operators with exponentially decreasing norm when $t \to \infty$. That means the following estimates hold:

$$\|\exp\{-tA\}\|_{E\mapsto E} \leq Me^{-\delta t}, \|tA\exp\{-tA\}\|_{E\mapsto E} \leq M, t > 0 \tag{1.5}$$

for some $M > 1$, $\delta > 0$. Let $B(t)$ be closed operators.

The strongly positive operator A defines the fractional spaces $E_\alpha = E_\alpha(A, E)$ $(0 < \alpha < 1)$ consisting of all $u \in E$ for which the following norms are finite:

$$\|u\|_{E_\alpha} = \sup_{\lambda > 0} \|\lambda^{1-\alpha} A \exp\{-\lambda A\} u\|_E.$$

As noted in [19], it is important to study the stability of solutions of the initial value problem (1.4) for delay differential equations and of difference schemes for approximate solutions of problem (1.4) under the assumption that

$$\|B(t)A^{-1}\|_{E\mapsto E} \leq 1 \tag{1.6}$$

holds for every $t \geq 0$. This assumption for delay differential equation (1.2) follows from assumption (1.1) in the case when $E = R^1$. Unfortunately, we have not been able to obtain the stability estimate for the solution of problem (1.4) in the arbitrary Banach space E. Nevertheless, in [20], the analogue of stability estimate for the solution of problem (1.4) was established, when the space E is replaced by the fractional spaces $E_\alpha(0 < \alpha < 1)$ which were defined above under the condition

$$\|B(t)A^{-1}\|_{E\mapsto E} \leq \frac{1-\alpha}{M2^{2-\alpha}} \tag{1.7}$$

for every $t \geq 0$, where M is the constant from (1.5). However, the condition (1.7) is stronger than (1.6) and $E \neq E_\alpha$. Finally, in papers [32]-[35], theorems on well-posedness of the initial value problem for the delay parabolic equation

$$\begin{cases} \varepsilon\frac{dv(t)}{dt} + Av(t) = B(t)v(t-\omega) + f(t), t \geq 0, \\ v(t) = g(t)(-\omega \leq t \leq 0) \end{cases} \tag{1.8}$$

in an arbitrary Banach space E with the small positive parameter ε in the high derivative and with the unbounded linear operators A and $B(t)$ in E with dense domains $D(A) \subseteq D(B(t))$ were established.

Applying the first and second order of accuracy implicit difference schemes for differential equations without the presence of delay, the first and second order of accuracy implicit difference schemes

$$\begin{cases} \frac{1}{\tau}(u_k - u_{k-1}) + Au_k = B_k u_{k-N} + \varphi_k, \varphi_k = f(t_k),\ B_k = B(t_k), t_k = k\tau, 1 \leq k, \\ N\tau = \omega, u_k = g(t_k), t_k = k\tau, -N \leq k \leq 0, \end{cases} \tag{1.9}$$

$$\begin{cases} \frac{1}{\tau}(u_k - u_{k-1}) + ASu_k = SB_k g(t_{k-N} - \frac{\tau}{2}), 1 \le k \le N, \\ \frac{1}{\tau}(u_k - u_{k-1}) + ASu_k = \frac{1}{2}SB_k(u_{k-N} + u_{k-N-1}) + \varphi_k, \varphi_k = Sf(t_k - \frac{\tau}{2}), \\ B_k = B(t_k - \frac{\tau}{2}), t_k = k\tau, N+1 \le k \end{cases} \tag{1.10}$$

are presented for approximate solutions of the initial value problem (1.4). Here, we will put $S = I + \frac{1}{2}\tau A$.

The main aim of the present paper is to study the stability of difference schemes (1.9) and (1.10). We establish the stability estimates in fractional spaces $E_\alpha(0 < \alpha < 1)$ under an assumption stronger than (1.6). In practice, the stability estimates in Hölder norms for the solutions of difference schemes for the approximate solutions of the mixed problem of delay parabolic equations are obtained.

The paper is organized as follows. In Section 2, main theorems on stability of difference schemes (1.9) and (1.10) are established. In Section 3, the stability estimates in Hölder norms for the solutions of difference schemes for the approximate solutions of delay parabolic equations are obtained. Finally, Section 4 is conclusion.

2. Theorems on Stability of Difference Schemes (1.9) and (1.10)

First, we consider the problem (1.4) when A^{-1} and $B(t)$ commute, i.e.

$$A^{-1}B(t)u = B(t)A^{-1}u, u \in D(A). \tag{2.1}$$

Theorem 2.1. *Assume that the condition (1.7) holds for every $t \ge 0$, where M is the constant from (1.5). Then for the solution of difference scheme (1.9), the estimate*

$$\|u_k\|_{E_\alpha} \le \max_{-N \le i \le 0} \|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k} \left\|\varphi_i\right\|_{E_\alpha} \tau \tag{2.2}$$

holds for any $k \ge 1$.

Proof. Let us consider $1 \le k \le N$. In this case

$$u_k = R^k g(0) + \sum_{j=1}^{k} R^{k-j+1} B_j g(t_{j-N})\tau + \sum_{j=1}^{k} R^{k-j+1} B_j \varphi_j \tau = v_k + w_k,$$

where

$$v_k = R^k g(0) + \sum_{j=1}^{k} R^{k-j+1} B_j g(t_{j-N})\tau,$$

$$w_k = \sum_{j=1}^{k} R^{k-j+1} B_j \varphi_j \tau, R = (I + \tau A)^{-1}.$$

The estimate

$$\|v_k\|_{E_\alpha} \le \max_{-N \le i \le 0} \|g(t_i)\|_{E_\alpha} \tag{2.3}$$

was proved in [12]. Therefore, we will estimate w_k. Using the formula

$$(I + \tau A)^{-k} = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \exp(-\tau t A) dt, k \ge 1, \tag{2.4}$$

estimate (1.5), we obtain

$$\lambda^{1-\alpha}\|A\exp\{-\lambda A\}w_k\|_E \leq \lambda^{1-\alpha}\sum_{j=1}^{k}\tau\frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\frac{dt}{(t\tau+\lambda)^{1-\alpha}}\|\varphi_j\|_{E_\alpha}$$

$$\leq \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau$$

for every k, $1 \leq k \leq N$ and $\lambda, \lambda > 0$. This shows that

$$\|w_k\|_{E_\alpha} \leq \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau \tag{2.5}$$

for every $k, 1 \leq k \leq N$. Using triangle inequality and estimates (2.3) and (2.5), we get

$$\|u_k\|_{E_\alpha} \leq \max_{-N\leq i\leq 0}\|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau. \tag{2.6}$$

Applying the mathematical induction, one can easily show that it is true for every k. Namely, assume that the estimate (2.6) is true for the $k, (n-1)N \leq k \leq nN$, for some $n = 1, 2, 3\cdots$. Letting $k = m + nN$, we have

$$\frac{1}{\tau}(u_{m+nN} - u_{m+nN-1}) + Au_{m+nN} = B_{m+nN}u_{m+nN-N} + \varphi_{m+nN}, 1 \leq m \leq N.$$

Using the estimate (2.6), we obtain

$$\|u_k\|_{E_\alpha} \leq \max_{1\leq m\leq N}\|u_{m+nN-N}\|_{E_\alpha} + \sum_{i=nN+1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau$$

$$\leq \max_{-N\leq i\leq 0}\|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{nN}\left\|\varphi_i\right\|_{E_\alpha}\tau + \sum_{m=nN+1}^{k}\left\|\varphi_m\right\|_{E_\alpha}\tau \leq \max_{-N\leq i\leq 0}\|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau$$

for every k, $nN \leq k \leq (n+1)N,\ n = 1, 2, 3, \cdots$, and $\lambda, \lambda > 0$.This shows that

$$\|u_k\|_{E_\alpha} \leq \max_{-N\leq i\leq 0}\|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau$$

for every $k, nN \leq k \leq (n+1)N,\ n = 1, 2, 3, \cdots$. Theorem 2.1 is proved.

Now, we consider the problem (1.4) when

$$A^{-1}B(t)x \neq B(t)A^{-1}x, x \in D(A)$$

for some $t \geq 0$. Recall that A is a strongly positive operator in a Banach spaces E iff its spectrum $\sigma(A)$ lies in the interior of the sector of angle $\varphi, 0 < 2\varphi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1 = [z = \rho\exp(i\varphi) : 0 \leq \rho < \infty]$ and $S_2 = [z = \rho\exp(-i\varphi) : 0 \leq \rho < \infty]$ and outside it the resolvent $(z - A)^{-1}$ is subject to the bound

$$\|(z - A)^{-1}\|_{E\to E} \leq \frac{M_1}{1 + |z|} \tag{2.7}$$

for some $M_1 > 0$. First of all let us give lemmas from the paper [12].

Lemma 2.1. For any z on the edges of the sector,

$$S_1 = [z = \rho\exp(i\varphi) : 0 \leq \rho < \infty]$$

and

$$S_2 = [z = \rho \exp(-i\varphi) : 0 \le \rho < \infty]$$

and outside it the estimate

$$\|A(z-A)^{-1}x\|_E \le \frac{M_1^{\alpha} M^{\alpha}(1+M_1)^{1-\alpha} 2^{(2-\alpha)\alpha}}{\alpha(1-\alpha)(1+|z|)^{\alpha}} \|x\|_{E_\alpha}$$

holds for any $x \in E_\alpha$. Here and in the future M and M_1 are the same constants of the estimates (1.5) and (2.7).

Lemma 2.2. Let for all $s \ge 0$ the operator $B(s)A^{-1} - A^{-1}B(s)$ with domain, which coincides with $D(A)$, permit the closure $Q = \overline{B(s)A^{-1} - A^{-1}B(s)}$ bounded in E. Then for all $\tau > 0$ the following estimate holds:

$$\begin{aligned} &\left\|A^{-1}[A\exp\{-\tau A\}B(s) - B(s)A\exp\{-\tau A\}]x\right\|_E \\ &\le \frac{e(\alpha+1)M^{\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}\|Q\|_{E\mapsto E}\|x\|_{E_\alpha}}{\tau^{1-\alpha}\pi\alpha^2(1-\alpha)}. \end{aligned}$$

Here $Q = \overline{A^{-1}(AB(s) - B(s)A)A^{-1}}$.

Suppose that

$$\begin{aligned} &\|\overline{A^{-1}(AB(t) - B(t)A)A^{-1}}\|_{E\mapsto E} \\ &\le \frac{\pi(1-\alpha)^2\alpha^2\varepsilon}{eM^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{2+\alpha-\alpha^2}(1+\alpha)} \end{aligned} \tag{2.8}$$

holds for every $t \ge 0$. Here and in the future ε is a some constant, $0 \le \varepsilon \le 1$.

Theorem 2.2. *Assume that the condition*

$$\|\overline{A^{-1}B(t)}\|_{E\mapsto E} \le \frac{(1-\alpha)(1-\varepsilon)}{M2^{2-\alpha}} \tag{2.9}$$

holds for every $t \ge 0$. Then for the solution of difference scheme (1.9), the estimate (2.2) holds.

Proof. Let us consider $1 \le k \le N$. The estimate

$$\|v_k\|_{E_\alpha} \le \max_{-N\le i\le 0} \|g(t_i)\|_{E_\alpha} \tag{2.10}$$

was proved in [12]. Therefore, using triangle inequality and estimates (2.10), and (2.5), we get

$$\|u_k\|_{E_\alpha} \le \max_{-N\le i\le 0} \|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k} \left\|\varphi_i\right\|_{E_\alpha} \tau.$$

In a similar manner with Theorem 2.1 applying the mathematical induction, one can easily show that it is true for every k. Theorem 2.2 is proved.

Now we consider the difference scheme (1.10). We have not been able to obtain the same result for the solution of the difference scheme (1.10) in spaces E_α under assumption (1.7). Nevertheless, for the solution of difference scheme (1.10) the stability estimate in the norm of the same fractional spaces $E_\alpha (0 < \alpha < 1)$ under an additional restriction of the operator A is established.

Theorem 2.3. *Suppose that the following estimates hold:*

$$\begin{aligned} \|(I+\tau A)(I+\tau AS)^{-1}\|_{E\mapsto E} &\le 1, \\ \|S(I+\tau A)(I+\tau AS)^{-1}\|_{E\mapsto E} &\le \frac{1+\sqrt{2}}{2} \end{aligned} \tag{2.11}$$

and

$$\|\overline{A^{-1}B(t)}\|_{E\mapsto E} \le \frac{(1-\alpha)}{M2^{1-\alpha}(1+\sqrt{2})},\ t \ge 0. \tag{2.12}$$

Then for the solution of difference scheme (1.10), estimate (2.2) holds.

Proof. Let us consider $1 \leq k \leq N$. In this case

$$u_k = R^k g(0) + \sum_{j=1}^{k} R^{k-j+1}(I + \frac{\tau A}{2})B_j(g(t_{j-N} + g(t_{j-N-1}))\tau$$

$$+ \sum_{j=1}^{k} R^{k-j+1}(I + \frac{\tau A}{2})\varphi_j \tau = v_k + w_k, \tag{2.13}$$

where

$$v_k = R^k g(0) + \sum_{j=1}^{k} R^{k-j+1}(I + \frac{\tau A}{2})B_j(g(t_{j-N} + g(t_{j-N-1}))\tau,$$

$$w_k = \sum_{j=1}^{k} R^{k-j+1}(I + \frac{\tau A}{2})\varphi_j \tau, R = (I + \tau A + \frac{(\tau A)^2}{2})^{-1}.$$

The estimate

$$\|v_k\|_{E_\alpha} \leq \max_{-N \leq i \leq 0} \|g(t_i)\|_{E_\alpha} \tag{2.14}$$

was proved in [12]. Therefore, we will estimate w_k. Using the formula (2.13), condition (2.11) and the estimate (1.5), we obtain

$$\lambda^{1-\alpha}\|A\exp\{-\lambda A\}w_k\|_E \leq \lambda^{1-\alpha}\sum_{j=1}^{k}\tau\|(I + \tau A)(I + \tau A + \frac{(\tau A)^2}{2})^{-1})^{k-j}\|_{E\mapsto E}$$

$$\times\|(I + \frac{\tau A}{2})(I + \tau A)(I + \tau A + \frac{(\tau A)^2}{2})^{-1}\|_{E\mapsto E}$$

$$\times\frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\frac{dt}{(t\tau + \lambda)^{1-\alpha}}\|\varphi_j\|_{E_\alpha}$$

$$\leq \lambda^{1-\alpha}\sum_{j=1}^{k}\tau\frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\frac{dt}{(t\tau + \lambda)^{1-\alpha}}\|\varphi_j\|_{E_\alpha} \leq \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau$$

for every k, $1 \leq k \leq N$ and $\lambda, \lambda > 0$. This shows that

$$\|w_k\|_{E_\alpha} \leq \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau \tag{2.15}$$

for every $k, 1 \leq k \leq N$. Using triangle inequality and estimates (2.14) and (2.15), we get

$$\|Au_k\|_{E_\alpha} \leq \max_{-N \leq i \leq 0}\|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k}\left\|\varphi_i\right\|_{E_\alpha}\tau. \tag{2.16}$$

In a similar manner with Theorem 2.1 applying the mathematical induction, one can easily show that it is true for every k. Theorem 2.3 is proved.

Now, we consider the difference scheme (1.10) when

$$A^{-1}B(t)x \neq B(t)A^{-1}x, x \in D(A)$$

for some $t \geq 0$. Suppose that the operator $B(t)A^{-1} - A^{-1}B(t)$ with domain, which coincides with $D(A)$, permits the closure bounded in E and the following estimate

$$\overline{\|A^{-1}(AB(t) - B(t)A)A^{-1}\|}_{E\mapsto E} \leq \frac{\pi(1-\alpha)^2\alpha^2(1+\alpha)^{-1}(1+\sqrt{2})^{-1}\varepsilon}{eM^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{2+\alpha-\alpha^2}}$$

holds for every $t \geq 0$ and some $\varepsilon\epsilon[0,1]$.

Theorem 2.4. *Assume that all conditions of Theorems 2.2 and 2.3 are satisfied. Then for the solution of difference scheme (1.10), estimate (2.2) holds.*

Proof. Let us consider $1 \leq k \leq N$. The estimate

$$\|v_k\|_{E_\alpha} \leq \max_{-N\leq i\leq 0} \|g(t_i)\|_{E_\alpha} \tag{2.17}$$

was proved in [12]. Therefore, using triangle inequality and estimates (2.17) and (2.15), we get

$$\|u_k\|_{E_\alpha} \leq \max_{-N\leq i\leq 0} \|g(t_i)\|_{E_\alpha} + \sum_{i=1}^{k} \left\|\varphi_i\right\|_{E_\alpha} \tau.$$

In a similar manner with Theorem 2.1 applying the mathematical induction, one can easily show that it is true for every k. Theorem 2.4 is proved.
Note that these abstract results are applicable to the study of stability of various delay parabolic equations with local and nonlocal boundary conditions with respect to space variables. However, it is important to study the structure of E_α for space operators in Banach spaces. The structure of E_α for some space differential and difference operators in Banach spaces has been investigated in papers (see, [21]-[29]). In Section 3, applications of Theorem 2.1 to the study of stability of difference schemes for delay parabolic equations are given.

3. Applications

First, the initial-boundary value problem for one dimensional delay differential equations of parabolic type is considered:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = b(t)\left(-a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2} + \delta u(t-\omega,x)\right) \\ +f(t,x), 0 < t < \infty, x \in (0,l), \\ u(t,x) = g(t,x), -\omega \leq t \leq 0, x \in [0,l], \\ u(t,0) = u(t,l) = 0, -\omega \leq t < \infty, \end{cases} \tag{3.1}$$

where $a(x)$, $b(t)$, $g(t,x)$, $f(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number. It will be assumed that $a(x) \geq a > 0$. The discretization of problem (3.1) is carried out in two steps. In the first step, let us define the grid space

$$[0,l]_h = \{x : x_r = rh, 0 \leq r \leq K, Kh = l\}.$$

We introduce the Banach space $C_h^\beta = C^\beta([0,l]_h)$ $(0 < \beta < 1)$ of the grid functions $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ defined on $[0,l]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{C_h^\beta} = \left\|\varphi^h\right\|_{C_h} + \sup_{1\leq k<k+\tau\leq K-1} \frac{|\varphi_{k+r} - \varphi_k|}{\tau^\beta},$$

where $C_h = C([0,l]_h)$ is the space of the grid functions $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ defined on $[0,l]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{C_h} = \max_{1\leq k\leq K-1} \left|\varphi_k\right|.$$

To the differential operator A^x generated by the problem (3.1), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \left\{-(a(x)\varphi_{\bar{x}})_{x,r} + \delta\varphi_r\right\}_1^{K-1},$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_0^K$ satisfying the conditions $\varphi_0 = \varphi_K = 0$. With the help of A_h^x, we arrive at the initial-boundary value problem

$$\begin{cases} \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = b(t)A_h^x u^h(t-\omega,x) + f^h(t,x), t \geq 0,\ x \in [0,1]_h, \\ u^h(t,x) = g^h(t,x) = g(t,x)(-\omega \leq t \leq 0), x \in [0,1]_h \end{cases} \tag{3.2}$$

for the system of ordinary differential equations. In the second step, we replace problem (3.2) by the first order of accuracy in t difference scheme

$$\begin{cases} \frac{1}{\tau}(u_k^h(x) - u_{k-1}^h(x)) + A_h^x u_k^h(x) = b(t_k)A_h^x u_{k-N}^h(x) + f_k^h(x), \\ f_k^h(x) = f^h(t_k,x), t_k = k\tau, 1 \leq k, N\tau = \omega, x \in [0,l]_h, \\ u_k^h(x) = g^h(t_k,x), t_k = k\tau, -N \leq k \leq 0, x \in [0,l]_h. \end{cases} \tag{3.3}$$

Theorem 3.1. *Assume that*

$$\sup_{0 \leq t < \infty} |b(t)| \leq \frac{1-\alpha}{M2^{2-\alpha}}. \tag{3.4}$$

Then for the solution of difference scheme (3.3) the following stability estimate

$$\sup_{1 \leq k < \infty} \left\|u_k^h\right\|_{C^{2\alpha}[0,1]_h} \leq M_3(\alpha)\left[\max_{-N \leq k \leq 0} \|g_k^h\|_{C^{2\alpha}[0,1]_h} + \sum_{k=1}^{\infty} \left\|f_k^h\right\|_{C^{2\alpha}[0,1]_h} \tau\right], 0 < \alpha < \frac{1}{2} \tag{3.5}$$

holds, where $M_3(\alpha)$ does not depend on g_k^h and f_k^h.

The proof of Theorem 3.1 is based on the estimate

$$\|\exp\{-t_k A_h^x\}\|_{C_h \to C_h} \leq M, k \geq 0,$$

and on the abstract Theorem 2.1, the positivity of the operator A_h^x in C_h^μ and on the following theorem on the structure of the fractional space $E_\alpha(C_h, A_h^x)$.

Theorem 3.2. *For any $0 < \alpha < \frac{1}{2}$ the norms in the spaces $E_\alpha\left(C_h, A_h^x\right)$ and $C_h^{2\alpha}$ are equivalent uniformly in h [22].*

Second, the initial nonlocal boundary value problem for one dimensional delay differential equations of parabolic type is considered:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = b(t)\left(-a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2} + \delta u(t-\omega,x)\right) \\ +f(t,x), 0 < t < \infty, x \in (0,l), \\ u(t,x) = g(t,x), -\omega \leq t \leq 0, x \in [0,l], \\ u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), -\omega \leq t < \infty, \end{cases} \tag{3.6}$$

where $a(x)$, $b(t)$, $g(t,x)$, $f(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number. It will be assumed that $a(x) \geq a > 0$. The discretization of problem (3.6) is carried out in two steps. In the first step, let us use the discretization in space variable x. To the differential operator A^x generated by the problem (3.6), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \left\{-(a(x)\varphi_{\bar{x}})_{x,r} + \delta\varphi_r\right\}_1^{K-1}, \tag{3.7}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_0^K$ satisfying the conditions $\varphi_0 = \varphi_K, \varphi_1 - \varphi_0 = \varphi_K - \varphi_{K-1}$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = b(t)A_h^x u^h(t-\omega,x) + f^h(t,x), t \geq 0, \ x \in [0,l]_h, \\ u^h(t,x) = g^h(t,x) = g(t,x)(-\omega \leq t \leq 0), x \in [0,l]_h \end{cases} \tag{3.8}$$

for the system of ordinary fractional differential equations. In the second step, we replace problem (3.8) by the first order of accuracy of difference scheme in t

$$\begin{cases} \frac{1}{\tau}(u_k^h(x) - u_{k-1}^h(x)) + A_h^x u_k^h(x) = b(t_k)A_h^x u_{k-N}^h(x) + f_k^h(x), \\ f_k^h(x) = f^h(t_k,x), t_k = k\tau, 1 \leq k, N\tau = \omega, x \in [0,l]_h, \\ u_k^h(x) = g^h(t_k,x), t_k = k\tau, -N \leq k \leq 0, x \in [0,l]_h. \end{cases} \tag{3.9}$$

Theorem 3.3. *Assume that all the conditions of Theorem 3.1 are satisfied. Then for the solution of difference scheme (3.9) the stability estimate (3.5) holds.*

The proof of Theorem 3.3 is based on the estimate

$$\| \exp\{-t_k A_h^x\}\|_{C_h \to C_h} \leq M, k \geq 0,$$

and on the abstract Theorem 2.1, the positivity of the operator A_h^x in C_h^μ and on the following theorem on the structure of the fractional space $E_\alpha(C_h, A_h^x)$.

Theorem 3.4. *For any $0 < \alpha < \frac{1}{2}$ the norms in the spaces $E_\alpha\left(C_h, A_h^x\right)$ and $C_h^{2\alpha}$ are equivalent uniformly in h [25].*

Third, the initial value problem on the range
$\{0 \leq t \leq 1, x = (x_1, \cdots, x_n) \in \mathbb{R}^n, r = (r_1, \cdots, r_n)\}$
for 2m-th order multidimensional delay differential equations of parabolic type is considered:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \sum\limits_{|r|=2m} a_\tau(x) \frac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta u(t,x) \\ = b(t)\left(\sum\limits_{|r|=2m} a_\tau(x) \frac{\partial^{|r|} u(t-\omega,x)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta u(t-\omega,x)\right) + f(t,x), 0 < t < \infty, x \in \mathbb{R}^n, \\ u(t,x) = g(t,x), -\omega \leq t \leq 0, x \in \mathbb{R}^n, |\, r \,| = r_1 + \dots + r_n, \end{cases} \tag{3.10}$$

where $a_r(x)$, $b(t)$, $g(t,x)$, $f(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number. It will be assumed that the symbol $[\xi = (\xi_1, \cdots, \xi_n) \epsilon \mathbb{R}^n]$

$$A_1^x(\xi) = \sum_{|r|=2m} a_r(x)\,(i\xi_1)^{r_1} \dots (i\xi_n)^{r_n}$$

of the differential operator of the form

$$A_1^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \tag{3.11}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \leq (-1)^m A_1^x(\xi) \leq M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$, where $|\ \xi\ |= \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}$. The discretization of problem (3.10) is carried out in two steps. In the first step the grid space $R_h^n\ (0 < h \leq h_0)$ is defined as the set of all points of the Euclidean space R^n whose coordinates are given by

$$x_k = s_k h, \qquad s_k = 0, \pm 1, \pm 2, \cdots, k = 1, \cdots, n.$$

The difference operator $A_h^x = B_h^x + \sigma I_h$ is assigned to the differential operator $A^x = B^x + \sigma I$, defined by (3.10). The operator

$$B_h^x = h^{-2m} \sum_{2m \leq |s| \leq S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} ... \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}}, \tag{3.12}$$

acts on functions defined on the entire space R_h^n. Here $s \in R^{\,2n}$ is a vector with nonnegative integer coordinates,

$$\Delta_{k\pm} f^h(x) = \pm \left(f^h(x \pm e_k h) - f^h(x) \right),$$

where e_k is the unit vector of the axis x_k.

An infinitely differentiable function $\varphi(x)$ of the continuous argument $x \in R^n$ that is continuous and bounded together with all its derivatives is said to be smooth. We say that the difference operator A_h^x is a λ-th order ($\lambda > 0$) approximation of the differential operator A^x if the inequality

$$\sup_{x \in R_h^n} \left| A_h^x \varphi(x) - A^x \varphi(x) \right| \leq M(\varphi) h^\lambda$$

holds for any smooth function $\varphi(x)$. The coefficients b_s^x are chosen in such a way that the operator A_h^x approximates in a specified way the operator A^x. It will be assumed that the operator A_h^x approximates the differential operator A^x with any prescribed order [30]-[31] .

The function $A^x(\xi h, h)$ is obtained by replacing the operator $\Delta_{k\pm}$ in the right-hand side of equality (3.12) with the expression $\pm(\exp\{\pm i\xi_k h\} - 1)$, respectively, and is called the symbol of the difference operator B_h^x.

It will be assumed that for $|\xi_k h| \leq \pi$ and fixed x the symbol $A^x(\xi h, h)$ of the operator $B_h^x = A_h^x - \sigma I_h$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geq M|\xi|^{2m}, |\arg A^x(\xi h, h)| \leq \phi < \phi_0 \leq \frac{\pi}{2}. \tag{3.13}$$

Suppose that the coefficient b_s^x of the operator $B_h^x = A_h^x - \sigma I_h$ is bounded and satisfies the inequalities

$$|b_s^{x+e_k h} - b_s^x| \leq M h^\varepsilon, x \in R_h^n, \varepsilon \in (0, 1]. \tag{3.14}$$

With the help of A_h^x we arrive at the initial value problem

$$\begin{cases} (u^h(t,x))' + A_h^x u^h(t,x) = b(t) A_h^x u^h(t-\omega, x) + f^h(t,x), t \geq 0, x \in \mathbb{R}_h^n, \\ \qquad u^h(t,x) = g^h(t,x) = g(t,x)(-\omega \leq t \leq 0), x \in \mathbb{R}_h^n, \end{cases} \tag{3.15}$$

for an infinite system of ordinary differential equations. Now, we replace problem (3.15) by the first order of accuracy of difference scheme in t

$$\begin{cases} \frac{1}{\tau}(u_k^h(x) - u_{k-1}^h(x)) + A_h^x u_k^h(x) = b(t_k) A_h^x u_{k-N}^h(x) + f_k^h(x), \\ f_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k, N\tau = \omega, x \in \mathbb{R}_h^n, \\ u_k^h(x) = g^h(t_k, x), t_k = k\tau, -N \leq k \leq 0, x \in \mathbb{R}_h^n. \end{cases} \tag{3.16}$$

To formulate the result, one needs to introduce the spaces $C_h = C(R_h^n)$ and $C_h^\beta = C^\beta(\mathrm{R}_h^n)$ of all bounded grid functions $u^h(x)$ defined on R_h^n, equipped with the norms

$$\|u^h\|_{C_h} = \sup_{x \varepsilon \mathrm{R}_h^n} |u^h(x)|,$$

$$\|u^h\|_{C_h^\beta} = \sup_{x \varepsilon \mathrm{R}_h^n} |u^h(x)| + \sup_{x,y \in \mathrm{R}_h^n} \frac{|u^h(x) - u^h(x+y)|}{|y|^\beta}.$$

Theorem 3.5. *Assume that the condition (3.4) holds. Then for the solution of difference scheme (3.16) the following stability estimate*

$$\sup_{1 \le k < \infty} \left\|u_k^h\right\|_{C^{2m\alpha}(\mathbb{R}_h^n)} \le M_2(\alpha) \left[\max_{-N \le k \le 0} \left\|g_k^h\right\|_{C^{2m\alpha}(\mathbb{R}_h^n)} \right.$$

$$\left. + \sum_{k=1}^{\infty} \tau \left\|f_k^h\right\|_{C^{2m\alpha}(\mathbb{R}_h^n)} \right], 0 < \alpha < \frac{1}{2m}$$

holds, where $M_2(\alpha)$ does not depend on g_k^h and f_k^h.

The proof of Theorem 3.5 is based on the estimate

$$\| \exp\{-t_k A_h^x\} \|_{C(\mathbb{R}_h^n) \to C(\mathbb{R}_h^n)} \le M, k \ge 0,$$

and on the abstract Theorem 2.1, the positivity of the operator A_h^x in $C(\mathbb{R}_h^n)$, and on the fact that the $E_\alpha = E_\alpha(A_h^x, C(\mathbb{R}_h^n))$–norms are equivalent to the norms $C^{2m\alpha}(\mathbb{R}_h^n)$ uniformly in h for $0 < \alpha < \frac{1}{2m}$. ([21], [24]).

4. Conclusion

In the present paper, the stability of difference schemes for the approximate solutions of the initial value problem for delay parabolic equations with unbounded operators acting on delay terms in an arbitrary Banach space is established. Theorems on stability of these difference schemes in fractional spaces are established. In practice, the stability estimates in Hölder norms for the solutions of difference schemes for the approximate solutions of the mixed problems for delay parabolic equations are obtained. Note that in the present paper $B(t)$ is a time-dependent unbounded space operator acting on the delay term. The delay w is a positive constant. In general, it is interesting to consider delay as $w(t)$, a function dependent on t. A well-known parabolic problem with delay used in population dynamics is the so-called Hutchinson equation where $B(t)$ is a time-dependent bounded nonlinear space operator acting on the delay term (see, [8],[9]). It would be an interesting case to consider when $B(t)$ is a nonlinear unbounded space operator acting on the delay term. Actually, it will be possible after establishing theorems on existence, uniqueness and stability of solutions and smoothness property of solutions and obtaining a suitable contractivity condition of the numerical solutions.

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