Filomat 26:8 (2014), 1167–1180
DOI 10.2298/FIL1406167L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Anti-Periodic Solutions for a Class of First Order Evolution Inclusions

Xiaoyou Liu[a], Yiliang Liu[b]

[a] *School of Mathematics and Physics, University of South China, Hengyang 421001, Hunan Province, P.R. China*
[b] *College of Sciences, Guangxi University for Nationalities, Nanning 530006, Guangxi Province, P. R. China*

Abstract. The existence of anti-periodic solutions for a class of first order nonlinear evolution inclusions defined in the framework of an evolution triple of spaces is considered. We study the problems under both convexity and nonconvexity conditions on the multivalued right-hand side. The main tools in our study are the maximal monotone property of the derivative operator with anti-periodic conditions, the surjectivity result for L-pseudomonotone operators and continuous extreme selection results from multivalued analysis. An example of a nonlinear parabolic problem is given to illustrate our results.

1. Introduction

In this paper, we study a class of first order nonlinear evolution inclusions defined in the framework of an evolution triple of spaces. Our aim is to get the existence results of anti-periodic solutions.

Let (V, H, V^*) be an evolution triple of spaces. $T = [0, b]$, here b is a positive real number. We consider the following problem (P1):

$$(P1) \quad \begin{cases} x'(t) + A(t, x(t)) + Bx(t) \in F(t, x(t)), \text{ a.e. on } T, \\ x(0) = -x(b), \end{cases}$$

where $A : T \times V \to V^*$ is a nonlinear operator, $B \in \mathcal{L}(V, V^*)$ and $F : T \times H \to 2^H \setminus \{\emptyset\}$ is a multivalued map. The time derivative of x is understood in the sense of vectorial distributions.

We also consider the following problem (P2):

$$(P2) \quad \begin{cases} x'(t) + A(t, x(t)) + Bx(t) \in F(t, x(t)) \bigcap \operatorname{ext}\overline{\operatorname{co}}\, F(t, x(t)), \text{ a.e. on } T, \\ x(0) = -x(b). \end{cases}$$

Anti-periodic problems arise naturally in the mathematical modeling of various physical processes. For this reason, existence of anti-periodic solutions to nonlinear evolution equations has been investigated by many authors in the last decades. For instance, Okochi [1, 2] studied the existence of anti-periodic

---

2010 *Mathematics Subject Classification*. Primary 35K90; Secondary 34G25
Keywords. Evolution inclusion, anti-periodic solution, maximal monotone operator, evolution triple, pseudomonotone operator, continuous extreme selection
Received: 16 May 2013; Accepted: 20 March 2014
Communicated by Dragan Djordjevic
Project supported by NNSF of China Grant Nos.11271087, 61263006
Email addresses: liuxiaoyou2002@hotmail.com (Xiaoyou Liu), yiliangliu100@126.com (Yiliang Liu)

solutions to evolution equations of subdifferential type in Hilbert space. Aizicovici, McKibben and Reich [3], Aizicovici and Pavel [4] considered anti-periodic problem for evolution equations governed by maximal monotone operator. Anti-periodic solutions for semilinear evolution equations were considered by Chen [5], Chen, O'Regan and Agarwal [6] and Liu [7]. Chen, Nieto and O'Regan [8] studied the anti-periodic problem for a nonlinear evolution equation where the nonlinear part is an odd maximal monotone mapping. Liu [9, 10] considered anti-periodic problems for nonlinear evolution equation with nonmonotone perturbations in a real reflexive Banach space. Furthermore, Liu and Migorski [11] recently constructed a new and important existence result and analyzed the controllability for differential inclusions with anti-periodic conditions in Banach spaces, which represented a great development in anti-periodic problems.

Evolution (or differential) inclusions enjoy a wide applications in the study of differential equations with discontinuous right hand sides, as well as in control theory, differential games, economic dynamics, etc. For the existence results of evolution inclusions, many works had been done on its Cauchy problem and periodic problem. For example, see Migórski [12–14], Papageorgiou, Papalini and Renzacci [15], Papageorgiou and Yannakakis [16], Papageorgiou, Papalini and Yannakakis [17], Tolstonogov [18, 19] and the references therein. Very recently, Park and Ha [20, 21] treated the anti-periodic solutions for evolution hemivariational inequalities.

Our method for treating the problems is as follows. We combine the anti-periodic condition with the derivative operator L, then we prove that in this case the derivative operator is maximal monotonicity, finally, we use the surjectivity result for operators which are pseudomonotone with respect to D(L) [15]. We mention that our work can be considered as the extension of the work of Liu [9] to the multivalued nonlinear nonmonotone perturbations case. From the properties of the Clarke subdifferential, we know that the results obtained in [20, 21] are also contained in our framework. To the authors knowledge, these problems are not considered before.

The rest of the paper is organized as follows. In section 2 we give some necessary notations and definitions. Two auxiliary results needed in the proof of our main results are presented in section 3. In section 4, we handle the existence results of problem (P1) with convex valued right-hand side. We study the nonconvex case of problem (P1) and the existence results of problem (P2) in section 5. In the final section 6, an example of a nonlinear parabolic problem with discontinuous right-hand side is considered.

2. Notations and Definitions

Let $T = [0, b]$ be an interval of the real axis with the Lebesgue measure μ and σ-algebra Σ of μ-measurable subsets of T and Y be a separable Banach space. We use the following notations:

$P_{(b)f(c)}(Y) = \{A \subseteq Y : A$ is nonempty, (bounded) closed (convex)$\}$,

$P_{(w)k(c)}(Y) = \{A \subseteq Y : A$ is nonempty, (weakly-) compact (convex)$\}$.

For a Banach space X the symbol ω-X stands for X equipped with the weak $\sigma(X, X^*)$ topology. The same notation will be used for subsets of X. In all other cases we assume that X and its subsets are equipped with the strong (normed) topology.

Given a set $A \subseteq Y$, $\operatorname{co} A$ ($\overline{\operatorname{co}}\, A$) denotes convex hull (closed convex hull) of A, $\operatorname{ext} A$ stands for the set of extreme points of A.

Let (X, τ) be a Hausdorff topological space and let $\{A_n\}_{n\geq 1} \subseteq 2^X \backslash \{\emptyset\}$. We define

$$\tau\text{-}\underline{\lim}_{n\to\infty} A_n = \{x \in X : x = \tau\text{-}\lim_{n\to\infty} x_n,\ x_n \in A_n,\ n \geq 1\},$$

$$\tau\text{-}\overline{\lim}_{n\to\infty} A_n = \{x \in X : x = \tau\text{-}\lim_{k\to\infty} x_{n_k},\ x_{n_k} \in A_{n_k},\ n_k < n_{k+1},\ k \geq 1\}.$$

Suppose V, Z are two Hausdorff topological spaces and $F : V \to 2^Z \backslash \{\emptyset\}$. We say that F is lower semicontinuous in the sense of Vietoris (l.s.c. for short) at a point $x_0 \in V$, if for any open set $W \subseteq Z$, $F(x_0) \cap W \neq \emptyset$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x) \cap W \neq \emptyset$ for all $x \in O(x_0)$. F is said to be upper semicontinuous in the sense of Vietoris (u.s.c. for short) at a point $x_0 \in V$, if for any open set $W \subseteq Z$, $F(x_0) \subseteq W$, there is a neighborhood $O(x_0)$ of x_0 such that $F(x) \subseteq W$ for all $x \in O(x_0)$. For the properties of l.s.c and u.s.c and further details about multivalued analysis, we can refer to the books [22, 23].

Let $F : T \to 2^Y \backslash \emptyset$ be a multifunction, for $1 \le p \le +\infty$, we define $S_F^p = \{f \in L^p(T, Y) : f(t) \in F(t)$ a.e. on $T\}$. On $P_{bf}(Y)$, we have a metric known as the "Hausdorff metric" and defined by

$$h(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(x, C)$ is the distance from a point x to a set C. We say a multivalued map is h-continuous if it is continuous in the Hausdorff metric $h(\cdot, \cdot)$.

We say that a multivalued map $F : T \to P_f(Y)$ is measurable if $F^{-1}(E) = \{t \in T : F(t) \cap E \neq \emptyset\} \in \Sigma$ for every closed set $E \subseteq Y$. If $F : T \times Y \to P_f(Y)$, then measurability of F means that $F^{-1}(E) \in \Sigma \otimes \mathcal{B}_Y$, where $\Sigma \otimes \mathcal{B}_Y$ is the σ-algebra of subsets in $T \times Y$ generated by the sets $A \times B$, $A \in \Sigma$, $B \in \mathcal{B}_Y$, and $\mathcal{B}_Y$ is the σ-algebra of the Borel sets in Y.

Let H be a separable Hilbert space and V a dense subspace of H carrying the structure of separable, reflexive Banach space. We assume that V is embedded continuously in H. Identifying H with its dual (pivot space), we have that H is embedded continuously and densely in V^*. The triple (V, H, V^*) is known as "evolution triple" or "Gelfand triple". By $|\cdot|_H$ (respectively, $\|\cdot\|_V$, $\|\cdot\|_{V^*}$), we denote the norm of H (respectively, V, V^*). Also, by $(\cdot, \cdot)$ we denote the inner product of H and by $\langle\cdot, \cdot\rangle$ the duality brackets of the pair (V^*, V). The two compatible in the sense that $\langle\cdot, \cdot\rangle_{H\times V} = (\cdot, \cdot)$. Here X^* stands for the topological dual space of X.

Given $1 < p < \infty$, we introduce the following function spaces $\mathcal{V} = L^p(T, V)$, $\mathcal{H} = L^p(T, H)$, $\mathcal{H}^* = L^q(T, H)$ and $\mathcal{V}^* = L^q(T, V^*)$ with $1/p + 1/q = 1$. We set $\mathcal{W} = \{x \in \mathcal{V} : x' \in \mathcal{V}^*\}$, where the time derivative is understood in the sense of vectorial distributions. The space $\mathcal{W}$ is a separable, reflexive Banach space furnished with the norm $\|x\|_{\mathcal{W}} = \|x\|_{\mathcal{V}} + \|x'\|_{\mathcal{V}^*}$. We have $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{H} \subseteq \mathcal{H}^* \subseteq \mathcal{V}^*$ with continuous embeddings. The pairing of $\mathcal{V}$ and $\mathcal{V}^*$ is denoted by $\langle\langle\cdot, \cdot\rangle\rangle$. It is well-known that the space $\mathcal{W}$ is embedded continuously in $C(T, H)$. Moreover, if V is embedded compactly in H, then so does $\mathcal{W}$ into $\mathcal{H}$ [24]. In the rest of this paper, we will assume that V is embedded compactly in H. The embedding operator is denoted by γ.

Next we recall some definitions on pseudomonotone operators. Let $P : Y \to Y^*$ be an operator on a real reflexive Banach space Y. P is called pseudomonotone if and only if $v_n \to v$ weakly in Y and $\limsup\langle Pv_n, v_n - v\rangle \le 0$ imply $\langle Pv, v - w\rangle \le \liminf\langle Pv_n, v_n - w\rangle$ for all $w \in Y$. The operator P is said to be demicontinuous if and only if it is continuous from Y to ω-Y^*. P is said to be hemicontinuous if and only if for all $v, w, z \in Y$, we have $\lambda \to \langle P(v + \lambda w), z\rangle$ is continuous on $[0, 1]$.

Let $G : Y \to 2^{Y^*}$ be a multivalued operator. G is said to be pseudomonotone if it satisfies the following:
(a) for every $y \in Y$, $Gy \in P_{wkc}(Y^*)$;
(b) G is u.s.c. from every finite dimensional subspace of Y into ω-Y^*;
(c) if $y_n \to y$ weakly in Y, $y_n^* \in Gy_n$, and $\limsup\langle y_n^*, y_n - y\rangle \le 0$, then for each $z \in Y$, there exists $y^*(z) \in Gy$ such that $\langle y^*(z), y - z\rangle \le \liminf\langle y_n^*, y_n - z\rangle$.

Let $L : D(L) \subseteq Y \to Y^*$ be a linear, maximal monotone operator. G is said to be pseudomonotone with respect to $D(L)$ if and only if the above items (a), (b) and the following hold:
(d) if $\{y_n\} \subseteq D(L)$ such that $y_n \to y$ weakly in Y, $Ly_n \to Ly$ weakly in Y^*, $y_n^* \in Gy_n$, $y_n^* \to y^*$ weakly in Y^*, and $\limsup\langle y_n^*, y_n - y\rangle \le 0$, then $y^* \in Gy$ and $\langle y_n^*, y_n\rangle \to \langle y^*, y\rangle$.

G is said to be bounded if it maps bounded subsets of Y into bounded subsets of Y^*. G is said to be coercive if there exists a function $c : \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$ such that $\langle y^*, y\rangle \ge c(\|y\|)\|y\|$ for every $y^* \in Gy$.

Definition 2.1. *A function $x \in \mathcal{W} \subseteq C(T, H)$ is called a solution to problem (P1) if and only if there exists $f \in \mathcal{V}^*$ such that $x(0) = -x(b)$, $f(t) \in F(t, x(t))$ a.e. on T and*

$$x'(t) + A(t, x(t)) + Bx(t) = f(t), \text{ a.e. on } T. \tag{1}$$

The solution of problem (P2) is defined likewise.

To conclude this section, we give two propositions which are used in proving our existence theorems.

Proposition 2.2 (see [15]). *If Y is a reflexive, strictly convex Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear maximal monotone operator and $G : Y \to 2^{Y^*}$ is bounded, coercive and pseudomonotone with respect to $D(L)$, then $R(L+G) = Y^*$, i.e. $L+G$ is surjective.*

A crucial point in our approach is the fact that the derivative operator with the anti-periodic condition is a maximal monotone linear operator.

We define

$$Lu = u', \quad D(L) = \{u \in \mathcal{W} : u(0) = -u(b)\}. \tag{2}$$

It is well-known that the derivative operator L with the conditions $u(0) = 0$ or $u(0) = u(b)$ is a maximal monotone operator (cf., [15, 24]). Now we give a result which says that the operator L defined by (2) is also a maximal monotone operator (see [9]).

Proposition 2.3. *The linear operator $L : D(L) \subseteq \mathcal{V} \to \mathcal{V}^*$ defined by (2) is maximal monotone.*

3. Existence for Convex Case

In this section we prove an existence theorem under the hypothesis that the multivalued nonlinearity F is convex valued. The precise hypotheses on the data of problem (P1) are the following:

H(A): $A : T \times V \to V^*$ is an operator such that:
(1) For every $x \in V$, $t \to A(t,x)$ is measurable;
(2) For a.e. $t \in T$, $x \to A(t,x)$ is demicontinuous, pseudomonotone;
(3) For a.e. $t \in T$ and all $x \in V$, we have $\|A(t,x)\|_{V^*} \le a(t) + c\|x\|_V^{p-1}$ with $a \in L^q(T)$, $c > 0$, $2 \le p < \infty$;
(4) For a.e. $t \in T$ and all $x \in V$, we have $\langle A(t,x), x\rangle \ge c_1\|x\|_V^p - a_1(t)$ with $a_1 \in L^1(T)$, $c_1 > 0$.

H(B): $B \in \mathcal{L}(V, V^*)$ and $\langle Bx, x\rangle \ge 0$ for all $x \in V$.

H(F)$_1$: $F : T \times H \to P_{fc}(H)$ is a multifunction such that:
(1) For every $x \in H$, $t \to F(t,x)$ is measurable;
(2) For a.e. $t \in T$, $x \to F(t,x)$ is sequentially closed in $H \times \omega$-H;
(3) For a.e. $t \in T$, all $x \in H$ and all $u \in F(t,x)$, $|u|_H \le a_2(t) + c_2|x|_H^{\frac{2}{q}}$, with $a_2 \in L^q(T)$, $c_2 > 0$.

H$_0$: $x(0) = -x(b) \in H$ (since $\mathcal{W} \subseteq C(T,H)$). If p=2, $2\|\gamma\|^2 c_2 < c_1$.

We denote by $\mathcal{A}$, $\mathcal{B}$: $\mathcal{V} \to \mathcal{V}^*$ the Nemitsky operators corresponding to $A(t, x(t))$, $Bx(t)$, respectively, i.e.,

$$(\mathcal{A}v)(t) = A(t, v(t)), \ (\mathcal{B}v)(t) = Bv(t), \ \text{a.e. on } T, \ \text{for } v \in \mathcal{V}. \tag{3}$$

Proposition 3.1 (see [16] Proposition 2). *If hypotheses H(A) hold, then $\mathcal{A}$ is demicontinuous and if $x_n \to x$ weakly in $\mathcal{W}$ and $\limsup\langle\langle \mathcal{A}x_n, x_n - x\rangle\rangle \le 0$, we have $\mathcal{A}x_n \to \mathcal{A}x$ weakly in $\mathcal{V}^*$ and $\langle\langle \mathcal{A}x_n, x_n\rangle\rangle \to \langle\langle \mathcal{A}x, x\rangle\rangle$.*

We start by deriving a priori bounds for the solutions of problem (P1).

Lemma 3.2. *Suppose that hypotheses H(A), H(B), H(F)$_1$ and H$_0$ hold and x is a solution to problem (P1), then there exists a constant $C > 0$ such that*

$$\|x\|_{\mathcal{W}} \le C.$$

Proof. Let x be a solution to problem (P1). Multiplying (1) by $x(t)$ and integrating over T, we have

$$\int_T \Big\langle x'(t) + A(t, x(t)) + Bx(t), x(t)\Big\rangle dt = \int_T \Big\langle f(t), x(t)\Big\rangle dt \tag{4}$$

with $x(0) = -x(b)$ and $f(t) \in F(t, x(t))$ a.e. on T. From the integration by parts formula and the anti-periodic condition, we get

$$\int_T \langle x'(t), x(t)\rangle dt = \frac{1}{2}\Big(|x(b)|_H^2 - |x(0)|_H^2\Big) = 0. \tag{5}$$

Combining H(A)(4), H(B) and (5) with (4), we obtain

$$c_1\|x\|^p_{\mathcal{V}} \le \|a_1\|_{L^1(T)} + \int_T \|f(t)\|_{V^*}\|x(t)\|_V dt. \tag{6}$$

Firstly, if $p > 2$, by $H(F)_1(3)$ and Young's inequality with ϵ, we can have

$$\begin{aligned}\int_T \|f(t)\|_{V^*}\|x(t)\|_V dt &\le \|\gamma\| \int_T \Big(a_2(t) + c_2|x(t)|_H^{\frac{2}{q}}\Big)\|x(t)\|_V dt\\ &\le \frac{2^q\|\gamma\|}{q\epsilon^q}\Big(\|a_2\|^q_{L^q(T)} + c_2^q\|\gamma\|^2 b^{\frac{p-2}{p}}\|x\|^2_{\mathcal{V}}\Big) + \frac{\|\gamma\|\epsilon^p}{p}\|x\|^p_{\mathcal{V}}.\end{aligned} \tag{7}$$

Inserting (7) into (6), we have

$$\Big(c_1 - \frac{\|\gamma\|\epsilon^p}{p}\Big)\|x\|^p_{\mathcal{V}} \le \|a_1\|_{L^1(T)} + \frac{2^q\|\gamma\|}{q\epsilon^q}\Big(\|a_2\|^q_{L^q(T)} + c_2^q\|\gamma\|^2 b^{\frac{p-2}{p}}\|x\|^2_{\mathcal{V}}\Big). \tag{8}$$

We now choose $\epsilon > 0$ such that $\Big(c_1 - \frac{\|\gamma\|\epsilon^p}{p}\Big) > 0$. For such ϵ and $p > 2$, we have that there is a suitable constant $C_1 > 0$ such that

$$\|x\|_{\mathcal{V}} \le C_1. \tag{9}$$

Secondly, if $p = 2$, we have

$$\begin{aligned}\int_T \|f(t)\|_{V^*}\|x(t)\|_V dt &\le \|\gamma\| \int_T \Big(a_2(t) + c_2|x(t)|_H\Big)\|x(t)\|_V dt\\ &\le \frac{\|\gamma\|}{2\epsilon^2}\|a_2\|^2_{L^2(T)} + \frac{\|\gamma\|\epsilon^2}{2}\|x\|^2_{\mathcal{V}} + c_2\|\gamma\|^2\|x\|^2_{\mathcal{V}}.\end{aligned} \tag{10}$$

Hence we have

$$\Big(c_1 - \frac{\|\gamma\|\epsilon^2}{2} - c_2\|\gamma\|^2\Big)\|x\|^2_{\mathcal{V}} \le \|a_1\|_{L^1(T)} + \frac{\|\gamma\|}{2\epsilon^2}\|a_2\|^2_{L^2(T)}. \tag{11}$$

We choose $\epsilon > 0$ such that $\Big(c_1 - \frac{\|\gamma\|\epsilon^2}{2}\Big) = \frac{c_1}{2}$. For such ϵ and from H_0, we obtain the inequality (9) is also valid in this case for suitable constant C_1.

To end the proof, it is enough to show the boundedness of $\|x'\|_{\mathcal{V}^*}$. Using equation (1), H(A)(3), H(B) and $H(F)_1(3)$, we can have the followings:

$$\|x'\|_{\mathcal{V}^*} \le \|f\|_{\mathcal{V}^*} + \|\mathcal{A}x\|_{\mathcal{V}^*} + \|\mathcal{B}x\|_{\mathcal{V}^*}, \tag{12}$$

$$\|\mathcal{A}x\|_{\mathcal{V}^*} \le \Big(\int_T \Big(a(t) + c\|x(t)\|_V^{p-1}\Big)^q dt\Big)^{1/q} \le 2^{\frac{q-1}{q}}\Big(\|a\|_{L^q} + c\|x\|^{p/q}_{\mathcal{V}}\Big), \tag{13}$$

$$\|\mathcal{B}x\|_{\mathcal{V}^*} \le \|B\|\Big(\int_T \|x(t)\|_V^q dt\Big)^{1/q} \le \|B\| b^{\frac{p-q}{pq}}\|x\|_{\mathcal{V}}, \tag{14}$$

$$\begin{aligned}\|f\|_{\mathcal{V}^*} &\le \Big(\int_T \Big[\|\gamma\|\Big(a_2(t) + c_2(\|\gamma\|\|x(t)\|_V)^{2/q}\Big)\Big]^q dt\Big)^{1/q}\\ &\le \|\gamma\| 2^{\frac{q-1}{q}}\Big(\|a_2\|_{L^q} + c_2\|\gamma\|^{\frac{2}{q}} b^{\frac{p-2}{pq}}\|x\|^{\frac{2}{q}}_{\mathcal{V}}\Big).\end{aligned} \tag{15}$$

From above inequalities (12)-(15) and (9), we can deduce that $\|x'\|_{\mathcal{V}^*} \le C_2$, here C_2 being a suitable constant. This lemma is proved. $\square$

Since $\mathcal{W}$ is embedded continuously into $C(T, H)$, from Lemma 3.2, we can assume that there exists a positive constant $M > 0$ such that for any solution x to problem (P1), $|x(t)|_H \leq M$ for all $t \in T$ (Proposition 23.23 [24]).

Let $Q = \{h \in H : |h|_H \leq M\}$. Considering the projection $\mathrm{pr} : H \to Q$ relating to each point $h \in H$ a unique point $\mathrm{pr}\, h \in Q$ such that $|\mathrm{pr}\, h - h|_H = \min\{|y - h|_H : y \in Q\}$, we know that $|\mathrm{pr}\, u - \mathrm{pr}\, v|_H \leq |u - v|_H$, for all u, $v \in H$. We define $F_1(t, x) = F(t, \mathrm{pr}\, x)$. Evidently F_1 satisfies $H(F)_1(1)$ and (2). Moreover, by the properties of pr, we have, for a.e. $t \in T$, all $x \in H$ and all $u \in F_1(t, x)$ such that

$$|u|_H \leq \varphi(t) = a_2(t) + c_2 M^{\frac{2}{q}}, \ \varphi \in L^q(T) \text{ and } |u|_H \leq a_2(t) + c_2 |x|_H^{\frac{2}{q}}. \tag{16}$$

Hence, Lemma 3.2 is still valid with $F(t, x)$ substituted by $F_1(t, x)$. Consequently, henceforth we assume without any loss of generality that

$$\text{For a.e. } t \in T, \text{ all } x \in H, \ u \in F(t, x), \ |u|_H \leq \varphi(t) \text{ with } \varphi \in L^q(T). \tag{17}$$

Theorem 3.3. *If hypotheses $H(A)$, $H(B)$, $H(F)_1$ and H_0 hold, then problem (P1) has at least one solution.*

Proof. Let us define an operator $G : \mathcal{V} \to 2^{\mathcal{H}^*}$ by

$$G(v) = S^q_{F(\cdot, v(\cdot))}, \text{ for all } v \in \mathcal{V}. \tag{18}$$

We claim that for all $v \in \mathcal{V}$, $G(v) \in P_{wkc}(\mathcal{H}^*)$. Because $H(F)_1(1)$ and $H(F)_1(2)$ in general do not imply that F is jointly measurable (see p.227 [23]), we need to show that $G(v) \neq \emptyset$. To this end, we consider step functions $\{r_n\}_{n \geq 1}$ such that $r_n(t) \to v(t)$ a.e. on T as $n \to \infty$. Since $t \to F(t, x)$ is measurable for all $x \in H$, for every $n \geq 1$, we have that $t \to F(t, r_n(t))$ is measurable. Invoking Yankov-von Neumann-Aumann selection theorem (Theorem 2.2.14 [23]), we can find $g_n : T \to H$ measurable function such that $g_n(t) \in F(t, r_n(t))$ a.e. on T. By (17), passing to a subsequence, if necessary, we may suppose that $g_n \to g$ weakly in $\mathcal{H}^*$. Then from Proposition 7.3.9 [23] (p.694), we have that for a.e. on T,

$$g(t) \in \overline{\mathrm{co}}\ \omega\text{-}\overline{\lim}\{g_n(t)\} \subseteq \overline{\mathrm{co}}\ \omega\text{-}\overline{\lim} F(t, r_n(t)) \subseteq F(t, v(t)). \tag{19}$$

The last inclusion is a consequence of the fact that F is $P_{fc}(H)$ valued and for a.e. $t \in T$, $x \to F(t, x)$ is sequentially closed in $H \times \omega$-H ($H(F)_1(2)$). It is clear that $g \in \mathcal{H}^*$ and $g \in G(v)$. As for verifying $G(v) \in P_{wkc}(\mathcal{H}^*)$, this can be deduced from F is $P_{fc}(H)$ valued and formula (17).

Let L, $\mathcal{A}$, $\mathcal{B}$ and G be defined by, respectively, (2), (3) and (18). We consider the following problem:

$$\text{Find } x \in D(L) \text{ such that } Lx + \mathcal{P}x \ni 0, \tag{20}$$

where the operator $\mathcal{P} : \mathcal{V} \to 2^{\mathcal{V}^*}$ is given by $\mathcal{P}x = \mathcal{A}x + \mathcal{B}x - Gx$ for $x \in \mathcal{V}$. Note that $x \in \mathcal{W}$ solves problem (P1) if and only if x solves (20). Next, we show the existence of a solution to (20) by using Proposition 2.2.

Claim 1: L defined by (2) is a linear maximal monotone operator.

From Proposition 2.3, we have the maximal monotonicity of L. It is easy to see that L is a linear and densely defined operator (also see Theorem 32.L [24]).

Claim 2: The operator $\mathcal{P}$ is: (i) bounded, (ii) coercive, (iii) pseudomonotone with respect to $D(L)$.

Proof of (i): This follows from (13), (14) and (17) and the continuity of the embedding $\mathcal{H}^* \subseteq \mathcal{V}^*$.

Proof of (ii): Let $v \in \mathcal{V}$ and $\eta \in \mathcal{P}v$. Thus $\eta = \mathcal{A}v + \mathcal{B}v - \xi$ with some $\xi \in Gv$. From H(A)(4), H(B) and (17), we have

$$\begin{aligned} \langle\langle \eta, v \rangle\rangle &= \langle\langle \mathcal{A}v, v \rangle\rangle + \langle\langle \mathcal{B}v, v \rangle\rangle - \langle\langle \xi, v \rangle\rangle \\ &\geq c_1 \|v\|^p_{\mathcal{V}} - \|a_1\|_{L^1(T)} - \|\gamma\| \|\varphi\|_{L^q(T)} \|v\|_{\mathcal{V}}. \end{aligned}$$

This implies that $\mathcal{P}$ is coercive.

Proof of (iii): The fact that for every $v \in \mathcal{V}$ the set $\mathcal{P}v \in P_{wkc}(\mathcal{V}^*)$ due to the property of the operator G defined by (18).

We show that $\mathcal{P}$ is u.s.c. in $\mathcal{V} \times \omega$-$\mathcal{V}^*$ topology. To this end, we prove that if a set D is weakly closed in $\mathcal{V}^*$, then the set $\mathcal{P}^-(D) = \{v \in \mathcal{V} : \mathcal{P}v \cap D \neq \emptyset\}$ is closed in $\mathcal{V}$. Let $\{v_n\}_{n\geq 1} \subseteq \mathcal{P}^-(D)$ be such that $v_n \to v$ in $\mathcal{V}$. Then there exists $\eta_n \in \mathcal{P}v_n \cap D$, so we have

$$\eta_n = \mathcal{A}v_n + \mathcal{B}v_n - \xi_n \tag{21}$$

with $\xi_n \in Gv_n$. Since $\{v_n\}_{n\geq 1}$ is bounded and $\mathcal{P}$ is a bounded operator. Passing to a subsequence, if necessary, we may assume that $\eta_n \to \eta$ weakly in $\mathcal{V}^*$, with $\eta \in D$, since D is closed in ω-$\mathcal{V}^*$. By the boundedness of $\{\xi_n\}_{n\geq 1}$ in $\mathcal{H}^*$, we again may assume that $\xi_n \to \xi$ weakly in $\mathcal{H}^*$. By the same reason as (19), we have $\xi \in Gv$. Now from Proposition 3.1 (see [16]), we obtain $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$ (since $\mathcal{A}$ is demicontinuous). It is obvious that $\mathcal{B}$ is a continuous linear operator, so we have $\mathcal{B}v_n \to \mathcal{B}v$ in $\mathcal{V}^*$. Passing to the limit in (21), we have

$$\eta = \mathcal{A}v + \mathcal{B}v - \xi.$$

Since $\xi \in Gv$ and $\eta \in D$, this implies that $\eta \in \mathcal{P}v \cap D$. Therefore, $\mathcal{P}^-(D)$ is closed in $\mathcal{V}$.

To conclude that $\mathcal{P}$ is pseudomonotone with respect to $D(L)$. It is enough to prove item (d) in the definition of the pseudomonotone operator. Let $\{v_n\}_{n\geq 1} \subseteq D(L)$, $v_n \to v$ weakly in $\mathcal{W}$, $\eta_n \in \mathcal{P}v_n$, $\eta_n \to \eta$ weakly in $\mathcal{V}^*$, and assume that

$$\limsup \langle\langle \eta_n, v_n - v\rangle\rangle \leq 0. \tag{22}$$

We have $\eta_n = \mathcal{A}v_n + \mathcal{B}v_n - \xi_n$ with $\xi_n \in Gv_n$. Since G is a bounded operator and $\{v_n\}_{n\geq 1}$ is bounded in $\mathcal{V}$, we may assume that

$$\xi_n \to \xi \text{ weakly in } \mathcal{H}^*. \tag{23}$$

Since the embedding $\mathcal{W} \subseteq \mathcal{H}$ is compact, we have

$$v_n \to v \text{ in } \mathcal{H}. \tag{24}$$

From (23) and (24), we deduce $\xi \in Gv$ (see (19)). It is clear that $\mathcal{B}v_n \to \mathcal{B}v$ weakly in $\mathcal{V}^*$ and $\mathcal{B}$ is monotone, so we have $\langle\langle \mathcal{B}v_n, v_n - v\rangle\rangle \geq \langle\langle \mathcal{B}v, v_n - v\rangle\rangle$. Taking limit of this inequality, we obtain

$$\liminf \langle\langle \mathcal{B}v_n, v_n - v\rangle\rangle \geq 0 \text{ and } \limsup -\langle\langle \mathcal{B}v_n, v_n - v\rangle\rangle \leq 0. \tag{25}$$

From (24) and $\{\xi_n\}_{n\geq 1}$ is bounded in $\mathcal{H}^*$, we have

$$\lim \langle\langle \xi_n, v_n - v\rangle\rangle = 0. \tag{26}$$

Combining (22), (25) and (26), we obtain

$$\begin{aligned}\limsup \langle\langle \mathcal{A}v_n, v_n - v\rangle\rangle &\leq \limsup \langle\langle \eta_n, v_n - v\rangle\rangle \\ &\quad + \limsup -\langle\langle \mathcal{B}v_n, v_n - v\rangle\rangle + \lim \langle\langle \xi_n, v_n - v\rangle\rangle \leq 0.\end{aligned}$$

From Proposition 3.1 (see [16]), we obtain $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$ and

$$\langle\langle \mathcal{A}v_n, v_n\rangle\rangle \to \langle\langle \mathcal{A}v, v\rangle\rangle. \tag{27}$$

Next we show

$$\langle\langle \mathcal{B}v_n, v_n\rangle\rangle \to \langle\langle \mathcal{B}v, v\rangle\rangle. \tag{28}$$

In fact, from (22), (26) and (27), we deduce

$$\begin{aligned}\limsup \langle\langle \mathcal{B}v_n, v_n - v\rangle\rangle &\leq \limsup \langle\langle \eta_n, v_n - v\rangle\rangle \\ &\quad + \limsup -\langle\langle \mathcal{A}v_n, v_n - v\rangle\rangle + \lim \langle\langle \xi_n, v_n - v\rangle\rangle \leq 0.\end{aligned}$$

This together with the first inequality of (25) implies that $\lim\langle\langle \mathcal{B}v_n, v_n - v\rangle\rangle = 0$ and (28). Passing to the limit in the equality

$$\langle\langle \eta_n, v_n\rangle\rangle = \langle\langle \mathcal{A}v_n, v_n\rangle\rangle + \langle\langle \mathcal{B}v_n, v_n\rangle\rangle - \langle\langle \xi_n, v_n\rangle\rangle,$$

and from (26), (27) and (28), we obtain $\lim\langle\langle \eta_n, v_n\rangle\rangle = \langle\langle \eta, v\rangle\rangle$. As before, we can show that $\eta \in \mathcal{P}v$. Thus $\mathcal{P}$ is pseudomonotone with respect to $D(L)$.

Since $\mathcal{V}$ is a strictly convex Banach space (this follows from the fact that in every reflexive Banach space there exists an equivalent norm such that the space is strictly convex, see p.256 [24]), by Proposition 2.2, we come to the conclusion that this theorem is proved. □

4. Existence for Nonconvex Case

In this section we prove two existence theorems for nonconvex problems. The first assumes l.s.c. of the multivalued nonlinearity, while the second, that is to say problem (P2), concerns "extremal solutions" which are important in optimal control theory, in connection with the "bang bang" control.

For the first result our hypotheses on $F(t, x)$ are the following:

H(F)$_2$: $F : T \times H \to P_f(H)$ is a multifunction such that:

(1) $(t, x) \to F(t, x)$ is measurable;

(2) For a.e. $t \in T$, $x \to F(t, x)$ is l.s.c.;

(3) For a.e. $t \in T$, all $x \in H$ and all $u \in F(t, x)$, $|u|_H \leq a_2(t) + c_2|x|_H^{\frac{2}{q}}$ with $a_2 \in L^q(T)$, $c_2 > 0$.

Theorem 4.1. *If hypotheses H(A), H(B), H(F)$_2$ and H$_0$ hold, then problem (P1) has at least one solution.*

Proof. From $H(F)_2(1)$, we know that the map $t \to F(t, x(t))$ is measurable for any measurable function $x : T \to H$ and its valus are closed. Let $G : \mathcal{V} \to 2^{\mathcal{H}^*}$ be defined by (18), we can easily get $Gv \in P_f(\mathcal{H}^*)$ for all $v \in \mathcal{V}$. Gv is also decomposable valued (i.e., if $D \in \Sigma$ and $f_1, f_2 \in Gv$, then $\chi_D f_1 + \chi_{D^c} f_2 \in Gv$).

We claim that $v \to Gv$ is a l.s.c. map. Let $v \in \mathcal{V}$, $h \in Gv$ and $v_n \in \mathcal{V}$, $n \geq 1$, be a sequence converging to v. Passing to a subsequence, if necessary, we may suppose that $v_n(t) \to v(t)$ a.e $t \in T$. It follows from the properties of measurable multivalued maps [25] that there is a sequence $h_n \in Gv_n$, $n \geq 1$, such that

$$|h(t) - h_n(t)|_H \leq d_H\Big(h(t), F(t, v_n(t))\Big) + \frac{1}{n}, \text{ a.e. } t \in T, \tag{29}$$

where $d_H(\cdot, \cdot)$ is the distance from a point to a set in H. Since the map $x \to F(t, x)$ is l.s.c. for a.e. $t \in T$, Proposition 1.2.26 in [23] implies that the map $y \to d_H(h(t), F(t, y))$ is u.s.c. for a.e. $t \in T$. Then from (29), we get

$$\lim_{n\to\infty} |h(t) - h_n(t)|_H = 0, \text{ a.e. } t \in T.$$

Combining this equality with (17) (since under the hypotheses $H(F)_2$, we also have lemma 3.2) and using Lebesgue's theorem on dominated convergence, we obtain that $h_n \to h$ in $\mathcal{H}^*$. Therefore the map $v \to Gv$ is l.s.c.

Theorem 2.8.7 in [23] (see also Proposition 2.2 [26]) enables us to have $g : \mathcal{V} \to \mathcal{H}^*$ a continuous map such that $g(v) \in Gv$ for all $v \in \mathcal{V}$. Consider the following problem:

$$\begin{cases} x'(t) + A(t, x(t)) + Bx(t) = g(x)(t) \text{ a.e. on } T, \\ x(0) = -x(b). \end{cases} \tag{30}$$

Arguing as in the proof of Theorem 3.3, we obtain a solution x to problem (30). Then by the definition of G, we deduce that x is a solution to problem (P1). The theorem is proved. □

In the following of this section, we will consider the other nonconvex problem, i.e. problem (P2):

$$(P2)\quad \begin{cases} x'(t) + A(t, x(t)) + Bx(t) \in F(t, x(t)) \bigcap \operatorname{ext}\overline{\operatorname{co}}\, F(t, x(t)), \text{ a.e. on } T, \\ x(0) = -x(b). \end{cases}$$

For a multivalud map F, we remark that even if $x \to F(t, x)$ has nice continuity properties, $x \to \operatorname{ext} F(t, x)$ need not be even closed valued. Moreover, in the general case, we have $\operatorname{ext}\overline{\operatorname{co}}\, F(t, x) \not\subseteq F(t, x)$. Hence, Tolstonogov in [19] recommended that when inclusions with nonconvex right-hand side $F(t, x)$ are considered, it is more natural to study inclusions with the right-hand side $F(t, x) \bigcap \operatorname{ext}\overline{\operatorname{co}}\, F(t, x)$, since the set $\operatorname{ext}\overline{\operatorname{co}}\, F(t, x)$ is determined by the set $\overline{\operatorname{co}}\, F(t, x)$ and the set $F(t, x)$ is essentially unclaimed.

In this case our hypotheses on the data are the following:

H(A)$_1$: $A : T \times V \to V^*$ is an operator such that H(A)(1), (3) and (4) hold and the following:
(2′) For a.e. $t \in T$, $x \to A(t, x)$ is hemicontinuous and strictly monotone.

H(F)$_3$: $F : T \times H \to P_f(H)$ is a multifunction such that:
(1) $(t, x) \to F(t, x)$ is measurable;
(2) For a.e. $t \in T$, $x \to \overline{\operatorname{co}}\, F(t, x)$ is h-continuous;
(3) For a.e. $t \in T$, all $x \in H$ and all $u \in F(t, x)$, $|u|_H \le a_2(t) + c_2|x|_H^{\frac{2}{q}}$ with $a_2 \in L^q(T)$, $c_2 > 0$.

From $H(F)_3(3)$ it follows that for every $x \in H$, a.e. $t \in T$, $\overline{\operatorname{co}}\, F(t, x)$ is a convex compact subset of ω-H. So, by the Krein-Milman theorem we have that $\operatorname{ext}\overline{\operatorname{co}}\, F(t, x)$ has nonempty values for a.e. $t \in T$. It should be mentioned that $\operatorname{ext}\overline{\operatorname{co}}\, F(t, x) \not\subseteq F(t, x)$ since $F(t, x)$ is only a closed subset of H, not a closed subset of ω-H. However, $F(t, x) \bigcap \operatorname{ext}\overline{\operatorname{co}}\, F(t, x) \neq \emptyset$. For example, strongly exposed points of $\overline{\operatorname{co}}\, F(t, x)$ belong to $F(t, x) \bigcap \operatorname{ext}\overline{\operatorname{co}}\, F(t, x)$ [26]. It means that problem (P2) is well defined.

We first consider the following equation:

$$\begin{cases} x'(t) + A(t, x(t)) + Bx(t) = f(t) \text{ a.e. on } T, \\ x(0) = -x(b). \end{cases} \tag{31}$$

Recall that if $A : X \to X^*$, here X is a real reflexive Banach space, is monotone and hemicontinuous then A is pseudomonotone (Proposition 27.6 [24]). If A is pseudomonotone and locally bounded, then A is demicontinuous (Proposition 27.7 [24]). Hence Theorem 3.3 implies that for every $f \in \mathcal{V}^*$ problem (31) has a solution.

Lemma 4.2. *Under the hypotheses $H(A)_1$ and $H(B)$, for every $f \in \mathcal{V}^*$, problem (31) has a unique solution.*

Proof. Let $x_1 \neq x_2$ be two solutions of problem (31), i.e. for $i = 1, 2$, we have

$$\begin{cases} x_i'(t) + A(t, x_i(t)) + Bx_i(t) = f(t) \text{ a.e. on } T, \\ x_i(0) = -x_i(b). \end{cases}$$

Subtracting these two equations, multiplying the result by $x_1(t) - x_2(t)$ and integrating over T, we obtain

$$\frac{1}{2}|x_1(b) - x_2(b)|_H^2 < \frac{1}{2}|x_1(0) - x_2(0)|_H^2, \tag{32}$$

since $x \to A(t, x)$ is strictly monotone and B is monotone. Due to the anti-periodic conditions $x_1(0) = -x_1(b)$ and $x_2(0) = -x_2(b)$, from (32), we have

$$|x_1(0) - x_2(0)|_H < |x_1(0) - x_2(0)|_H.$$

This is a contradiction. The proof is complete. □

It is obviously that the a priori estimate Lemma 3.2 is still valid in this case. Hence we can assume $H(F)_3$ satisfies (17). We put

$$S_\varphi = \{f \in \mathcal{H}^* : |f(t)|_H \le \varphi(t) \text{ a.e. on } T\},\ \varphi \in L^q(T).$$

The following result concerns the solution map for (31) and plays an important role in the proof of the existence result for problem (P2).

Lemma 4.3. *Under assumptions $H(A)_1$, $H(B)$ and H_0, the map P which to every right-hand side $f \in S_\varphi$ assigns the unique solution $x = P(f)$ of problem (31) is continuous from ω-S_φ into $C(T, H)$.*

Proof. Since S_φ is a metrizable convex compact subset of the space ω-$\mathcal{H}^*$, we need only to prove the sequential continuity of the map $f \to P(f)$.

Let $\{f_n\}_{n\geq 1} \subseteq S_\varphi$ be such that

$$f_n \to f \text{ in } \omega\text{-}\mathcal{H}^* \tag{33}$$

and $x_n = P(f_n)$, $n \geq 1$. From the compactness of the embedding $\mathcal{W} \hookrightarrow \mathcal{H}$ and the continuity of the embedding $\mathcal{W} \hookrightarrow C(T, H)$ it follows that there exist a subsequence x_{n_k}, $k \geq 1$ of the sequence x_n, $n \geq 1$, which we will denote in the sequel by x_k, $k \geq 1$, and $x \in \mathcal{W}$ with the following properties:

a) $f_k \to f$ in ω-$\mathcal{H}^*$ and ω-$\mathcal{V}^*$;

b) $x_k \to x$ in ω-$\mathcal{V}$;

c) $x'_k \to x'$ in ω-$\mathcal{V}^*$;

d) $x_k \to x$ in $\mathcal{H}$;

e) $x_k \to x$ in ω-$C(T, H)$;

f) $x_k(t) \to x(t)$ in H a.e. on T.

From e) it follows that $x(0) = -x(b)$ and thus

$$x \in D(L). \tag{34}$$

Taking account of (34) we rewrite an evident equality

$$\langle\langle x'_k, x_k - x\rangle\rangle + \langle\langle \mathcal{A}x_k, x_k - x\rangle\rangle + \langle\langle \mathcal{B}x_k, x_k - x\rangle\rangle = \langle\langle f_k, x_k - x\rangle\rangle$$

in the form

$$\begin{aligned}&\langle\langle Lx_k - Lx, x_k - x\rangle\rangle + \langle\langle x', x_k - x\rangle\rangle + \langle\langle \mathcal{A}x_k, x_k - x\rangle\rangle \\ &\qquad + \langle\langle \mathcal{B}x_k - \mathcal{B}x, x_k - x\rangle\rangle + \langle\langle \mathcal{B}x, x_k - x\rangle\rangle = \langle\langle f_k, x_k - x\rangle\rangle.\end{aligned}$$

From this equality, the monotonicity of L and the fact that $\langle\cdot,\cdot\rangle_{H\times V} = (\cdot,\cdot)$ we infer that

$$\langle\langle \mathcal{A}x_k, x_k - x\rangle\rangle \leq \langle\langle x', x - x_k\rangle\rangle + \langle\langle \mathcal{B}x, x - x_k\rangle\rangle + \|f_k\|_{\mathcal{H}^*}\|x_k - x\|_{\mathcal{H}}. \tag{35}$$

From (35) and a), b), d) we obtain

$$\limsup_{k\to\infty}\langle\langle \mathcal{A}x_k, x_k - x\rangle\rangle \leq 0.$$

Therefore, according to Proposition 3.1 (see [16]) we have

$$\mathcal{A}x_k \to \mathcal{A}x \text{ in } \omega\text{-}\mathcal{V}^*. \tag{36}$$

From a), b), c), (36) and (34) it follows that

$$\begin{aligned} x' + \mathcal{A}x + \mathcal{B}x &= f, \\ x(0) &= -x(b), \end{aligned} \tag{37}$$

i.e. $x = P(f)$.

From f) it follows that there exists $s \in T$ such that

$$x_k(s) \to x(s) \text{ in } H. \tag{38}$$

From the equality $\langle\cdot,\cdot\rangle_{H\times V} = (\cdot,\cdot)$, the monotonicity of A and B and the known integration by parts formula we infer that

$$\sup_{s\leq t\leq b}\frac{1}{2}|x_k(t) - x(t)|_H^2 \leq \frac{1}{2}|x_k(s) - x(s)|_H^2 + \|f_k - f\|_{\mathcal{H}^*}\|x_k - x\|_{\mathcal{H}}, \tag{39}$$

$$\sup_{0\le t\le s}\frac{1}{2}|x_k(t)-x(t)|_H^2 \le \frac{1}{2}|x_k(0)-x(0)|_H^2 + \|f_k - f\|_{\mathcal{H}^*}\|x_k - x\|_{\mathcal{H}}. \tag{40}$$

Using a), d), (38) and passing to the limit in (39), (40) we see that

$$x_k \to x \text{ in } C(T, H).$$

We have thus proved that if $f_n \to f$ in ω-$\mathcal{H}^*$, then there exits a subsequence f_{n_k}, $k \ge 1$ of the sequence f_n, $n \ge 1$ such that

$$x_k = P(f_{n_k}) \to x = P(f) \text{ in } C(T, H).$$

Using the well-known arguments to prove by contradiction and the uniqueness of a solution of equation (37) we infer that

$$x_n = P(f_n) \to x = P(f) \text{ in } C(T, H).$$

Lemma 4.3 is proved. □

Now it is time to give the existence result for problem (P2).

Theorem 4.4. *Under assumptions $H(A)_1$, H(B), $H(F)_3$ and H_0, then problem (P2) has a solution.*

Proof. Let $\Gamma = \{P(f) : f \in S_\varphi\}$ be the solution set of equation (31) with $f \in S_\varphi$. Since S_φ is a convex compact subset of ω-$\mathcal{H}^*$, then, according to Lemma 4.3, Γ is compact in ω-$\mathcal{W}$ and in $C(T, H)$.

From $H(F)_3(1)$, it follows that for every $x(\cdot) \in C(T, H)$ the multifunction $F(t, x(t))$ is measurable and $t \to F(t, x)$ is measurable for all $x \in H$. Then by Theorem 9.1 in [25], we have $t \to \overline{\text{co}}\, F(t, x)$ is measurable for all $x \in H$. Hence according to $H(F)_3(2)$, the multifunction $\overline{\text{co}}\, F(t, x)$ is of Carathéodory type. Now taking into consideration Proposition 8.2 of [26], we have that there exists a continuous function $g : \Gamma \subseteq C(T, H) \to \mathcal{H}^*$ such that, for every $x(\cdot) \in \Gamma$ and a.e. on T,

$$g(x)(t) \in F(t, x(t)) \text{ and } g(x)(t) \in \text{ext}\overline{\text{co}}\, F(t, x(t)), \tag{41}$$

i.e. $g(x) \in S^q_{F(\cdot,x(\cdot))} \cap S^q_{\text{ext}\overline{\text{co}}\, F(\cdot,x(\cdot))}$ for every $x \in \Gamma$.

Let an operator $\mathcal{S} : S_\varphi \to \mathcal{H}^*$ defined by $\mathcal{S}(f) = g \cdot P(f)$ which is continuous from ω-S_φ into $\mathcal{H}^*$ and hence from ω-S_φ into ω-$\mathcal{H}^*$ (by lemma 4.3 and the continuity of g). As mentioned above, the a priori estimate Lemma 3.2 is still valid in this situation. Hence by the definition of φ (see (16)), we have $\mathcal{S}(f) \in S_\varphi$ for every $f \in S_\varphi$. Since S_φ is a metrizable convex compact subset of the space ω-$\mathcal{H}^*$, and $\mathcal{S}$ is a continuous map from ω-S_φ into ω-S_φ, then by Schauder's fixed point theorem, we deduce that there exists $f_* \in S_\varphi$ such that $f_* = \mathcal{S}(f_*) = g \cdot P(f_*)$. We put $x_* = P(f_*)$, then $f_* = g(x_*)$. From (41), we have

$$f_*(t) = g(x_*)(t) \in F(t, x_*(t)) \bigcap \text{ext}\overline{\text{co}}\, F(t, x_*(t)) \text{ a.e. on } T.$$

This together with $x_* = P(f_*)$ implies that problem (P2) has a solution x_*. The theorem is proved. □

5. An Example

In this final section, we give an example to illustrate our abstract results.

Let $T = [0, b]$ and Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. We consider the following nonlinear parabolic problem with a discontinuous right-hand side:

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{k=1}^{N} D_k a_k(x, t, u, Du) + a_0(x, t, u, Du) - \sum_{i,j=1}^{N} D_i(a_{ij}(x) D_j u) \\ = f(x, t, u), \text{ a.e. on } \Omega \times T, \\ u(x, t)|_{\partial\Omega\times T} = 0, \text{ and } u(x, 0) = -u(x, b) \text{ a.e. on } \Omega. \end{aligned} \tag{42}$$

Here $D_k = \frac{\partial}{\partial x_k}$, D =gradient. Since $f(x,t,\cdot)$ is not continuous, problem (42) need not have solutions. To obtain an existnece theorem for problem (42) we pass to a multivalued problem by filling in the gaps at the discontinuity points of $f(x,t,\cdot)$. We introduce the functions $f_1(x,t,u)$ and $f_2(x,t,u)$ defined by

$$f_1(x,t,u) = \underline{\lim}_{z\to u} f(x,t,z) = \sup_{\epsilon>0} \inf_{|z-u|<\epsilon} f(x,t,z),$$

$$f_2(x,t,u) = \overline{\lim}_{z\to u} f(x,t,z) = \inf_{\epsilon>0} \sup_{|z-u|<\epsilon} f(x,t,z).$$

Put $\hat{f}(x,t,u) = [f_1(x,t,u), f_2(x,t,u)] = \{v \in \mathbb{R} : f_1(x,t,u) \le v \le f_2(x,t,u)\}$. Then, instead of (42), we study the following multivalued problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \sum_{k=1}^{N} D_k a_k(x,t,u,Du) + a_0(x,t,u,Du) - \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j u) \\ &\qquad\qquad \in \hat{f}(x,t,u), \text{ a.e. on } \Omega\times T, \\ &u(x,t)|_{\partial\Omega\times T} = 0, \text{ and } u(x,0) = -u(x,b) \text{ a.e. on } \Omega. \end{aligned} \tag{43}$$

The hypotheses on the data of this problem are the following.

H(a): $a_k : \Omega\times T\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, $k = 1,2,\cdots,N$, are functions such that:
(1) Carathéodory and growth condition: Each $a_k(x,t,u,\xi)$ satisfies, for every $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$, $(x,t) \to a_k(x,t,u,\xi)$ is measurable, for a.e. $(x,t) \in \Omega\times T$, $(u,\xi) \to a_k(x,t,u,\xi)$ is continuous. A constant $c_1 > 0$ and a function $\beta_1 \in L^q(\Omega\times T)$ exist such that

$$|a_k(x,t,u,\xi)| \le \beta_1(x,t) + c_1(|u|^{p-1} + \|\xi\|^{p-1}) \tag{44}$$

for a.e. $(x,t) \in \Omega\times T$ and all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$, with $\|\xi\|$ denoting the Euclidian norm of the vector ξ.
(2) Monotonicity type condition: for a.e. $(x,t) \in \Omega\times T$, all $u \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \ne \xi'$, we have

$$\sum_{k=1}^{N} \Big(a_k(x,t,u,\xi) - a_k(x,t,u,\xi')\Big)(\xi_k - \xi'_k) > 0.$$

(3) Coercivity type condition: for a.e. $(x,t) \in \Omega\times T$ and all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with some constant $c_2 > 0$ and some function $\beta_2 \in L^1(\Omega\times T)$, we have

$$\sum_{k=1}^{N} a_k(x,t,u,\xi)\xi_k \ge c_2\|\xi\|^p - \beta_2(x,t).$$

H(a$_0$): $a_0 : \Omega\times T\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a function which satisfies H(a)(1). Here, of couuse, the corresponding formula similar to (44) may have different constants β_1 and c_1.

H(a$_1$): Let $i,j = 1,2,\cdots,N$, $a_{ij} \in L^\infty(\Omega)$ such that, for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge 0$.

H(f): f_1, f_2 are N-measurable, i.e., for all $u : \Omega\times T \to \mathbb{R}$ measurable, $(x,t) \to f_i(x,t,u(x,t))$ is measurable for $i = 1,2$ and

$$|f(x,t,u)| \le \beta_3(x,t) + c_3|u|,$$

for a.e. $(x,t) \in \Omega\times T$ and for all $u \in \mathbb{R}$ with $\beta_3 \in L^q(T, L^2(\Omega))$ and $c_3 > 0$.

Theorem 5.1. *Asume that the hypotheses H(a), H(a$_0$), H(a$_1$) and H(f) hold true, then problem (43) has a solution $u \in L^p(T, W_0^{1,p}(\Omega)) \bigcap C(T, L^2(\Omega))$ such that $\frac{\partial u}{\partial t} \in L^q(T, W^{-1,q}(\Omega))$.*

Proof. In the problem under consideration, the evolution triple is $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $V^* = W^{-1,q}(\Omega)$ (with $2 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). From Sobolev embedding theorems, all embeddings are compact. Let $A : T \times V \to V^*$ be the operator defined by

$$\langle A(t,v), w\rangle = \int_\Omega \sum_{k=1}^N a_k(x,t,v,Dv) D_k w dx + \int_\Omega a_0(x,t,v,Dv) w dx$$

for all $w \in V$. Using hypotheses H(a) and H(a_0), one can easily check that $A(t,v)$ satisfies hypothesis H(A). The pseudomonotonicity of $A(t,\cdot)$ follows the results of Gossez and Mustoven [27].

Let $B \in \mathcal{L}(V, V^*)$ be defined by, for all $w \in V$,

$$\langle Bv, w\rangle = \int_\Omega \sum_{i.j=1}^N a_{ij}(x) D_j v D_i w dx.$$

Because of hypothesis H(a_1), we know that H(B) holds.

Let $F : T \times H \to P_{fc}(H)$ be defined by

$$F(t,v) = \{h \in L^2(\Omega) = H : f_1(x,t,v(x)) \leq h(x) \leq f_2(x,t,v(x)) \text{ a.e. } \Omega \times T\}.$$

We note that $f_1(x,t,\cdot)$ is l.s.c. and $f_2(x,t,\cdot)$ is u.s.c. (see Proposition 1 [28]). So Example 1.2.8 in [23] implies that $\hat{f}(x,t,\cdot)$ is u.s.c. Hypothesis $H(f)$ implies that H(F)$_1$ is satisfied. Then, we rewrite equivalently (43) as problem (P1) with A, B and F defined as above. Finally, we apply Theorem 3.3. This completes the proof. □

References

[1] H. Okochi, On the existence of periodic solutions to nonlinear abstract parabolic equations, Journal of the Mathematical Society of Japan 40 (1988) 541–553.

[2] H. Okochi, On the existence of anti-periodic solutions to a nonlinear evolution equation associated with odd subdifferential operators, Journal of Functional Analysis 91 (1990) 246–258.

[3] S. Aizicovici, M. McKibben, S. Reich, Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities, Nonlinear Analysis: Theory, Methods & Applications 43 (2001) 233–251.

[4] S. Aizicovici, N. H. Pavel, Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space, Journal of Functional Analysis 99 (1991) 387–408.

[5] Y. Q. Chen, Anti-periodic solutions for semilinear evolution equations, Journal of Mathematical Analysis and Applications 315 (2006) 337–348.

[6] Y. Q. Chen, D. O'Regan, R. P. Agarwal, Anti-periodic solutions for semilinear evolution equations in Banach spaces, Journal of Applied Mathematics and Computing 38 (1-2) (2012) 63–70.

[7] Q. Liu, Existence of anti-periodic mild solutions for semilinear evolution equations, Journal of Mathematical Analysis and Applications 377 (2011) 110–120.

[8] Y. Q. Chen, J. J. Nieto, D. O'Regan, Anti-periodic solutions for evolution equations associated with maximal monotone mappings, Applied Mathematics Letters 24 (2011) 302–307.

[9] Z. H. Liu, Anti-periodic solutions to nonlinear evolution equations, Journal of Functional Analysis 258 (2010) 2026–2033.

[10] J. B. Liu, Z. H. Liu, On the existence of anti-periodic solutions for implicit differential equations, Acta Mathematica Hungarica 132(3) (2011) 294–305.

[11] Z.H. Liu, S. Migorski, Analysis and control of differential inclusions with anti-periodic conditions, Proceedings of the Royal Society of Edinburgh,144A(3)(2014)457-467.

[12] S. Migórski, Existence, variational and optimal control problems for nonlinear second order evolution inclusions, Dynamic Systems & Applications 4 (1995) 513–528.

[13] S. Migórski, Existence and relaxation results for nonlinear evolution inclusions revisited, Journal of Applied Mathematics and Stochastic Analysis 8(2) (1995) 143–149.

[14] S. Migórski, Some remarks on extremal solutions of nonlinear first order evolution inclusions, Dynamic Systems & Applications 6 (1997) 21–25.

[15] N. S. Papageorgiou, F. Papalini, F. Renzacci, Existence of solutions and periodic solutions for nonlinear evolution inclusions, Rendiconti del Circolo Matematico di Palermo 48 (1999) 341–364.

[16] N. S. Papageorgiou, N. Yannakakis, Second order nonlinear evolution inclusions I: Existence and relaxation results, Acta Mathematica Sinica, English Series 21(5) (2005) 977–996.

[17] N. S. Papageorgiou, F. Papalini, N. Yannakakis, Nonmonotone, nonlinear evolution inclusions, Mathematical and Computer Modelling 32 (2000) 1345–1366.
[18] A. A. Tolstonogov, D. A. Tolstonogov, On the "bang-bang" principle for nonlinear evolution inclusions, Nonlinear Differential Equations and Applications NoDEA 6 (1999) 101–118.
[19] A. A. Tolstonogov, Properties of solutions to second order evolution control systems. I, Siberian Mathematical Journal 43(4) (2002) 731–745.
[20] J. Y. Park, T. G. Ha, Existence of antiperiodic solutions for hemivariational inequalities, Nonlinear Analysis: Theory, Methods & Applications 68 (2008) 747–767.
[21] J. Y. Park, T. G. Ha, Existence of anti-periodic solutions for quasilinear parabolic hemivariational inequalities, Nonlinear Analysis: Theory, Methods & Applications 71 (2009) 3203–3217.
[22] J. P. Aubin, A. Cellina, Differential Inclusions: Set-Valued Maps and Viability Theory, Springer-Verlag, Berlin, New York, Tokyo, 1984.
[23] S. C. Hu, N. S. Papageorgiou, Handbook of Multivalued Analysis, Volume I: Theory. Kluwer, Dordrecht, The Netherlands, 1997.
[24] E. Zeidler, Nonlinear Functional Analysis and Its Applications, IIA and IIB, Springer, New York, 1990.
[25] C. J. Himmelberg, Measurable relations, Fundamenta Mathematicae 87 (1975) 53–72.
[26] A. A. Tolstonogov, D. A. Tolstonogov, Lp-continuous extreme selectors of multifunctions with decomposable values: Existence theorems, Set-Valued Analysis 4 (1996) 173–203.
[27] J. P. Gossez, V. Mustonen, Pseudo-monotonicity and the Leray-Lions condition, Differential and Integral Equations 6 (1993) 37–45.
[28] K. C. Chang, The obstacle problem and partial differential equations with discontinuous nonlinearities, Communications on Pure and Applied Mathematics 33(2) (1980) 117–146.