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Fixed Point Results for Ćirić Type Weak Contraction in Metric Spaces with Applications to Partial Metric Spaces

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Abstract. Partial metric spaces are generalizations of metric spaces which allow for non-zero self-distances. The need for such a definition was felt in the domain of computer science. Fixed point theory has rapidly developed on this space in recent times. Here we define a Ćirić type weak contraction mapping with the help of discontinuous control functions and show that in a complete metric space such a function has a fixed point. Our main result has several corollaries and is supported with examples. One of the examples shows that the corollaries are properly contained in the theorem. We give applications of our results in partial metric spaces.

1. Introduction

The introduction of the mathematics of nonzero self-distance was felt necessary in the domain of computer science. It was felt that in order to make possible a metric approach to certain problems of denotational semantics [34] the concept of metric needed to be extended. Such a generalization was proposed by Matthews in [23, 24] where he introduced the notion of partial metric space in which the distance of a point from itself may not be zero. Our interest is in fixed point problems.

Independent of the consideration in computer science, the structure is mathematically interesting. There are several natural examples of these spaces. The topology here is T_0 - topology in general. Particularly, fixed point theory on this space has developed rapidly in recent times. Several aspects of this study are discussed in works like [1, 3–5, 20, 21, 29, 33, 37].

In metric spaces we find a lot of efforts to generalize the Banach's contraction mapping principle as, for instances, in [6, 10–13, 25, 35]. Particularly, weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber et al. in Hilbert spaces [2] and subsequently extended to metric spaces by Rhoades [32]. A weak contraction is intermediate to a Banachs contraction and a non-expansive mapping. The weak contraction principle established by Rhoades [32] was further generalized by several authors like Dutta and Choudhury [15], Popescu [30], Choudhury and Kundu [8] etc. There are several other fixed point results of weakly contractive mappings and their generalizations. Some instances of these works are noted in [7, 9, 14, 26].

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In recent years fixed point results has experienced a rapid development in partially ordered metric spaces. An early result in this direction is due to Turinici [36] in which fixed point problems were studied in partially ordered uniform spaces. Later, this branch of fixed point theory has developed through a number of works some of which are in [17, 18, 27, 28, 31]. Particularly, Harjani et. al have established a generalized weak contraction principle in partially ordered metric spaces [17].

The purpose of this paper is to introduce a Ćirić type weak contraction mapping with the help of discontinuous control functions and to establish that such a function has a fixed point in a complete partially ordered metric space. Our main result has several corollaries and is supported with examples. One of the examples shows that the corollaries are properly contained in the theorem. We give applications of our results in partial metric spaces. Our approach is a blending of analytic and order theoretic methods.

2. Mathematical Preliminaries

Definition 2.1 ([23]). A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

$(P1)$ $x = y \iff p(x, x) = p(y, y) = p(x, y)$,
$(P2)$ $p(x, x) \leq p(x, y)$,
$(P3)$ $p(x, y) = p(y, x)$,
$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then (X, p) is called a partial metric space. From $(P1)$ and $(P2)$, it is clear that $p(x, y) = 0$ implies $x = y$. But for $x = y$, $p(x, y)$ may not be 0. Each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p - balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) \leq p(x, x)+\epsilon\}$ for all $x \in X$ and $\varepsilon > 0$ [23]. If p be a partial metric on X, then $d_p : X\times X \to \mathbb{R}^+$ defined as $d_p(x, y) = 2\,p(x, y)-p(x, x)-p(y, y)$ is a metric on X [23]. In fact, $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,\, m\to\infty} p(x_n, x_m)$. Let (X, p) be a partial metric space. A sequence $\{x_n\}$ converges to a point $x \in X$ with respect to τ_p if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. Also, $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,\, m\to\infty} p(x_n, x_m)$ exists and is finite. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,\, m\to\infty} p(x_n, x_m)$. A mapping $T : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $T(B_p(x_0, \delta)) \subseteq B_p(fx_0, \epsilon)$. This definition implies that if a function $T : X \to X$ is continuous then $Tx_n \to Tx$ whenever $x_n \to x$ as $n \to \infty$.

Definition 2.2 ([20]). A sequence $\{x_n\}$ in a partial metric space (X, p) is called 0-Cauchy if $\lim_{n,\, m\to\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if each 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

Note that, each 0-Cauchy sequence in (X, p) is Cauchy in (X, d_p) and every complete partial metric space is 0-complete.

Lemma 2.3 ([23]). *Let (X, p) be a partial metric space. Then:*
(i) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).
(ii) (X, p) is complete if and only if the metric space (X, d_p) is complete.

Proposition 2.4 ([16, 19]). *Let (X, p) be a partial metric space. Then the function $d : X \times X \longrightarrow [0, \infty)$ defined by $d(x, y) = 0$ whenever $x = y$ and $d(x, y) = p(x, y)$ whenever $x \neq y$, is a metric on X such that $\tau_{d_p} \subseteq \tau_d$. Moreover, (X, d) is complete if and only if (X, p) is 0-complete.*

Let (X, p) be a partial metric space, T a self map on X, d the constructed metric in Proposition 2.4 and $x, y \in X$. We define

$$M_d(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(y, Tx) + d(x, Ty)}{2}\right\},$$

and

$$M_p(x,\ y) = \max\left\{p(x,\ y),\ \frac{p(x,\ Tx) + p(y,\ Ty)}{2},\ \frac{p(y,\ Tx) + p(x,\ Ty)}{2}\right\}.$$

Lemma 2.5. *$M_d(x,\ y) = M_p(x,\ y)$, for all $x,\ y \in X$ with $x \neq y$.*

The proof of the lemma is almost identical with that of Lemma 2.2 in [16]. We do not give the details of proof here. Instead, we refer it to [16].

Khan et al. [22] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

Definition 2.6 ([22]). A function $\psi : [0,\ \infty) \longrightarrow [0,\ \infty)$ is called an altering distance function if the following properties are satisfied:
(i) ψ is monotone increasing and continuous;
(ii) $\psi(t) = 0$ if and only if $t = 0$.

In our results in the following sections we will use the following class of functions.
$\Psi = \{\psi\colon [0,\infty) \to [0,\infty) : \psi$ an altering distance function$\}$ and
$\Theta = \{\alpha\colon [0,\infty) \to [0,\infty) : \alpha$ is bounded on any bounded interval in $[0,\ \infty)\}$.

3. Main Results

Theorem 3.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\varphi,\ \theta \in \Theta$ such that*

$$\psi(x) \leq \varphi(y) \implies x \leq y, \tag{1}$$

for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$,

$$\psi(t) - \overline{\lim}\ \varphi(x_n) + \underline{\lim}\ \theta(x_n) > 0, \tag{2}$$

and for all $x,\ y \in X$ with $x \succeq y$,

$$\psi(d(Tx,\ Ty)) \leq \varphi(M(x,\ y)) - \theta(M(x,\ y)), \tag{3}$$

where

$$M(x,\ y) = \max\left\{d(x,\ y),\ \frac{d(x,\ Tx) + d(y,\ Ty)}{2},\ \frac{d(y,\ Tx) + d(x,\ Ty)}{2}\right\}.$$

Also suppose that
(a) T is continuous or
(b) X has the following properties:
(i) if a nondecreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all $n \geq 0$;
(ii) if a nonincreasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all $n \geq 0$.

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. Starting with x_0, we construct the sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n, \text{ for all } n \geq 0. \tag{4}$$

Since T is non-decreasing and $x_0 \preceq Tx_0$, we have

$$x_0 \preceq Tx_0 = x_1 \preceq Tx_1 = x_2 \preceq \ldots \preceq Tx_{n-1} = x_n \preceq Tx_n = x_{n+1} \preceq \ldots. \tag{5}$$

Let $R_n = d(x_{n+1}, x_n)$, for all $n \geq 0$.
Since $x_{n+1} \geq x_n$, from (3) and (4), we have

$$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(Tx_{n+1}, Tx_n)) \leq \varphi(M(x_{n+1}, x_n)) - \theta(M(x_{n+1}, x_n)), \tag{6}$$

where

$$M(x_{n+1}, x_n) = \max\left\{d(x_{n+1}, x_n), \frac{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})}{2}, \frac{d(x_n, x_{n+2})}{2}\right\}.$$

Since $\frac{d(x_n, x_{n+2})}{2} \leq \frac{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})}{2}$, we have

$$M(x_{n+1}, x_n) = \max\left\{d(x_{n+1}, x_n), \frac{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})}{2}\right\}.$$

If $M(x_{n+1}, x_n) = d(x_{n+1}, x_n)$, then it follows from (6) that

$$\psi(d(x_{n+2}, x_{n+1})) \leq \varphi(d(x_{n+1}, x_n)) - \theta(d(x_{n+1}, x_n)),$$

that is,

$$\psi(R_{n+1}) \leq \varphi(R_n) - \theta(R_n), \tag{7}$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi(R_n)$, which by (1) implies that $R_{n+1} \leq R_n$, for all positive integer n.

If $M(x_{n+1}, x_n) = \frac{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})}{2} = \frac{R_{n+1} + R_n}{2} = Q_n$ (*say*), then it follows from (6) that

$$\psi(R_{n+1}) \leq \varphi(Q_n) - \theta(Q_n), \tag{8}$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi(Q_n) = \varphi\left(\frac{R_{n+1} + R_n}{2}\right)$, which by (1) implies that $R_{n+1} \leq \frac{R_{n+1} + R_n}{2}$, that is, $R_{n+1} \leq R_n$, for all positive integer n.

From the above discussion, $\{R_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geq 0$ such that

$$R_n = d(x_{n+1}, x_n) \longrightarrow r \text{ as } n \longrightarrow \infty. \tag{9}$$

Then by (9),

$$Q_n = \frac{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})}{2} \longrightarrow r \text{ as } n \longrightarrow \infty. \tag{10}$$

Taking limit supremum in both sides of (7), using (9), the continuity of ψ and the property of φ and θ, we obtain

$$\psi(r) \leq \overline{\lim}\, \varphi(R_n) + \overline{\lim}\, (-\theta(R_n)).$$

Since $\overline{\lim}\, (-\theta(R_n)) = -\underline{\lim}\, \theta(R_n)$, it follows that

$$\psi(r) \leq \overline{\lim}\, \varphi(R_n) - \underline{\lim}\, \theta(R_n),$$

that is,

$$\psi(r) - \overline{\lim}\, \varphi(R_n) + \underline{\lim}\, \theta(R_n) \leq 0,$$

which, by (2) and (9), is a contradiction unless $r = 0$.

Arguing similarly as mentioned above, from (8) and (10), we have

$$\psi(r) - \overline{\lim}\ \varphi(Q_n) + \underline{\lim}\ \theta(Q_n) \leq 0,$$

which, by (2) and (10), is a contradiction unless $r = 0$. Hence, we have

$$R_n = d(x_{n+1},\ x_n) \longrightarrow 0, \text{ as } n \longrightarrow \infty \tag{11}$$

and

$$Q_n = \frac{d(x_{n+1},\ x_{n+2}) + d(x_n,\ x_{n+1})}{2} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{12}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)},\ x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k,\ d(x_{m(k)},\ x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)},\ x_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{n(k)}),$$

that is,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) \leq \epsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \longrightarrow \infty$ in the above inequality and using (11), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon. \tag{13}$$

Again,

$$d(x_{m(k)+1},\ x_{n(k)+1}) \leq d(x_{m(k)+1},\ x_{m(k)}) + d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)+1},\ x_{m(k)}) + d(x_{m(k)+1},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)}).$$

Letting $k \longrightarrow \infty$ in above inequalities, using (11) and (13), we have

$$\lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1}) = \epsilon. \tag{14}$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)+1}) \leq d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Letting $k \longrightarrow \infty$ in the above inequalities and using (11) and (13), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon. \tag{15}$$

Similarly,

$$\lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)+1}) = \epsilon. \tag{16}$$

As $n(k) > m(k)$, $x_{n(k)} \geq x_{m(k)}$, from (3) and (4), we have

$$\psi(d(x_{n(k)+1}, x_{m(k)+1})) = \psi(d(Tx_{n(k)}, Tx_{m(k)})) \qquad \leq \varphi(M(x_{n(k)}, x_{m(k)})) - \theta(M(x_{n(k)}, x_{m(k)})), \tag{17}$$

where

$$M(x_{n(k)},\ x_{m(k)}) = \max\left\{d(x_{n(k)},\ x_{m(k)}),\ \frac{d(x_{n(k)},\ x_{n(k)+1}) + d(x_{m(k)},\ x_{m(k)+1})}{2}, \frac{d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)},\ x_{m(k)+1})}{2}\right\}.$$

By (11), (13), (15) and (16),

$$\lim_{k\to\infty} M(x_{n(k)},\ x_{m(k)}) = \epsilon. \tag{18}$$

Taking limit supremum in both side of the inequality (17), using (14), (18), the continuity of ψ and the property of φ and θ, we obtain

$$\psi(\epsilon) \leq \overline{\lim}\ \varphi(M(x_{n(k)},\ x_{m(k)})) + \overline{\lim}\ (-\ \theta(M(x_{n(k)},\ x_{m(k)}))).$$

Since $\overline{\lim}\ (-\ \theta(M(x_{n(k)},\ x_{m(k)}))) = -\ \underline{\lim}\ \theta(M(x_{n(k)},\ x_{m(k)}))$, it follows that

$$\psi(\epsilon) \leq \overline{\lim}\ \varphi(M(x_{n(k)},\ x_{m(k)})) - \underline{\lim}\ \theta(M(x_{n(k)},\ x_{m(k)})),$$

that is,

$$\psi(\epsilon) - \overline{\lim}\ \varphi(M(x_{n(k)},\ x_{m(k)})) + \underline{\lim}\ \theta(M(x_{n(k)},\ x_{m(k)})) \leq 0,$$

which, by (2) and (18), is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $z \in X$ such that

$$x_n \longrightarrow z \ \text{ as } \ n \longrightarrow \infty. \tag{19}$$

Let the condition (a) holds.
The continuity of T implies that $Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$, that is, z is a fixed point of T
Next we suppose that the condition (b) holds.
By (5) and (19), we have $x_n \leq z$ for all $n \geq 0$. Then applying (3) and using (4), we have

$$\psi(d(Tz,\ x_{n+1})) = \psi(d(Tz,\ Tx_n)) \leq \varphi(M(z,\ x_n)) - \theta(M(z,\ x_n)), \tag{20}$$

where

$$M(z,\ x_n) = \max\left\{d(z,\ x_n),\ \frac{d(z,\ Tz) + d(x_n,\ x_{n+1})}{2}, \frac{d(x_n,\ Tz) + d(z,\ x_{n+1})}{2}\right\}.$$

By (19), we have

$$\lim_{n\to\infty} M(z,\ x_n) = \frac{d(z,\ Tz)}{2}. \tag{21}$$

Taking limit supremum in both side of the inequality (20), using (19), (21), the properties of ψ and the property of φ and θ, we obtain

$$\psi(\frac{d(Tz,\ z)}{2}) \leq \psi(d(Tz,\ z)) \leq \overline{\lim}\ \varphi(M(z,\ x_n)) + \overline{\lim}\ (-\ \theta(M(z,\ x_n))).$$

Arguing similarly as discussed above, we have

$$\psi\left(\frac{d(Tz,\ z)}{2}\right) - \overline{\lim}\ \varphi(M(z,\ x_n)) + \underline{\lim}\ \theta(M(z,\ x_n)) \leq 0,$$

which, by (2) and (21), is a contradiction unless $d(Tz,\ z) = 0$, that is, $z = Tz$, that is, z is a fixed point of T.

Considering ψ to be the identity mapping and $\theta(t) = 0$ for all $t \in [0,\ \infty)$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\varphi \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\overline{\lim}\ \varphi(x_n) < t$ and for all $x,\ y \in X$ with $x \succeq y$,*

$$d(Tx,\ Ty) \leq \varphi(M(x,\ y)),$$

where $M(x,\ y)$ same as in Theorem 3.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 3.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering φ to be identical with the function ψ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \theta(x_n) > 0$ and for all $x,\ y \in X$ with $x \succeq y$,*

$$\psi(d(Tx,\ Ty)) \leq \psi(M(x,\ y)) - \theta(M(x,\ y)),$$

where $M(x,\ y)$ same as in Theorem 3.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 3.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

If ψ and φ are the identity mappings in Theorem 3.1, we have the following corollary.

Corollary 3.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \theta(x_n) > 0$ and for all $x,\ y \in X$ with $x \succeq y$,*

$$d(Tx,\ Ty) \leq M(x,\ y) - \theta(M(x,\ y)),$$

where $M(x,\ y)$ same as in Theorem 3.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 3.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering ψ and φ to be the identity mappings and $\theta(t) = (1 - k)t$, where $0 \leq k < 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.5. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Assume that there exists $k \in [0,\ 1)$ such that for all $x,\ y \in X$ with $x \succeq y$,*

$$d(Tx,\ Ty) \leq k\, M(x,\ y),$$

where $M(x,\ y)$ same as in Theorem 3.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 3.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

In the following, our aim is to prove the existence and uniqueness of the fixed point in Theorem 3.1.

Theorem 3.6. *In addition to the hypotheses of Theorem 3.1, suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.*

Proof. From Theorem 3.1, the set of fixed points of T is non-empty. We shall show that if x and y are two fixed points of T, that is, if $x = Tx$ and $y = Ty$, then $x = y$.

By the assumption, there exists $u_0 \in X$ such that $u_0 \preceq x$ and $u_0 \preceq y$. Then, similarly as in the proof of Theorem 3.1, we define the sequence $\{u_n\}$ such that

$$u_{n+1} = Tu_n = T^{n+1}u_0, \quad n = 0, 1, 2, ... \tag{22}$$

Monotonicity of T implies that

$$T^n u_0 = u_n \preceq x = T^n x \text{ and } T^n u_0 = u_n \preceq y = T^n y.$$

Let $G_n = d(x, u_n)$. Since $u_n \preceq x$, using the contractive condition (3), we have

$$\psi(d(x, u_{n+1})) = \psi(d(Tx, Tu_n)) \leq \varphi(M(x, u_n)) - \theta(M(x, u_n)), \tag{23}$$

where

$$\begin{aligned} M(x, u_n) &= \max\left\{d(x, u_n), \frac{d(x, Tx) + d(u_n, u_{n+1})}{2}, \frac{d(u_n, Tx) + d(x, u_{n+1})}{2}\right\} \\ &= \max\left\{d(x, u_n), \frac{d(u_n, u_{n+1})}{2}, \frac{d(u_n, x) + d(x, u_{n+1})}{2}\right\}. \end{aligned}$$

Since $\frac{d(u_n, u_{n+1})}{2} \leq \frac{d(u_n, x) + d(x, u_{n+1})}{2}$, it follows that

$$M(x, u_n) = \max\left\{d(x, u_n), \frac{d(u_n, x) + d(x, u_{n+1})}{2}\right\}.$$

If $M(x, u_n) = d(x, u_n) = G_n$, then it follows from (23) that

$$\psi(d(x, u_{n+1})) \leq \varphi(d(x, u_n)) - \theta(d(x, u_n)),$$

that is,

$$\psi(G_{n+1}) \leq \varphi(G_n) - \theta(G_n),$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(G_{n+1}) \leq \varphi(G_n)$, which, by (1), implies that $G_{n+1} \leq G_n$, for all positive integer n.

If $M(x, u_n) = \frac{d(u_n, x) + d(x, u_{n+1})}{2} = \frac{G_n + G_{n+1}}{2} = H_n$ (*say*), then it follows from (23) that

$$\psi(G_{n+1}) \leq \varphi(H_n) - \theta(H_n),$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(G_{n+1}) \leq \varphi(H_n)$, which, by (1), implies that $G_{n+1} \leq H_n = \frac{G_n + G_{n+1}}{2}$, that is, $G_{n+1} \leq G_n$, for all positive integer n.

From the above discussion, $\{G_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Arguing similarly as in the proof of Theorem 3.1, we have

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} d(x, u_n) = 0. \tag{24}$$

Similarly, we show that

$$\lim_{n\to\infty} d(y, u_n) = 0. \tag{25}$$

By the triangle inequality, and using (24) and (25), we have

$$d(x, y) \leq [d(x, u_n) + d(u_n, y)] \longrightarrow 0, \text{ as } n \longrightarrow \infty,$$

which implies that $x = y$, that is, T has a unique fixed point in X. □

Example 3.7. Let $X = \{0,\ 1,\ 2,\ 3,\ 4, ...\}$. We define a partial ordering '$\preceq$ ' in X as $x \succeq y$ if and only if $x \leq y$ and $(y - x)$ is divisible by 2, for all $x, y \in \{2, 3, 4, ...\}$ and $0 \succeq 1,\ \ 1 \succeq 2$.

We define the metric d on X as

$$d(x,\ y) = \begin{cases} x + y + 2, \text{ if } x \neq y, \\ 0, \text{ if } x = y. \end{cases}$$

Let $T : X \to X$ be defined as follows

$$Tx = \begin{cases} x - 2, & \text{if } x \geq 2, \\ 0, & \text{if } \ x = 0, 1. \end{cases}$$

Let $\psi, \varphi\ , \theta : [0, \infty) \to [0, \infty)$ be defined as follows:

$$\psi(t) = t, \quad \varphi(t) = \begin{cases} t - \dfrac{1}{t}, & \text{for } t > 1, \\ t^2, & \text{for } t \in [0, 1] \end{cases} \qquad \text{and} \quad \theta(t) = \begin{cases} \dfrac{1}{t}, & \text{for } \ t > 1, \\ 2t, & \text{for } t \in [0, 1]. \end{cases}$$

For $x_0 = 10$, $Tx_0 = 8$. Then we have $x_0 \preceq Tx_0$.

Now, we will verify that (3) is satisfied for all $x,\ y \in X$ with $x \succeq y$. With out loss of generality we assume that $y > x$. Then the following cases are possible.

Case I $x \in \{0, 1, 2\}$ and $y \in \{1, 2\}$ then

$$d(Tx,\ Ty) = 0,\ \ d(x,\ y) = \begin{cases} 3, & \text{if } x = 0,\ y = 1, \\ 5, & \text{if } \ x = 1,\ y = 2, \end{cases}$$

$$\frac{d(x,\ Tx) + d(y,\ Ty)}{2} = \frac{d(y,\ Tx) + d(x,\ Ty)}{2} = \begin{cases} \frac{3}{2}, & \text{if } x = 0,\ y = 1, \\ \frac{7}{2}, & \text{if } \ x = 1,\ y = 2. \end{cases}$$

Case II $x \in \{0, 1, 2\}$, $y \in \{3, 4, 5, ...\}$ then

$$d(Tx,\ Ty) = y,\ \ d(x,\ y) = x + y + 2 \geq 5,$$

$$\frac{d(x,\ Tx) + d(y,\ Ty)}{2} = \begin{cases} y, & \text{if } \ x = 0, \\ \frac{x+2y+2}{2}, & \text{if } x \neq 0, \end{cases}$$

and

$$\frac{d(y,\ Tx) + d(x,\ Ty)}{2} = \begin{cases} \frac{5}{2}, & \text{if } x = 1,\ y = 3, \\ 3, & \text{if } x = 2,\ y = 4, \\ \frac{x+2y+2}{2}, & \text{otherwise.} \end{cases}$$

Cases III $x \in \{3, 4, 5...\}$, $y \in \{4, 5, ...\}$ then

$$d(Tx,\ Ty) = x + y - 2,\ \ d(x,\ y) = x + y + 2 \geq 7,\ \ \frac{d(x,\ Tx) + d(y,\ Ty)}{2} = x + y$$

and

$$\frac{d(y,\ Tx) + d(x,\ Ty)}{2} = \begin{cases} y - 1, & \text{if } x = y - 2, \\ x + y, & \text{otherwise.} \end{cases}$$

Cases IV If $x = y$, then

$$d(Tx,\ Ty) = 0,\ \ d(x,\ y) = 0$$

and

$$\frac{d(x,\ Tx) + d(y,\ Ty)}{2} = \frac{d(y,\ Tx) + d(x,\ Ty)}{2} = \begin{cases} 0, & \text{if } x = 0, \\ x + 2, & \text{if } x \in \{1, 2\}, \\ 2x, & \text{if } x \in \{3, 4, 5...\}. \end{cases}$$

In all the cases the inequality (3) is satisfied for all $x,\ y \in X$ with $x \succeq y$. Hence the required conditions of Theorem 3.1 are satisfied and it is seen that 0 is a fixed point of T.

Remark 3.8. In the above example, $\theta(t) \neq 0$, for all $t \in [0,\ \infty)$. Therefore, corollary 3.2 is not applicable to this example and hence theorem 3.1 properly contains its corollary 3.2.

Remark 3.9. In the above example φ is not identical with the function ψ. Therefore, corollary 3.3 is not applicable to this example and hence theorem 3.1 properly contains its corollary 3.3.

Remark 3.10. In the above example, φ is not the identity mapping. Therefore, corollaries 3.4 and 3.5 are not applicable to the above example. Hence theorem 3.1 properly contains its corollaries 3.4 and 3.5.

In the following example, the function T is discontinuous.

Example 3.11. *Let $X = [1.5,\ 2]$ with usual partial order " $\leq$ " and usual metric "d" be a partially ordered metric space. Let $T : X \longrightarrow X$ be defined as follows:*

$$Tx = \begin{cases} 1.81, & \textit{if} \quad 1.5 \leq x < 1.75, \\ x + \frac{1}{x} - \frac{1}{2}, & \textit{if} \quad 1.75 \leq x \leq 2. \end{cases}$$

Let $\psi, \varphi\ , \theta : [0, \infty) \to [0, \infty)$ be given respectively by the formulas

$$\psi(t) = \varphi(t) = t, \quad \theta(t) = \begin{cases} \frac{[t]}{1000}, & \text{if } 3 < t < 4, \\ \frac{t}{1000}, & \text{otherwise,} \end{cases}$$

where $[t]$ denotes the greatest integer not exceeding t.

Here all the conditions of Theorems 3.1 and 3.6 are satisfied and $x = 2$ is the unique fixed point of T.

4. Application to Partial Metric Space

In this section we obtain some fixed point results in partial metric spaces by application of the corresponding results of previous section.

Theorem 4.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that $(X,\ p)$ is a complete partial metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\varphi,\ \theta \in \Theta$ such that (1) and (2) are satisfied and for all $x,\ y \in X$ with $x \succeq y$,*

$$\psi(p(Tx,\ Ty)) \leq \varphi(M_p(x,\ y)) - \theta(M_p(x,\ y)),$$

where

$$M_p(x,\ y) = \max\left\{p(x,\ y),\ \frac{p(x,\ Tx) + p(y,\ Ty)}{2},\ \frac{p(y,\ Tx) + p(x,\ Ty)}{2}\right\}.$$

Also suppose that
(a) T is continuous or
(b) X has the following properties:
(i) if a nondecreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all $n \geq 0$;
(ii) if a nonincreasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all $n \geq 0$.

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. By using Proposition 2.4, $(X,\ d)$ is a complete metric space, where d is the constructed metric.

If $x = y = Tx$, then we have nothing to prove. Otherwise, by Lemma 2.5, we have

$$M_d(x,\ y) = M_p(x,\ y).$$

Thus, for all $x,\ y \in X$ with $x \geq y$, we obtain

$$\psi(d(Tx,\ Ty)) \leq \psi(p(Tx,\ Ty)) \leq \varphi(M_p(x,\ y)) - \theta(M_p(x,\ y)) = \varphi(M_d(x,\ y)) - \theta(M_d(x,\ y)),$$

where

$$M_d(x,\ y) = \max\left\{d(x,\ y),\ \frac{d(x,\ Tx) + d(y,\ Ty)}{2},\ \frac{d(y,\ Tx) + d(x,\ Ty)}{2}\right\}.$$

Then by using Theorem 3.1, T has a fixed point.

Corollary 4.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that $(X,\ p)$ is a complete partial metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\varphi \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\overline{\lim}\ \varphi(x_n) < t$ and for all $x,\ y \in X$ with $x \geq y$,*

$$p(Tx,\ Ty) \leq \varphi(M_p(x,\ y)),$$

where $M_p(x,\ y)$ same as in Theorem 4.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 4.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Corollary 4.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that $(X,\ p)$ is a complete partial metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \theta(x_n) > 0$ and for all $x,\ y \in X$ with $x \geq y$,*

$$\psi(p(Tx,\ Ty)) \leq \psi(M_p(x,\ y)) - \theta(M_p(x,\ y)),$$

where $M_p(x,\ y)$ same as in Theorem 4.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 4.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Corollary 4.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that $(X,\ p)$ is a complete partial metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \theta(x_n) > 0$ and for all $x,\ y \in X$ with $x \geq y$,*

$$p(Tx,\ Ty) \leq M_p(x,\ y) - \theta(M_p(x,\ y)),$$

where $M_p(x,\ y)$ same as in Theorem 4.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 4.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Corollary 4.5. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that $(X,\ p)$ is a complete partial metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Assume that there exists $k \in [0,\ 1)$ such that for all $x,\ y \in X$ with $x \geq y$,*

$$p(Tx,\ Ty) \leq k\, M_p(x,\ y),$$

where $M_p(x,\ y)$ same as in Theorem 4.1. Also suppose that the condition (a) or (b) (mentioned in Theorem 4.1) holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Theorem 4.6. *In addition to the hypotheses of Theorem 4.1, suppose that for every $x,\ y \in X$ there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.*

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References

[1] T. Abdeljawad, E. Karapinar, K. Tas, Existence and uniqueness of common fixed point on partial metric spaces, Appl. Math. Lett. 24 (2011) 1900–1904.

[2] Ya. I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, New Results in Operator Theory, in : I. Gohberg, Yu. Lyubich (Eds.), Advances and Appl. Vol. 98, Birkhäuser, Basel, 1997, 7–22.

[3] I. Altun, H. Simsek, Some fixed point theorems on dualistic partial metric spaces, J. Adv. Math. Stud. 1 (2008) 1–8.

[4] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157 (2010) 2778–2785.

[5] I. Altun, A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces, Fixed Point Theory Appl. 2011 (2011), Article ID 508730, 10 pages.

[6] A. D. Arvanitakis, A proof of the generalized Banach contraction conjecture, Proc. Amer. Math. Soc. 131 (2003) 3647–3656.

[7] C. E. Chidume, H. Zegeye, S. J. Aneke, Approximation of fixed points of weakly contractive nonselfmaps in Banach spaces, J. Math. Anal. Appl. 270 (2002) 189–199.

[8] B. S. Choudhury, A. Kundu, (ψ, α, β)-weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25 (2012) 6–10.

[9] B. S. Choudhury, P. Konar, B. E. Rhoades, N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlinear Anal. 74 (2011) 2116–2126.

[10] L. B. Ćirić, Generalized contractions and fixed point theorems, Publications De L'Institut Mathématique 12 (26) (1971) 19–26.

[11] L. B. Ćirić, Fixed point theorems for mappings with a generalized contractive iterate at a point, Publications De L'Institut Mathématique 13 (27) (1972) 11–16.

[12] L. B. Ćirić, On a family of contractive maps and fixed points, Publications De L'Institut Mathématique 17 (31) (1974) 45–51.

[13] L. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974) 267–273.

[14] D. Doric, Common fixed point for generalized $(\phi,\ \psi)$-weak contractions, Appl. Math. Lett. 22 (2009) 1896–1900.

[15] P. N. Dutta, B. S. Choudhury, A generalisation of contraction principle in metric spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 406368.

[16] R. H. Haghi, Sh. Rezapour, N. Shahzad, Be careful on partial metric fixed point results, Topology and its Applications 160 (2013) 450–454.

[17] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009) 3403–3410.

[18] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 1188–1197.

[19] P. Hitzler, A. Seda, Mathematical Aspects of Logic Programming Semantics, Chapman and Hall/CRC Studies in Informatic Series, CRC Press, 2011.

[20] D. Ilić, V. Pavlović , V. Rakočević, Some new extensions of Banach's contraction principle to partial metric space, Appl. Math. Lett. 24 (2011) 1326–1330.

[21] E. Karapinar, I. M. Erhan, Fixed point theorems for operators on partial metric spaces, Appl. Math. Lett. 24 (2011) 1894 –1899.

[22] M. S. Khan, M. Swaleh, S. Sessa, Fixed points theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984) 1–9.

[23] S. G. Matthews, Partial metric topology, In, General Topology and its Applications, Proc. 8th Summer Conf., Queen's College (1992), Annals of the New York Academy of Sciences, 728 (1994) 183–197.

[24] S. G. Matthews, Partial Metric spaces, 8th British Colloquium for Theoretical Computer Science, 1992, Research report 212, Dept. of computer Science, University of Warwick.

[25] J. Merryfield, B. Rothschild, J. D. Stein Jr., An application of Ramsey's theorem to the Banach contraction principle, Proc. Amer. Math. Soc. 130 (2002) 927–933.

[26] H. K. Nashine, B. Samet, Fixed point results for mappings satisfying $(\psi,\ \phi)$-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 2201–2209.

[27] H. K. Nashine, B. Samet, C. Vetro, Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces, Math. Comput. Modelling 54 (2011) 712–720.

[28] J. J. Nieto, R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.

[29] S. Oltra, O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istid. Math. Univ. Trieste 36 (2004) 17–26.

[30] O. Popescu, Fixed points for (ψ, ϕ)- weak contractions, Appl. Math. Lett. 24 (2011) 1–4.

[31] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.

[32] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (200l) 2683–2693.

[33] A. I. Rus, Fixed point theory in partial metric spaces, Anale Uninerstatti de Vest, Timisoara Seria Matematica- Informatica XLVI (2)(2008) 149 – 160.

[34] J. E. Stoy, Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory, MIT Press, Cambridge, Mass., 1977.

[35] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861–1869.

[36] M. Turinici, Abstract comparison principles and multivariable Gronwall- Bellman inequalities, J. Math. Anal. Appl. 117 (1986) 100–127.

[37] O. Valero, On Banach fixed point theorems for partial metric spaces, Appl. General Topology 6 (2005) 229–240.