Filomat 28:7 (2014), 1345–1352
DOI 10.2298/FIL1407345C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Levitin-Polyak Well-Posedness for Set-Valued Optimization Problems with Constraints

Jiawei Chen[a], Yeol Je Cho[b], Xiaoqing Ou[c]

[a]*School of Mathematics and Statistics, Southwest University, Chongqing 400715, China*
[b]*Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Korea,*
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[c]*College of Management, Chongqing College of Humanities, Science & Technology, Chongqing 401524, China*

Abstract. In this paper, Levitin-Polyak well-posedness for set-valued optimization problems with constraints is introduced. Some sufficient and necessary conditions for the Levitin-Polyak well-posedness of these problems are established under some suitable conditions. The equivalence between the well-posedness of optimization problems with constraints and the existence and uniqueness of their solutions are proved. Finally, we give some examples to illustrate the presented results.

1. Introduction

The well-posedness plays an important role in the stability analysis and numerical methods for optimization theory and applications and nonlinear operator equations. The well-posedness for minimization problems (shortly, (MP)) was first introduced and studied by Levitin and Polyak [16] and Tykhonov [21], respectively. These are so-called the Levitin-Polyak and Tykhonov well-posedness, respectively. The well-posedness of (MP) implies the existence and uniqueness of solutions of (MP). In practical situations, the solutions of (MP) are usually more than one. In this case, the notion of the well-posedness in the generalized sense which implies the existence of solutions of (MP) was introduced. Since then, many authors investigated the well-posedness and generalized well-posedness for optimization problems, variational inequality problems and equilibrium problems (see, for example, [3–6, 8, 11, 13–15, 17, 19, 20, 23] and references therein).

In [12], Hu et al. studied some sufficient and necessary conditions for the Levitin-Polyak type well-posedness of variational inequality problems and optimization problems with variational inequality constraints and obtained the relationships between the Levitin-Polyak well-posedness of the problems and the existence and uniqueness of its solutions. Fang et al. [9, 10] considered the well-posedness by perturbations for mixed variational inequality problems in Banach spaces, established the equivalence between the

2010 *Mathematics Subject Classification*. Primary 49J40; Secondary 49J27, 90C33
Keywords. Set-valued optimization problems with constraints, Levitin-Polyak well-posedness, Metric characterization, System of general variational inclusion, System of general variational disclusion
Received: 12 July 2013; Accepted: 18 March 2014
Communicated by Naseer Shahzad
Research supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100), the Natural Science Foundation of China(11401487), the Fundamental Research Funds for the Central Universities (SWU113037,XDJK2014C073).
Email addresses: J.W.Chen713@163.com (Jiawei Chen), yjcho@gnu.ac.kr (Yeol Je Cho), qingxiaoou@163.com (Xiaoqing Ou)

well-posedness of mixed variational inequality problems and that of the corresponding inclusion problems and fixed point problems and obtained the relationship between the well-posedness by perturbations and the existence and uniqueness of its solutions. They also pointed out that it is deserved to consider the well-posedness for the inclusion problem in [9]. Lin and Chuang [18] investigated the well-posedness in the generalized sense for variational inclusion problems and variational disclusion problems, the well-posedness for optimization problems with variational inclusion problems, variational disclusion problems and scalar equilibrium problems as constraints. In 2012, Wang and Huang [22] studied the necessary and sufficient conditions for the Levitin-Polyak well-posedness of generalized quasi-variational inclusion and disclusion problems and for optimization problems with constraints in Hausdorff topological vector spaces. In many practical problems, their constraints appear in the form of systems. To the best of our knowledge, there are very few results concerning the Levitin-Polyak well-posedness for set-valued optimization problems with systems of general variational inclusions and disclusion constraints.

Inspired and motivated by the above works, the aim of this paper is devoted to study the Levitin-Polyak well-posedness for set-valued optimization problems with systems of general variational inclusion and disclusion constraints, characterize the sufficient and necessary conditions for the Levitin-Polyak well-posedness of these problems and establish the equivalence between the well-posedness of optimization problems with constraints and the existence and uniqueness of their solutions.

2. Preliminaries

Throughout this paper, without other specifications, let I be a finite index set, R be the set of real numbers, C be a closed convex pointed cone of a Hausdorff topological vector spaces P with $\mathrm{int}C \neq \emptyset$, Λ_1 and Λ_2 be nonempty closed subsets of a normed linear space $\bigwedge$, Z_i be Hausdorff topological vector space, H_i and K_i be nonempty closed convex subsets of normed linear spaces X_i and Y_i for each $i \in I$, respectively. Let $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$, $H = \prod_{i\in I} H_i$, $K = \prod_{i\in I} K_i$ and $X^{-i} = \prod_{j\in I, j\neq i} X_j$. Denote the element of X^{-i} by x^{-i} and so $x \in X$ denoted by $x = (x_i)_{i\in I} = (x^{-i}, x_i) \in X^{-i} \times X_i$. We always denote 2^X by the family of all nonempty subsets of X. Let $C_i : H \to 2^{Z_i}$ be a set-valued mapping such that, for each $i \in I, x \in H$, $C_i(x)$ is a closed convex and pointed cone of Z_i and let $e_i : H \to Z_i$ be a continuous vector-valued mapping such that $e_i(x) \in -\mathrm{int}C_i(x)$ for all $x \in H$, $M_1, M_2 : \bigwedge \times H \to 2^P$, $\Gamma_i : H \to 2^{H_i}, T_i : H \to 2^{K_i}$, $\Psi_i : H \times H_i \to 2^{Z_i}$ and $F_i : H \times K \times H_i \to 2^{Z_i}$ be set-valued mappings for each $i \in I$.

Consider the following set-valued optimization problems with system of general variational inclusion (shortly, (SOPSGVI)) and disclusion constraints (shortly, (SOPSGVDI)), respectively:

$$\begin{aligned} &\min\ M_1(p, x) \\ \text{subject to}\ \ &p \in \Lambda_1,\ x \in S(p), \end{aligned}$$

and

$$\begin{aligned} &\min\ M_2(p, x) \\ \text{subject to}\ \ &p \in \Lambda_2,\ x \in S^d(p), \end{aligned}$$

where $S(p)$ and $S^d(p)$ are solutions sets of the following system of general variational inclusion (SGVI) and system of general variational disclusion (SGVDI) involving set-valued mappings, respectively:

Find $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$0 \in F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \tag{1}$$

for all $x_i \in \Gamma_i(x^*)$ and

Find $x^* \in H$ such that for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$0 \notin F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \tag{2}$$

for all $x_i \in \Gamma_i(x^*)$.

Denote the feasible solutions sets of (SOPSGVI) and (SOPSGVDI) by $\mathcal{N}_1 = \{(p,x) \in \Lambda_1 \times H : x \in S(p)\}$ and $\mathcal{N}_2 = \{(p,x) \in \Lambda_2 \times H : x \in S^d(p)\}$, respectively. In the sequel, we always assume that $\mathcal{N}_1$ and $\mathcal{N}_2$ are two nonempty closed sets.

Definition 2.1. For each $j = 1, 2$, let $\nu \in M_j(\mathcal{N}_j)$. ν is said to be a *minimal point* of $M_j(\mathcal{N}_j)$ if

$$M_j(\mathcal{N}_j) \cap (\nu - C \setminus \{0\}) = \emptyset.$$

Definition 2.2. A point $(p,x) \in \mathcal{N}_1$ (resp.,$\mathcal{N}_2$) is said to be an *efficient solution* of (SOPSGVI) (resp., (SOPS-GVDI)) if there exists $\nu \in M_1(p,x)$ (resp.,$M_2(p,x)$) such that ν is a minimal point of $M_1(\mathcal{N}_1)$ (resp.,$M_2(\mathcal{N}_2)$).

We present the following examples which are (SOPSGVI) and (SOPSGVDI), respectively.

Example 2.3. Let the index set I be a singleton, $\Lambda_1 = [-1, 1]$, $X = Y = Z = P = (-\infty, +\infty)$, $H = K = [4, +\infty)$ and $C = [0, +\infty)$. For each $p \in \Lambda_1, x, z \in H$ and $y \in K$, let $M_1(p,x) = [0, |p|x]$, $\Gamma(x,p) = [x, x+p^2]$, $T(x,p) = [p-x, x+1]$, $F(x,y,z) = [-y-1-p, z-x]$ and $\Psi(x,z) = [2x-6, 2+z-x]$. Simple computation allows that, for each $p \in \Lambda_1$, $S(p) = [4, 8+p]$ and the efficient solutions set of (SOPSGVI) is $\{(p,x) \in \Lambda_1 \times H : x \in S(p)\}$.

Example 2.4. Let the index set I be a singleton, $\Lambda_2 = [-\frac{1}{2}, \frac{1}{2}]$, $X = Y = Z = (-\infty, +\infty)$, $C = [0, +\infty)$ and $H = K = [1, 10]$. For each $p \in \Lambda_2$, $x, z \in H$ and $y \in K$, let $M_2(p,x) = [0, p^2x]$, $\Gamma(x,p) = [x, x+p^2]$, $T(x,p) = [p-x, x+1]$, $F(x,y,z) = (-\infty, -y-1-p) \cup (z-x, +\infty)$ and $\Psi(x,z) = (1+p, 1+p+\frac{x}{2})$. After computation, we obtain that, for each $p \in \Lambda_2$, $S^d(p) = H$. It is easy to see that the efficient solution set of (SOPSGVDI) is $\{(p,x) : p \in [-1,1], x \in H\}$.

Definition 2.5. ([1, 2]) Let $\bigvee$ be a Hausdorff topological vector space and E be a locally convex Hausdorff topological vector space. A mapping $\psi : \bigvee \to 2^E$ is said to be:

(1) *upper semi-continuous* (usc) at $\upsilon_0 \in \bigvee$ if, for each open set V with $\psi(\upsilon_0) \subset V$, there exists $\delta > 0$ such that $\psi(\upsilon) \subset V$ for all $\upsilon \in B(\upsilon_0, \delta)$;

(2) *lower semi-continuous* (lsc) at $\upsilon_0 \in \bigvee$ if, for each open set V with $\psi(\upsilon_0) \bigcap V \neq \emptyset$, there exists $\delta > 0$ such that $\psi(\upsilon) \cap V \neq \emptyset$ for all $\upsilon \in B(\upsilon_0, \delta)$;

(3) *closed* if its graph is closed, i.e., $Gr(\psi) = \{(\upsilon, \zeta) \in \bigvee \times E : \upsilon \in \bigvee, \zeta \in \psi(\upsilon)\}$ is closed;

(4) *opened* if its graph is opened.

We say that ψ is lsc (resp., usc) on $\bigvee$ if it is lsc (resp., usc) at each $\upsilon \in \bigvee$. ψ is said to be *continuous* on $\bigvee$ if it is both lsc and usc on $\bigvee$.

Lemma 2.6. ([1, 2]) (1) *ψ is lsc at $\upsilon_0 \in \bigvee$ if and only if, for any net $\{\upsilon_\alpha\} \subseteq \bigvee$ with $\upsilon_\alpha \to \upsilon_0$ and $\zeta_0 \in \psi(\upsilon_0)$, there exists a net $\{\zeta_\alpha\} \subseteq E$ with $\zeta_\alpha \in \psi(\upsilon_\alpha)$ for all α such that $\zeta_\alpha \to \zeta_0$.*

(2) *If ψ is compact-valued, then ψ is usc at $\upsilon_0 \in \bigvee$ if and only if, for any net $\{\upsilon_\alpha\} \subseteq \bigvee$ with $\upsilon_\alpha \to \upsilon_0$ and for any net $\{\zeta_\alpha\} \subseteq E$ with $\zeta_\alpha \in \psi(\upsilon_\alpha)$ for all α, there exist $\zeta_0 \in \psi(\upsilon_0)$ and a subnet $\{\zeta_\beta\}$ of $\{\zeta_\alpha\}$ such that $\zeta_\beta \to \zeta_0$.*

(3) *If ψ is usc and closed-valued, then ψ is closed. Conversely, if ψ is closed and E is compact, then ψ is usc.*

3. Main Results

In this section, we introduce and study the Levitin-Polyak type well-posedness for (SOPSGVI) and (SOPSGVDI), characterize the sufficient and necessary conditions for the Levitin-Polyak well-posedness of these problems under some suitable conditions and prove the equivalence between the well-posedness of optimization problems with constraints and the existence and uniqueness of their solutions. In order to characterize the Levitin-Polyak type well-posedness for (SOPSGVI) and (SOPSGVDI), we introduce the following approximating solutions sets for (SOPSGVI) and (SOPSGVDI).

For each $\nu, u \in P$ and $\lambda, \epsilon > 0$, let

$$\mathscr{E}_1(\nu) = \{(p,x) \in \Lambda_1 \times H : x \in S(p), \nu \in M_1(p,x), M_1(\mathscr{N}_1) \cap (\nu - C \setminus \{0\}) = \emptyset\},$$

$$\mathscr{E}_2(\nu) = \{(p,x) \in \Lambda_2 \times H : x \in S^d(p), \nu \in M_2(p,x), M_2(\mathscr{N}_2) \cap (\nu - C \setminus \{0\}) = \emptyset\}$$

and

$$\mathscr{K}_j(\nu, u, \lambda, \epsilon) = \{(p,x) \in \Lambda_j \times H : M_j(p,x) \cap (\nu + \lambda u - C) \neq \emptyset\} \bigcap Q_j(\epsilon), j \in \{1,2\},$$

where

$$\begin{aligned} Q_1(\epsilon) \quad = \quad & \{(p,x) \in \Lambda_1 \times H : \forall i \in I, d_i(x_i, \Gamma_i(x,p)) \leq \epsilon \exists\, y_i \in T_i(x,p), \text{ s.t.} \\ & 0 \in F_i(x,y,\omega_i) + \Psi_i(x,\omega_i) + \{0,\epsilon\}e_i(x), \ \forall \omega_i \in \Gamma_i(x,p)\} \end{aligned}$$

and

$$\begin{aligned} Q_2(\epsilon) \quad = \quad & \{(p,x) \in \Lambda_2 \times H : \forall i \in I, d_i(x_i, \Gamma_i(x,p)) \leq \epsilon \exists\, y_i \in T_i(x,p), \text{ s.t.} \\ & 0 \notin F_i(x,y,\omega_i) + \Psi_i(x,\omega_i) + \{0,\epsilon\}e_i(x), \ \forall \omega_i \in \Gamma_i(x,p)\}. \end{aligned}$$

Clearly, for each $\nu \in P$, $u \in \mathrm{int}C$, $\lambda_1, \lambda_2, \epsilon_1, \epsilon_2 > 0$ and $\lambda_1 \leq \lambda_2$, $\epsilon_1 \leq \epsilon_2$, we have $\mathscr{N}_j \subseteq Q_j(\epsilon_1) \subseteq Q_j(\epsilon_2)$ and $\mathscr{K}_j(\nu, u, \lambda_1, \epsilon_1) \subseteq \mathscr{K}_j(\nu, u, \lambda_2, \epsilon_2)$ for each $j \in \{1,2\}$.

In the following, we give the definition of the Levitin-Polyak well-posedness for (SOPSGVI) and (SOPSGVDI), respectively.

Definition 3.1. Let $\{a_n\} \subseteq P$ with $a_n \to 0$ and $(p^\star, x^\star)$ be an efficient solution of (SOPSGVI). A sequence $\{(p^n, x^n)\} \subseteq \Lambda_1 \times H$ is said to be the *Levitin-Polyak* (for short, LP) *approximating solution sequence* of (SOPSGVI) at $(p^\star, x^\star)$ corresponding to $\{a_n\}$ if the following conditions hold:

(a) there exists $\nu \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$ such that

$$M_1(p^n, x^n) \cap (\nu + a_n - C) \neq \emptyset$$

for all $n \in N$;

(b) there exists a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ such that $(p^n, x^n) \in Q_1(\epsilon_n)$ for all $n \in N$.

Similarly, we can define the LP approximating solution sequence for (SOPSGVDI).

Definition 3.2. Let $u \in \mathrm{int}C$, $\{a_n\} \subseteq P$ with $a_n \to 0$ and $(p^\star, x^\star)$ be an efficient solution of (SOPSGVI) (resp., (SOPSGVDI)) is said to be *LP well-posed* at $(p^\star, x^\star)$ if each LP approximating solution sequence of (SOPSGVI) (resp., (SOPSGVDI)) at $(p^\star, x^\star)$ corresponding to $\{a_n\}$ converges strongly to $(p^\star, x^\star)$.

Lemma 3.3. ([7]) *Let $\{a_n\} \subseteq P$ with $a_n \to 0$ and $u \in \mathrm{int}C$. Then there exists a sequence $\{\lambda_n\}$ of positive real numbers with $\lambda_n \to 0$ such that $\lambda_n u - a_n \in \mathrm{int}C$ for all $n \in N$.*

Lemma 3.4. *Let $u \in \mathrm{int}C$. Assume that $(p^\star, x^\star)$ is an efficient solution of (SOPSGVI) and $\nu \in M_1(p^\star, x^\star)$ is a minimal point of $M_1(\mathscr{N}_1)$. Then $(p^\star, x^\star) \in \mathscr{K}_1(\nu, u, \lambda, \epsilon)$ for all $\lambda, \epsilon > 0$.*

Proof. Since $(p^\star, x^\star)$ is an efficient solution of (SOPSGVI) and $\nu \in M_1(p^\star, x^\star)$ is a minimal point of $M_1(\mathscr{N}_1)$, we have

$$x^\star \in S(p^\star), \quad M_1(\mathscr{N}_1) \cap (\nu - C \setminus \{0\}) = \emptyset.$$

Then $x^\star \in H$ and, for each $i \in I$, $x_i^\star \in \Gamma_i(x^\star, p^\star)$, there exists $y_i^\star \in T_i(x^\star, p^\star)$ such that

$$0 \in F_i(x^\star, y^\star, x_i) + \Psi_i(x^\star, x_i)$$

for all $x_i \in \Gamma_i(x^\star, p^\star)$ and so

$$M_1(p^\star, x^\star) \cap (v - C) = \{v\}. \tag{3}$$

For each $\lambda, \epsilon > 0$, $\mathscr{N}_1 \subseteq Q_1(\epsilon)$, one has $(p^\star, x^\star) \in Q_1(\epsilon)$. By $u \in \text{int}C$, it follows that $-C \subseteq \lambda u - C$ and

$$M_1(p^\star, x^\star) \cap (v - C) \subseteq M_1(p^\star, x^\star) \cap (v + \lambda u - C). \tag{4}$$

It follows from (3) and (4) that

$$M_1(p^\star, x^\star) \cap (v + \lambda u - C) \neq \emptyset.$$

Therefore, from $(p^\star, x^\star) \in Q_1(\epsilon)$, it follows that $(p^\star, x^\star) \in \mathscr{K}_1(v, u, \lambda, \epsilon)$ for all $\lambda, \epsilon > 0$. This completes the proof.

Lemma 3.5. *Let $u \in intC$. Assume that $(p^\star, x^\star)$ is an efficient solution of (SOPSGVDI) and $v \in M_2(p^\star, x^\star)$ is a minimal point of $M_2(\mathscr{N}_2)$. Then $(p^\star, x^\star) \in \mathscr{K}_2(v, u, \lambda, \epsilon)$ for all $\lambda, \epsilon > 0$.*

Proof. The proof is similar to that of Lemma 3.4 and so it is omitted here. This completes the proof.

Theorem 3.6. *Let $u \in intC$ and $(p^\star, x^\star)$ be an efficient solution of (SOPSGVI). Then (SOPSGVI) is LP well-posed at $(p^\star, x^\star)$ if and only if, for any $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$,*

$$diam\mathscr{K}_1(v, u, \lambda, \epsilon) \to 0 \text{ as } (\lambda, \epsilon) \to (0, 0). \tag{5}$$

Proof. Let (SOPSGVI) be LP well-posed at $(p^\star, x^\star)$. Taking $v \in M_1(p^\star, x^\star)$ arbitrarily which is a minimal point of $M_1(\mathscr{N}_1)$. Then there exist $\sigma > 0$, two sequences of positive real numbers $\{\lambda_n\}, \{\epsilon_n\}$ with $(\lambda_n, \epsilon_n) \to (0, 0)$ and $\{(p^n, x^n)\}, \{(\bar{p}^n, \bar{x}^n)\} \subseteq \mathscr{K}_1(v, u, \lambda_n, \epsilon_n)$ such that

$$d((p^n, x^n), (\bar{p}^n, \bar{x}^n)) = \|(p^n, x^n) - (\bar{p}^n, \bar{x}^n)\| > \sigma. \tag{6}$$

Again, from $\{(p^n, x^n)\}, \{(\bar{p}^n, \bar{x}^n)\} \subseteq \mathscr{K}_1(v, u, \lambda_n, \epsilon_n)$, one has

$$(p^n, x^n) \in Q_1(\epsilon_n),\ M_1(p^n, x^n) \cap (v + \lambda_n u - C) \neq \emptyset$$

and

$$(\bar{p}^n, \bar{x}^n) \in Q_1(\epsilon_n),\ M_1(\bar{p}^n, \bar{x}^n) \cap (v + \lambda_n u - C) \neq \emptyset.$$

Since $\lambda_n \to 0, \lambda_n u \to 0$. Therefore, $\{(p^n, x^n)\}$ and $\{(\bar{p}^n, \bar{x}^n)\}$ are two LP approximating solution sequences of (SOPSGVI) corresponding to $\{\lambda_n u\}$. By the LP well-posedness of (SOPSGVI) at $(p^\star, x^\star)$, $(p_n, x^n) \to (p^\star, x^\star)$ and $(\bar{p}_n, \bar{x}^n) \to (p^\star, x^\star)$. Consequently, one has

$$\|(p^n, x^n) - (\bar{p}^n, \bar{x}^n)\| \leq \|(p_n, x^n) - (p^\star, x^\star)\| + \|(\bar{p}_n, \bar{x}^n) - (p^\star, x^\star)\| \to 0,$$

which contradicts (6).

Conversely, let $\{a_n\} \subseteq P$ with $a_n \to 0$ and $\{(p^n, x^n)\}$ be a LP approximating solution sequence of (SOPSGVI) at $(p^\star, x^\star)$ corresponding to $\{a_n\}$. Then there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$ such that $(p^n, x^n) \in Q_1(\epsilon_n)$ and $M_1(p^n, x^n) \cap (v + a_n - C) \neq \emptyset$. This implies that there exists $v_n \in M_1(p^n, x^n)$ such that $v_n \in v + a_n - C$. Since $\{a_n\} \subseteq P$ with $a_n \to 0$ and $u \in \text{int}C$, by Lemma 3.3, there exists a sequence $\{\lambda_n\}$ of positive real numbers with $\lambda_n \to 0$ such that $a_n \in \lambda_n u - \text{int}C$ for all $n \in N$. In view of $v_n \in v + a_n - C$, one has

$$v_n \in v + \lambda_n u - \text{int}C - C \subseteq v + \lambda_n u - \text{int}C.$$

Moreover, $v_n \in M_1(p^n, x^n) \cap (v + \lambda_n u - \text{int}C)$. This together with $(p^n, x^n) \in Q_1(\epsilon_n)$ yields that $(p^n, x^n) \in \mathscr{K}_1(v, u, \lambda_n, \epsilon_n)$. Note that $(p^\star, x^\star) \in \mathscr{N}_1 \subseteq Q_1(\epsilon_n)$ and $v \in M_1(p^\star, x^\star) \cap (v + \lambda_n u - C)$. Thus we have

$$(p^\star, x^\star) \in \mathscr{K}_1(v, u, \lambda_n, \epsilon_n)$$

for all $n \in N$ and so

$$\|(p^n, x^n) - (p^*, x^*)\| \leq \text{diam}(\mathscr{K}_1(v, u, \lambda_n, \epsilon_n))$$

for all $n \in N$. It follows from (5) that $(p^n, x^n) \to (p^\star, x^\star)$. Therefore, (SOPSGVI) is LP well-posed at $(p^\star, x^\star)$. This completes the proof.

Theorem 3.7. *Let $u \in intC$ and $(p^\star, x^\star)$ be an efficient solution of (SOPSGVDI). Then (SOPSGVDI) is LP well-posed at $(p^\star, x^\star)$ if and only if, for any $v \in M_2(p^\star, x^\star)$ which is a minimal point of $M_2(\mathscr{N}_2)$,*

$$diam\mathscr{K}_2(v, u, \lambda, \epsilon) \to 0 \text{ as } (\lambda, \epsilon) \to (0, 0). \tag{7}$$

Proof. The proof is similar to that of Theorem 3.6 and so it is omitted here.

Theorem 3.8. *Let $\bigwedge$ and X be finite dimensional and $u \in intC$ and $(p^\star, x^\star)$ be an efficient solution of (SOPSGVI). For each $i \in I$, let $e_i : H \to Z_i$ be a continuous mapping, the mappings $F_i : H \times K \times H_i \to 2^{Z_i}$, $\Psi_i : H \times H_i \to 2^{Z_i}$ be closed, $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ be closed-valued and continuous and $T_i : H \times \bigwedge \to 2^{K_i}$, $M_1 : \bigwedge \times X \to 2^P$ be usc and compact-valued. Assume that, for each $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$, there exist $\lambda_0, \epsilon_0 > 0$ such that $\mathscr{K}_1(v, u, \lambda_0, \epsilon_0)$ is nonempty and bounded. Then (SOPSGVI) is LP well-posed at $(p^\star, x^\star)$ if and only if, for any $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$, $\mathscr{E}_1(v) = \{(p^\star, x^\star)\}$.*

Proof. Let (SOPSGVI) be LP well-posedness at $(p^\star, x^\star)$. For any $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$, we get $(p^\star, x^\star) \in \mathscr{E}_1(v)$. Suppose to the contrary that $\mathscr{E}_1(v) \neq \{(p^\star, x^\star)\}$. Then there exists $(\tilde{p}, \tilde{x}) \in \mathscr{E}_1(v)$ such that

$$\|(\tilde{p}, \tilde{x}) - (p^\star, x^\star)\| > 0. \tag{8}$$

Moreover, $\tilde{x} \in S(\tilde{p})$, $v \in M_1(\tilde{p}, \tilde{x})$ and $M_1(\mathscr{N}_1) \cap (v - C \setminus \{0\}) = \emptyset$, which show that $(\tilde{p}, \tilde{x}) \in \mathscr{N}_1 \subseteq Q_1(\epsilon)$ for any $\epsilon > 0$ and $M_1(\tilde{p}, \tilde{x}) \cap (v - C) = v$. For each $n \in N$, let $a_n = 0$, $p^n = \tilde{p}$ and $x^n = \tilde{x}$. Then $\{(p^n, x^n)\}$ is a LP approximating solution sequence of (SOPSGVI) at $(p^\star, x^\star)$ corresponding to $\{a_n\}$. By the LP well-posedness of (SOPSGVI) at $(p^\star, x^\star)$, one has $(p^n, x^n) \to (p^\star, x^\star)$, i.e.,

$$\|(\tilde{p}, \tilde{x}) - (p^\star, x^\star)\| = \|(p^n, x^n) - (p^\star, x^\star)\| \to 0,$$

which contradicts (8).

Conversely, suppose that for any $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$, $\mathscr{E}_1(v) = \{(p^\star, x^\star)\}$. Let $\{a_n\} \subseteq P$ with $a_n \to 0$ and $\{(p^n, x^n)\}$ be a LP approximating solution sequence of (SOPSGVI) at $(p^\star, x^\star)$ corresponding to $\{a_n\}$. Then there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $v \in M_1(p^\star, x^\star)$ which is a minimal point of $M_1(\mathscr{N}_1)$ such that $(p^n, x^n) \in Q_1(\epsilon_n)$ and $M_1(p^n, x^n) \cap (v + a_n - C) \neq \emptyset$. As in the proof of Theorem 3.6, there exists a sequence $\{\lambda_n\}$ of positive real numbers with $\lambda_n \to 0$ such that $(p^n, x^n) \in \mathscr{K}_1(v, u, \lambda_n, \epsilon_n)$. Since $\mathscr{K}_1(v, u, \lambda_0, \epsilon_0)$ is nonempty and bounded, there exists $n_0 \in N$ such that $\lambda_n \leq \lambda_0$ and $\epsilon_n \leq \epsilon_0$ for $n \geq n_0$ and so

$$(p^n, x^n) \in \mathscr{K}_1(v, u, \lambda_n, \epsilon_n) \subseteq \mathscr{K}_1(v, u, \lambda_0, \epsilon_0).$$

Therefore, $\{(p^n, x^n)\}$ is bounded. For any subsequence $\{(p^{n_k}, x^{n_k})\}$ of $\{(p^n, x^n)\}$ with $(p^{n_k}, x^{n_k}) \to (\tilde{p}, \tilde{x}) \in \Lambda_1 \times H \subseteq \bigwedge \times X$, since $\bigwedge$ and X are finite dimensional, we have

$$M_1(p^{n_k}, x^{n_k}) \cap (v + \lambda_{n_k} u - C) \neq \emptyset, \quad (p^{n_k}, x^{n_k}) \in Q_1(\epsilon_{n_k}). \tag{9}$$

Moreover, there exists $v_{n_k} \in P$ such that

$$v_{n_k} \in M_1(p^{n_k}, x^{n_k}) \cap (v + \lambda_{n_k} u - C),$$

that is,

$$\nu_{n_k} \in M_1(p^{n_k}, x^{n_k}),\ \nu_{n_k} \in \nu + \lambda_{n_k} u - C. \tag{10}$$

Since $M_1 : \bigwedge \times X \to 2^P$ is usc and compact-valued, from Lemma 2.6 (2), there exists $\tilde{\nu} \in M_1(\tilde{p}, \tilde{x})$ such that a subsequence of $\{\nu_{n_k}\}$ strongly converges to $\tilde{\nu}$. Without loss of generality, let $\nu_{n_k} \to \tilde{\nu} \in M_1(\tilde{p}, \tilde{x})$. It follows from (10) that $\tilde{\nu} - \nu \in -C$, namely, $\tilde{\nu} \in \nu - C$. This together with Definition 2.1 and $\nu \in M_1(p^\star, x^\star)$ derives that

$$\tilde{\nu} \in M_1(\tilde{p}, \tilde{x}) \cap (\nu - C) \subseteq M_1(\mathscr{N}_1) \cap (\nu - C) = \{\nu\}, \tag{11}$$

that is, $\tilde{\nu} = \nu$. Again, from $(p^{n_k}, x^{n_k}) \in Q_1(\epsilon_{n_k})$, it follows that, for each $i \in I$,

$$d_i(x_i^{n_k}, \Gamma_i(x^{n_k}, p^{n_k})) \le \epsilon_{n_k} \tag{12}$$

and there exists $y_i^{n_k} \in T_i(x^{n_k}, p^{n_k})$ such that

$$0 \in F_i(x^{n_k}, p^{n_k}, \omega_i) + \Psi_i(x^{n_k}, \omega_i) + \{0, \epsilon_{n_k}\} e_i(x^{n_k}) \tag{13}$$

for all $\omega_i \in \Gamma_i(x^{n_k}, p^{n_k})$. Note that, for each $i \in I$, $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is closed-valued and continuous. By (12), we have

$$d_i(\tilde{x}_i, \Gamma_i(\tilde{x}, \tilde{p})) \le 0,$$

i.e., $\tilde{x}_i \in \Gamma_i(\tilde{x}, \tilde{p})$. Since $\epsilon_{n_k} \to 0$, it follows that, for each $i \in I$, $e_i : H \to Z_i$ is continuous, the mappings $F_i : H \times K \times H_i \to 2^{Z_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ are closed and $T_i : H \times \bigwedge \to 2^{K_i}$ is usc and compact-valued, it follows from (13) that there exists $\tilde{y}_i \in T_i(\tilde{x}, \tilde{p})$ such that

$$0 \in F_i(\tilde{x}, \tilde{p}, \omega_i) + \Psi_i(\tilde{x}, \omega_i)$$

for all $\omega_i \in \Gamma_i(\tilde{x}, \tilde{p})$. Hence $\tilde{x} \in S(\tilde{p})$ and so $(\tilde{p}, \tilde{x}) \in \mathscr{N}_1$. As a consequence, $(\tilde{p}, \tilde{x}) \in \mathscr{E}_1(\nu)$. In the light of $\mathscr{E}_1(\nu) = \{(p^\star, x^\star)\}$ and $(\tilde{p}, \tilde{x}) = (p^\star, x^\star)$. This implies that (p_n, x^n) converges to $(p^\star, x^\star)$. Therefore, (SOPSGVI) is LP well-posed at $(p^\star, x^\star)$. This completes the proof.

We give the following example to illustrate Theorems 3.6 and 3.8.

Example 3.9. Let the index set I be a singleton, $\Lambda_1 = [-1, 1]$, $X = Y = Z = P = (-\infty, +\infty]$, $H = K = [0, +\infty]$ and $C = [0, +\infty]$. For each $p \in \Lambda_1$, $x, z \in H$ and $y \in K$, let $e(x) = -1$, $M_1(p, x) = [-|p| - x, 0]$, $\Gamma(x, p) = [x, x + p]$, $T(x, p) = [p + x, x]$, $F(x, y, z) = [y - p, z - x]$ and $\Psi(x, z) = [x - z, 2 + z - x]$. Simple computation allows that

$$S(p) = \begin{cases} \{0\}, & \text{if } p = 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$\mathscr{N}_1 = \{(0, 0)\}$ and the efficient solution set of (SOPSGVI) is $\{(0, 0)\}$. $0 \in M_1(0, 0)$ is the uniquely minimal point of $M_1(\mathscr{N}_1)$. Let $u = 1 \in \text{int}C$. There exist $\lambda_0 = 1, \epsilon_0 = \frac{1}{2}$ such that $\mathscr{K}_1(0, 1, \lambda_0, \epsilon_0) = \{0\} \times [0, \frac{1}{2}]$ is nonempty and bounded. This shows that all the conditions of Theorem 3.8 are satisfied. It is easy to check that $\mathscr{K}_1(0, 1, \lambda, \epsilon) = \{0\} \times [0, \epsilon] \to (0, 0)$ as $(\lambda, \epsilon) \to (0, 0)$ and $\mathscr{E}_1(0) = \{(0, 0)\}$. So, $\text{diam}\mathscr{K}_1(0, 1, \lambda, \epsilon) \to 0$ as $(\lambda, \epsilon) \to (0, 0)$. By Theorem 3.6, (SOPSGVI) is LP well-posed at $(0, 0)$.

Theorem 3.10. *Let $\bigwedge$, X be finite dimensional, $u \in intC$ and let $(p^\star, x^\star)$ be an efficient solution of (SOPSGVDI). For each $i \in I$, let $e_i : H \to Z_i$ be a continuous vector valued mapping and the mappings $F_i : H \times K \times H_i \to 2^{Z_i}$, $\Psi_i : H \times H_i \to 2^{Z_i}$ be set-valued such that $F_i + \Psi_i$ is opened, $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ be closed and lsc, $T_i : H \times \bigwedge \to 2^{K_i}$ and $M_2 : \bigwedge \times X \to 2^P$ be usc and compact-valued. Assume that, for each $\nu \in M_2(p^\star, x^\star)$ which is a minimal point of $M_2(\mathscr{N}_2)$, there exist $\lambda, \epsilon > 0$ such that $\mathscr{K}_2(\nu, u, \lambda, \epsilon)$ is nonempty and bounded. Then (SOPSGVDI) is LP well-posed at $(p^\star, x^\star)$ if and only if, for any $\nu \in M_2(p^\star, x^\star)$ which is a minimal point of $M_2(\mathscr{N}_2)$, $\mathscr{E}_2(\nu) = \{(p^\star, x^\star)\}$.*

Proof. The proof is similar to that of Theorem 3.8 and so it is omitted here.

Example 3.11. Let I be a singled index set, $\Lambda_2 = [-1, 1]$, $X = Y = Z = (-\infty, +\infty)$, $C = [0, +\infty)$ and $H = K = [0, 10]$. For each $p \in \Lambda_2$, $x, z \in H$ and $y \in K$, let $e(x) = -1$, $M_2(p, x) = [p^2, x]$, $\Gamma(x, p) = [x, x + p^2]$, $T(x, p) = [p-x, x+1]$, $F(x, y, z) = (-\infty, -y-1-p)\cup(z-x, +\infty)$ and $\Psi(x, z) = (x-z, 2+p+2x)$. It is easy to see that $\mathscr{N}_2 = \Lambda_2 \times \{0\}$ and $S^d(p) = \{0\}$ for each $p \in \Lambda_2$. After simple computation, we know that the efficient solution set of (SOPSGVDI) is $\{(0, 0)\}$. $0 \in M_2(0, 0)$ is the uniquely minimal point of $M_2(\mathscr{N}_2)$. Let $u = 1 \in \text{int}C$. There exist $\lambda_0 = 1$ and $\epsilon_0 = \frac{1}{2}$ such that $\mathscr{K}_2(0, 1, 1, \frac{1}{2}) = [-1, 1] \times \{0\}$ is nonempty and bounded. This shows that all the conditions of Theorem 3.10 are satisfied. It is easy to check that $\mathscr{K}_2(0, 1, \lambda, \epsilon) \to (0, 0)$ as $(\lambda, \epsilon) \to (0, 0)$ and $\mathscr{E}_2(0) = \{(0, 0)\}$. So, $\text{diam}\mathscr{K}_2(0, 1, \lambda, \epsilon) \to 0$ as $(\lambda, \epsilon) \to (0, 0)$. By Theorem 3.7, (SOPSGVDI) is LP well-posed at $(0, 0)$.

References

[1] J.P. Aubin, I. Ekeland, Applied Nonlinear Analysis, John Wiley and Sons, New York, 1984.
[2] C. Berge, Topological Spaces, Oliver and Boyd, London, 1963.
[3] L.C. Ceng, H. Gupta, C.F. Wen, Well-posedness by perturbations of variational-hemivariational inequalities with perturbations, Filomat 26 (2012) 881–895.
[4] J.W. Chen, Y.C. Liou, Systems of parametric strong quasi-equilibrium problems: existence and well-posedness aspects, Taiwan. J. Math. 18(2014) 337–355.
[5] J.W. Chen, Z. Wan, Y.J. Cho, Levitin-Polyak well-posedness by perturbations for systems of set-valued vector quasi-equilibrium problems, Math. Meth. Oper. Res. 77(2013) 33–64.
[6] J.W. Chen, Z. Wan, Y.J. Cho, The existence of solutions and well-posedness for bilevel mixed equilibrium problems in Banach spaces, Taiwan. J. Math. 17(2013) 725–748.
[7] M. Durea, Scalarization for pointwise well-posed vectorial problems, Math. Meth. Oper. Res. 66(2007) 409–418.
[8] Y.P. Fang, R. Hu, N.J. Huang, Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints, Comput. Math. Appl. 55(2008) 89–100.
[9] Y.P. Fang, N.J. Huang, J.C. Yao, Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems, J. Glob. Optim. 41 (2008) 117–133.
[10] Y.P. Fang, N.J. Huang, J.C. Yao, Well-posedness by perturbations of mixed variational inequalities in Banach spaces, Euro. J. Oper. Res. 201(2010) 682–692.
[11] M. Furi, A. Vignoli, About well-posed optimization problems for functions in metric spaces, J. Optim. Theory Appl. 5(1970) 225–229.
[12] R. Hu, Y.P. Fang, N.J. Huang, Levitin-Polyak well-posedness for variational inequalities and for optimization problems with variational inequalities, J. Ind. Manag. Optim. 6(2010) 465–481.
[13] X.X. Huang, X.Q. Yang, Generalized Levitin-Polyak well-posedness in constrained optimization, SIAM J. Optim. 17(2006) 243–258.
[14] A.S. Konsulova, J.P. Revalski, Constrained convex optimization problems well-posedness and stability, Numer. Funct. Anal. Optim. 7-8 (1994) 889–907.
[15] C.S. Lalitha, G. Bhatia, Well-posedness for parametric quasivariational inequality problems and for optimizations problems with quasivariational inequality constraints, Optim. 59(2010) 997–1011.
[16] E.S. Levitin, B.T. Polyak, Convergence of minimizing sequences in conditional extremum problems, Soviet Math. Doklady 7 (1966) 764–767.
[17] S.J. Li, M.H. Li, Levitin-Polyak well-posedness of vector equilibrium problems, Math. Meth. Oper. Res. 69(2009) 125–140.
[18] L.J. Lin, C.S. Chuang, Well-posedness in the generalized sense for variational inclusion and disclusion problems and well-posedness for optimization problems with constraint, Nonlinear Anal. 70 (2009) 3609–3617.
[19] M.B. Lignola, J. Morgan, Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution, J. Glob. Optim. 16(2000) 57–67.
[20] R. Lucchetti, F. Patrone, A characterization of Tykhonov well-posedness for minimum problems with applications to variational inequalities, Numer. Funct. Anal. Optim. 3 (1981) 461–476.
[21] A.N. Tykhonov, On the stability of the functional optimization problem, USSR J. Comput. Math. Math. Phys. 6 (1966) 631–634.
[22] S. Wang, N.J. Huang, Levitin-Polyak well-posedness for generalized quasi-variational inclusion and disclusion problems and optimization problems with constraints, Taiwan. J. Math. 16(2012) 237–257.
[23] Z. Xu, D.L. Zhu, X.X. Huang, Levitin-Polyak well-posedness in generalized vector variational inequality problem with functional constraints, Math. Meth. Oper. Res. 67(2008) 505–524.