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Properties (S) and (gS) for Bounded Linear Operators

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Abstract. An operator T acting on a Banach space $\mathscr{X}$ obeys property (R) if $\pi_a^0(T) = E^0(T)$, where $\pi_a^0(T)$ is the set of all left poles of T of finite rank and $E^0(T)$ is the set of all isolated eigenvalues of T of finite multiplicity. In this paper we introduce and study two new properties (S) and (gS) in connection with Weyl type theorems. Among other things, we prove that if T is a bounded linear operator acting on a Banach space, then T satisfies property (R) if and only if T satisfies property (S) and $\pi^0(T) = \pi_a^0(T)$, where $\pi^0(T)$ is the set of poles of finite rank. Also we show if T satisfies Weyl theorem, then T satisfies property (S). Analogous results for property (gS) are given. Moreover, these properties are also studied in the frame of polaroid operator.

1. Introduction and Preliminary

Throughout this paper, $\mathscr{X}$ denotes an infinite-dimensional complex Banach space, $\mathscr{L}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$. For $T \in \mathscr{L}(\mathscr{X})$, let T^*, $\sigma(T)$, $\sigma_p(T)$, $\sigma_s(T)$ and $\sigma_a(T)$ denote respectively the *adjoint*, the *spectrum*, the *point spectrum*, the *surjectivity spectrum* and the *approximate point spectrum* of T. Let $\mathbb{C}$ denote the set of *complex numbers*. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $\operatorname{iso} K$ is the set of isolated points of K. For an operator $T \in \mathscr{L}(\mathscr{X})$ we shall denote by $\alpha(T)$ the dimension of the *kernel* $\ker(T)$, and by $\beta(T)$ the codimension of the *range* $\Re(T)$. Recall that the operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *upper semi-Fredholm*, $T \in SF_+(\mathscr{X})$, if the range of $T \in \mathscr{L}(\mathscr{X})$ is closed and $\alpha(T) < \infty$, while $T \in \mathscr{L}(\mathscr{X})$ is said to be *lower semi-Fredholm*, $T \in SF_-(\mathscr{X})$, if $\beta(T) < \infty$. An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *semi-Fredholm* if $T \in SF_+(\mathscr{X}) \cup SF_-(\mathscr{X})$ and *Fredholm* if $T \in F(\mathscr{X}) = SF_+(\mathscr{X}) \cap SF_-(\mathscr{X})$. If T is semi-Fredholm then the *index* of T is defined by $\mathrm{ind}\,(\mathrm{T}) = \alpha(\mathrm{T}) - \beta(\mathrm{T})$.

A bounded linear operator T acting on a Banach space $\mathscr{X}$ is *Weyl*, $T \in W(\mathscr{X})$, if it is Fredholm

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of index zero. Define $SF_+^-(\mathscr{X}) = \{T \in SF_+(\mathscr{X}) : \text{ind}\,(T) \le 0\}$ and $SF_-^+(\mathscr{X}) = \{T \in SF_-(\mathscr{X}) : \text{ind}\,(T) \ge 0\}$. The classes of operators defined above generate the following spectra: The *Weyl spectrum* $\sigma_w(T)$ is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$, the *Weyl essential approximate spectrum* is given by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(X)\}$, while the *Weyl essential surjective spectrum* is given by $\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_-^+(X)\}$. According to Coburn [17], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$. According to Rakočević [25], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\,\sigma_\text{a}(\text{T}) : 0 < \alpha(\text{T} - \lambda\text{I}) < \infty\}$. It is known [25] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

Recall that the *ascent*, $a(T)$, of an operator $T \in \mathscr{L}(\mathscr{X})$ is the smallest non negative integer p such that $\ker(T^p)$ $=\ker(T^{p+1})$ and if such integer does not exist we put $a(T) = \infty$. Analogously the *descent*, $d(T)$, of an operator $T \in \mathscr{L}(\mathscr{X})$ is the smallest non negative integer q such that $R(T^q) = R(T^{q+1})$ and if such integer does not exist we put $d(T) = \infty$.

The class of all *upper semi-Browder* operators is defined by $B_+(\mathscr{X}) = \{T \in SF_+(\mathscr{X}) : a(T) < \infty\}$ and the class of Browder operators is defined by $B(\mathscr{X}) = \{T \in F(\mathscr{X}) : a(T) < \infty \,\text{and}\, d(T) < \infty\}$. The *Browder spectrum* of $T \in \mathscr{L}(\mathscr{X})$ is defined by $\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_(\mathscr{X})\}$ and the *upper Browder spectrum* is defined by $\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_+(\mathscr{X})\}$.

For $T \in B(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ into $\Re(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an *upper (a lower) semi- Fredholm operator*, then T is called an *upper* (a *lower*) *semi-B-Fredholm operator*. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [7]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm operator.* A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a } B\text{-Weyl operator}\}$.

An operator $T \in \mathscr{L}(\mathscr{X})$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \,\text{is not a Drazin invertible}\}$. Define also the set $LD(\mathscr{X})$ by $LD(\mathscr{X}) = \{T \in \mathscr{L}(\mathscr{X}) : a(T) < \infty \,\text{and}\, \Re(T^{a(T)+1}) \,\text{is closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin LD(\mathscr{X})\}$. Following [11], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *left Drazin invertible* if $T \in LD(\mathscr{X})$. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda \in LD(\mathscr{X})$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. Let $\pi_a(T)$ denotes the set of all left poles of T and let $\pi_a^0(T)$ denotes the set of all left poles of T of finite rank. From Theorem 2.8 of [11] it follows that if $T \in \mathscr{L}(\mathscr{X})$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less than or equal to 0.

Let $\pi(T)$ be the set of all poles of the resolvent of T and let $\pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\pi^0(T) = \{\lambda \in \pi(T) : \alpha(T - \lambda) < \infty\}$. According to [21], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\{a(T - \lambda), d(T - \lambda)\} < \infty$. Moreover, if this is true then

$a(T-\lambda) = d(T-\lambda)$. According also to [21], the space $\Re((T-\lambda)^{a(T-\lambda)+1})$ is closed for each $\lambda \in \pi(T)$. Hence we have always $\pi(T) \subset \pi_a(T)$ and $\pi^0(T) \subset \pi_a^0(T)$. We say that *Browders theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta(T) = \pi^0(T)$, or equivalently $\sigma_b(T) = \sigma_w(T)$ and that *a-Browders theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta_a(T) = \pi_a^0(T)$, or equivalently $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$. Following [10], we say that *generalized Weyl's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : \alpha(T-\lambda) > 0\}$ is the set of all isolated eigenvalues of T, and that *generalized Browder's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta^g(T) = \pi(T)$. It is proved in Theorem 2.1 of [5] that generalized Browder's theorem is equivalent to Browder's theorem. In [11, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \pi(T)$, it is proved in Theorem 2.9 of [13] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(\mathscr{X})$ be the class of all *upper semi-B-Fredholm operators*, $SBF_+^-(\mathscr{X}) = \{T \in SBF_+(\mathscr{X}) : \operatorname{ind}(T) \leq 0\}$. The *upper B-Weyl spectrum* of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin SBF_+^-(\mathscr{X})\}$, while the *lower B-Weyl spectrum* of T is defined by $\sigma_{SBF_-^+}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin SBF_-^+(\mathscr{X})\}$. We say that *generalized a-Weyl's theorem* holds for $T \in \mathscr{L}(\mathscr{X})$ if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \alpha(T-\lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathscr{L}(\mathscr{X})$ obeys *generalized a-Browders theorem* if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [5, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [11, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [13, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Following [24], we say that $T \in \mathscr{L}(\mathscr{X})$ possesses *property* (w) if $\Delta_a(T) = E^0(T)$. The property (w) has been studied in [1]. In Theorem 2.8 of [2], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in \mathscr{L}(\mathscr{X})$ possesses *property* (gw) if $\Delta_a^g(T) = E(T)$. Property (gw) has been introduced and studied in [6, 27]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [6] that an operator possessing property (gw) possesses property (w) but the converse is not true in general. According to [14], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to possess *property* (gb) if $\Delta_a^g(T) = \pi(T)$, and is said to possess *property* (b) if $\Delta_a(T) = \pi^0(T)$. It is shown in Theorem 2.3 of [14] that an operator possessing property (gb) possesses property (b) but the converse is not true in general, see also [26]. Following [4], we say an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be satisfies *property* (R) if $\pi_a^0(T) = E^0(T)$. In Theorem 2.4 of [4], it is shown that T satisfies property (w) if and only if T satisfies a-Browder's theorem and T satisfies property (R).

Following [20] we say that $T \in \mathscr{L}(\mathscr{X})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathscr{X}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathscr{L}(\mathscr{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathscr{L}(\mathscr{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T

has SVEP at every isolated point of $\sigma(T)$, see [23]. In [22, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

In this paper we shall consider properties which are related to Weyl type theorem for bounded linear operators $T \in \mathscr{L}(\mathscr{X})$, defined on a complex Banach space $\mathscr{X}$. These properties, that we call *property* (S), means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues of finite multiplicity are exactly those points λ of the spectrum for which $T-\lambda$ is Browder (see Definition 2.1) and we call *property* (gS), means that the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T-\lambda$ is Drazin invertible (see Definition 2.1). Properties (S) and (gS) are related to Weyl's theorem and generalized Weyl's theorem, respectively. We shall characterize properties (S) and (gS) in several ways and we shall also describe the relationships of it with the other variants of Weyl type theorems. Our main tool is localized version of the single valued extension property. Also, we consider the properties (S) and (gS) in the frame of polaroid type operators. In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems, extending a similar diagram given in [28].

2. Properties (S) and (gS)

For $T \in \mathscr{L}(\mathscr{X})$, let $\Delta_b(T) = \sigma(T) \setminus \sigma_b(T)$, $\Delta_b^g(T) = \sigma(T) \setminus \sigma_D(T)$.

Definition 2.1. *Let $T \in \mathscr{L}(\mathscr{X})$. T is said to be satisfy property (S) if $\Delta_b(T) = E^0(T)$ and is said to be satisfy property (gS) if $\Delta_b^g(T) = E(T)$.*

Theorem 2.2. *Let $T \in \mathscr{L}(\mathscr{X})$. If T satisfies property (gS), then T satisfies property (S).*

Proof. Suppose that T satisfies property (gS), then $\Delta_b^g(T) = E(T)$. If $\lambda \in \Delta_b(T)$, then $\lambda \in \Delta_b^g(T) = E(T)$. Since $T-\lambda$ is a Browder operator, then $0 < \alpha(T-\lambda) < \infty$. So $\lambda \in E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$ be arbitrary. Then λ is an eigenvalue of T isolated in $\sigma(T)$. Since T satisfies property (gS), it follows that $\lambda \in \Delta_b^g(T) = E(T)$ and $T-\lambda$ is a Drazin invertible operator. Hence $\lambda \in \pi(T)$ and so $a(T-\lambda) = d(T-\lambda) < \infty$. As $\alpha(T-\lambda) < \infty$, then $\lambda \in \pi^0(T) = \Delta_b(T)$. $\square$

The converse of Theorem 2.2 does not hold in general as shown by the following example.

Example 2.3. *Let Q be defined for each $x = \{\xi_i\} \in \ell^1(\mathbb{N})$ by*

$$Q(\xi_1, \xi_2, \cdots) = (0, \alpha_1\xi_2, \alpha_2\xi_3, \cdots, \alpha_{k-1}\xi_k, \cdots),$$

where $\{\alpha_i\}$ is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$.

Define T on $\mathscr{X} = \ell^1(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ by $T = Q \oplus 0$. Then $\sigma(T) = \sigma_a(T) = \{0\}, E(T) = \{0\}, E^0(T) = \emptyset$. It follows from Example 3.12 of [11] that $\Re(T^n)$ is not closed for all $n \in \mathbb{N}$. This implies that $\sigma_b(T) = \sigma_{SBF_+^-}(T) = \sigma_{LD}(T) = \{0\}, \pi(T) = \emptyset$ and $\emptyset = \pi_a(T)$. We then have $\Delta_b^g(T) = \emptyset \neq E(T) = \{0\}$ and $\Delta_b(T) = E^0(T)$. Hence T satisfies property (S), but T does not satisfies property (gS).

Theorem 2.4. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following assertions hold.*

(i) T satisfies Weyl's theorem if and only if T satisfies property (S) and $\sigma_b(T) = \sigma_w(T)$.

(ii) T satisfies generalized Weyl's theorem if and only if T satisfies property (gS) and $\sigma_{BW}(T) = \sigma_D(T)$.

Proof. (i). Assume that T satisfies Weyl's theorem, then T satisfies Browder's theorem and $E^0(T) = \pi^0(T) = \Delta_b(T)$, i.e., T satisfies property (S). Since T satisfies Browder's theorem, then $\sigma_w(T) = \sigma_b(T)$.

Conversely, if T satisfies property (S) and $\sigma_w(T) = \sigma_b(T)$. Then

$$E^0(T) = \sigma(T) \setminus \sigma_b(T) = \sigma(T) \setminus \sigma_w(T).$$

That is, T satisfies Weyl's theorem.

(ii). Assume that T satisfies generalized Weyl's theorem, then it follows from Corollary 2.1 of [5] that T satisfies generalized Browder's theorem and $E(T) = \pi(T)$. Hence $\Delta_b^g(T) = E(T)$, i.e, T satisfies property (gS). Moreover, since T satisfies generalized Browder's theorem, we have $\sigma_{BW}(T) = \sigma_D(T)$.

Conversely, if T satisfies property (gS) and $\sigma_{BW}(T) = \sigma_D(T)$. Then

$$E(T) = \sigma(T) \setminus \sigma_D(T) = \sigma(T) \setminus \sigma_{BW}(T).$$

That is, T satisfies generalized Weyl's theorem. □

Theorem 2.5. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following are equivalent.*

(i) T satisfies property (gS);

(ii) $\sigma_{BW}(T) \subseteq \sigma(T) \setminus E(T)$.

Proof. (i)⇒(ii). Assume that T satisfies property (gS), then $\Delta_b^g(T) = \pi(T) = E(T)$. Let $\lambda \in E(T)$, then $T - \lambda$ is a B-Fredholm of index 0. Therefore, $\lambda \notin \sigma_{BW}(T)$ and so $\sigma_{BW}(T) \subseteq \sigma(T) \setminus E(T)$.

(ii)⇒(i). Suppose that $\sigma_{BW}(T) \subseteq \sigma(T) \setminus E(T)$ and let $\lambda \in E(T)$. Then λ is isolated in $\sigma(T)$ and $\lambda \notin \sigma_{BW}(T)$. So $T - \lambda$ is a B-Fredholm of index 0. It follows from Theorem 4.2 of [9] that $T - \lambda$ is Drazin invertible and so $\lambda \in \pi(T)$. As we have always $\pi(T) \subseteq E(T)$. Hence, $\pi(T) = E(T)$ and so T satisfies property (gS). □

Theorem 2.6. *Let $T \in \mathscr{L}(\mathscr{X})$. Then the following assertions hold.*

(i) If T satisfies property (w), then T satisfies property (S).

(ii) If T satisfies property (gw), then T satisfies property (gS).

Proof. (i). It follows from Theorem 2.8 of [2] that T satisfies Weyl's theorem, so the result follows now from part(i) of Theorem 2.4.

(ii). It follows from Theorem 2.4 of [6] that T satisfies generalized Weyl's theorem, so the result follows now from part(ii) of Theorem 2.4. □

The converse of Theorem 2.6 doe not not hold in general as shown by the following example.

Example 2.7. *Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and*

$$U(x_1, x_2, \cdots) := (0, x_2, x_3, \cdots) \text{ for all } (x_n) \in \ell^2(\mathbb{N}).$$

If $T := R \oplus U$ then $\sigma(T) = \sigma_w(T) = \sigma_D(T) = \sigma_{BW}(T) = \sigma_b(T) = D(0,1)$, where $D(0,1)$ is the unit disc of $\mathbb{C}$. Hence $iso\sigma(T) = \emptyset$, then $E^0(T) = E(T) = \emptyset$. Then T satisfies property S, since $\Delta_b(T) = E^0(T)$ and T satisfies property (gS), Since $\Delta_b^g(T) = E(T)$. Moreover, $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbb{C}$. Hence $\sigma_{SF_+^-}(T) = \sigma_{LD}(T) = \sigma_{SBF_+^-}(T) = C(0,1)$. Then T does not satisfy property (w), since $\Delta_a(T) = \{0\}$. Also, T does not satisfies property (gw), since $\Delta_a^g(T) = \{0\}$.

Definition 2.8. *Let $T \in \mathscr{L}(\mathscr{X})$. T is said to be satisfy property (gR) if $\sigma_a(T) \setminus \sigma_{LD}(T) = E(T)$, or equivalently $\pi_a(T) = E(T)$.*

Theorem 2.9. *Let $T \in \mathscr{L}(\mathscr{X})$. If T satisfies property (gR), then T satisfies property (R).*

Proof. Suppose that T satisfies property (gR), then $\sigma_a(T) \setminus \sigma_{LD}(T) = E(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T) = E(T)$. Since $T - \lambda \in B_+(\mathscr{X})$, so $\lambda \in \pi_a^0(T)$, then $\alpha(T-\lambda) < \infty$. As $\lambda \in E(T)$, then $\lambda \in$ isoσ(T) and $\alpha(T-\lambda) > 0$. Therefore, $\lambda \in E^0(T)$. To show the opposite inclusion, let $\lambda \in E_0(T)$ be arbitrary. Then λ is an eigenvalue of T isolated in $\sigma(T)$. Since T satisfies property (gR), it follows that $\lambda \in \sigma_a(T) \setminus \sigma_{LD}(T)$ and $T - \lambda$ is left Drazin invertible. As $\alpha(T-\lambda)$ is finite, we conclude that $\lambda \in \pi_a^0(T)$ and so $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$. □

Theorem 2.10. *Let $T \in \mathscr{L}(\mathscr{X})$. Then*

(i) T satisfies property (R) if and only if T satisfies property (S) and $\pi^0(T) = \pi_a^0(T)$.

(ii T satisfies property (gR) if and only if T satisfies property (gS) and $\pi(T) = \pi_a(T)$.

Proof. (i). Suppose that T satisfies property (R), then $\pi_a^0(T) = E^0(T)$. To show that T satisfies property (S) it suffices to show that $E^0(T) = \pi^0(T)$. If $\lambda \in E^0(T)$, then $\lambda \in \sigma_a(T)$. Since $0 < \alpha(T-\lambda) < \infty$ we know that both T and T^* has SVEP at λ. From the equality $E^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T)$ we see that $\lambda \notin \sigma_{ub}(T)$ and hence $T - \lambda \in B_+(\mathscr{X})$. The SVEP for T and T^* by Remark 1.2 of [2] that $a(T-\lambda) = d(T-\lambda) < \infty$. From Theorem 3.4 of [1] we then obtain that $\alpha(T-\lambda) = \beta(T-\lambda) < \infty$, so $\lambda \in \pi^0(T)$. Hence $E^0(T) \subseteq \pi^0(T)$, and since the other inclusion holds for every $T \in \mathscr{L}(\mathscr{X})$ we conclude that $\pi^0(T) = E^0(T)$.

Conversely assume that T satisfies property (S) and $\pi^0(T) = \pi_a^0(T)$. Then $E^0(T) = \pi_a^0(T)$. That is, T satisfies property (R).

(ii). Suppose that T satisfies property (gR), then $\pi_a(T) = E(T)$. To show that T satisfies property (gS) it suffices to show that $E(T) = \pi(T)$. Observe first that the inclusion $\pi(T) \subseteq E(T)$ holds for all $T \in \mathscr{L}(\mathscr{X})$. To show the opposite inclusion, suppose that T satisfies property (gR) and let $\lambda \in E(T) = \pi_a(T)$. Then $a(T-\lambda) < \infty$, and since $\lambda \in iso\sigma(T)$, then it follows from Theorem 2.8 of [11] that $\lambda \notin \sigma_{SBF_+^-}(T)$. Since $a(T-\lambda)$ is finite and $\lambda \in iso\sigma(T)$, then T^* has SVEP at λ and so $\lambda \notin \sigma_{BW}(T)$. Since $\lambda \in iso\sigma(T)$, then $\lambda \notin \sigma_D(T)$. Therefore, $\lambda \in \pi(T)$. □

The following example shows the property (gS) does not imply property (gR).

Example 2.11. *Let R be the unilateral right shift defined on $\ell^2(\mathbb{N})$. Define T on the Banach space $\mathscr{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then $\sigma(T) = \sigma_b(T) = \sigma_w(T) = \sigma_{BW}(T) = \sigma_D(T) = D(0,1)$, where $D(0,1)$ is the closed unit disc of $\mathbb{C}$, $\sigma_a(T) = C(0,1) \cup \{0\}$. Hence $iso\sigma(T) = \emptyset$, $\sigma_{SF_+^-}(T) = \sigma_{ub}(T) = C(0,1) \cup \{0\}$, $\sigma_{SBF_+^-}(T) = C(0,1) = \sigma_{LD}(T)$, where $C(0,1)$ is the unit circle of $\mathbb{C}$, $\pi_a(T) = \{0\}$, $\pi_a^0(T) = \emptyset$, $\pi(T) = \emptyset, E(T) = \pi^0(T) = \emptyset = E_a^0(T) = E^0(T)$ and $E_a(T) = \{0\}$. Hence $\Delta_b^g(T) = E(T)$, $\Delta_b(T) = E^0(T)$. So T satisfies property (gS) and hence property (S). But T do not satisfy property (gR), since $\pi_a(T) \neq E(T)$.*

Following [28], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to be satisfies property (Bgw) if $\Delta_a^g(T) = E^0(T)$.

Theorem 2.12. *Let $T \in \mathscr{L}(\mathscr{X})$. If T satisfies property (Bgw), then T satisfies property (S).*

Proof. We conclude from Theorem 2.12 of [28] that T satisfies property (w) and hence T satisfies property (S) by Theorem 2.6. □

As a consequence of [4, Theorem 2.4] and [28, Theorem 2.12], we have

Theorem 2.13. *Let $T \in \mathscr{L}(\mathscr{X})$. If T satisfies property (Bgw), then T satisfies property (R).*

The converse of Theorem 2.12 and Theorem 2.13 are not true in general, see Example 2.5 of [4].

Theorem 2.14. *Let $T \in \mathscr{L}(\mathscr{X})$. If T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$. Then property (w), property (b), property (R), property (S), Weyl's theorem and a-Weyl's theorem are equivalent for T.*

Proof. We conclude from [1, Corollary 2.5], [1, Corollary 3.53] and Theorem 2.19 of [4] that

$$\sigma(T) = \sigma_a(T), \sigma_w(T) = \sigma_b(T) = \sigma_{SF_+^-}(T) = \sigma_{ub}(T),$$

and

$$\pi^0(T) = E^0(T), \pi_a^0(T) = E_a^0(T), E^0(T) = \pi_a^0(T).$$

Hence

$$\begin{aligned} E_a^0(T) &= \Delta_a(T) = \Delta(T) \\ &= \sigma_a(T) \setminus \sigma_{ub}(T) = \Delta_b(T) = E^0(T). \end{aligned}$$

Therefore, Then property (w), property (b), property (R), property (S), Weyl's theorem and a-Weyl's theorem are equivalent for T. □

Theorem 2.15. *Let $T \in \mathscr{L}(\mathscr{X})$. If T has SVEP at every $\lambda \notin \sigma_{SF_-^+}(T)$. Then property (w), property (b), property (R), property (S), Weyl's theorem and a-Weyl's theorem are equivalent for T^*.*

Proof. We conclude from [1, Corollary 2.5], [1, Corollary 3.53] and Theorem 2.20 of [4] that

$$\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*),$$

$$\sigma_w(T^*) = \sigma_w(T) = \sigma_b(T^*) = \sigma_b(T) = \sigma_{SF_+^-}(T^*) = \sigma_{SF_-^+}(T) = \sigma_{lb}(T) = \sigma_{ub}(T^*),$$

and

$$\pi^0(T) = \pi^0(T^*) = E^0(T^*) = E^0(T), \pi_a^0(T^*) = E_a^0(T^*), E^0(T^*) = \pi_a^0(T^*).$$

Hence

$$\begin{aligned} E_a^0(T^*) &= \Delta_a(T^*) = \Delta(T^*) = \Delta(T) \\ &= \sigma_a(T^*) \setminus \sigma_{ub}(T^*) = \Delta_b(T^*) = E^0(T^*). \end{aligned}$$

Therefore, Then property (w), property (b), property (R), property (S), Weyl's theorem and a-Weyl's theorem are equivalent for T^*. □

Theorem 2.16. *Suppose that T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. Then the following assertions are equivalent:*

(i) $E(T) = \pi(T)$*;*

(ii) $E_a(T) = \pi_a(T)$*;*

(iii) $E(T) = \pi_a(T)$.

Consequently, property (gR), property (gw), property (gS), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T.

Proof. Suppose that T^* has SVEP at every $\lambda \notin \sigma_{SBF_+^-}(T)$. We prove first the equality $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. If $\lambda \notin \sigma_{SBF_+^-}(T)$ then $T - \lambda$ is an upper semi-B-Fredholm operator and $ind(T - \lambda) \leq 0$. As T^* has SVEP, then it follows from Corollary 2.8 of [12] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{BW}(T)$. Therefore, $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T)$. Since the other inclusion is always verified, we have the equality. Now we prove that $\sigma_D(T) = \sigma_{BW}(T)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{SF_+^-}(T)$ is always verified. Then T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$. This implies that T satisfies Browder's theorem. As we know from Theorem 2.1 of [5] that Browder's theorem is equivalent to generalized Browder's theorem, we have $\sigma_{BW}(T) = \sigma_D(T)$. On the other hand, as T^* has SVEP at every $\lambda \notin \sigma_{SF_+^-}(T)$, then $\sigma(T) = \sigma_a(T)$. From this we deduce that $E(T) = E_a(T)$ and

$$\pi_a(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_D(T) = \pi(T),$$

from which the equivalence of (i), (ii) and (iii) easily follows. To show the last statement observed that the SVEP of T^* at the points $\lambda \notin \sigma_{SBF_+^-}(T)$ entails that generalized a-Browder's theorem (and hence generalized Browder's theorem) holds for T, see [15, Corollary 2.7]. By Corollary 2.1 of [5] and Theorem 2.4 then property (gR), property (gw), property (gS), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T. □

Dually, we have

Theorem 2.17. *Suppose that T has SVEP at every $\lambda \notin \sigma_{SBF_{-}^{+}}(T)$. Then the following assertions are equivalent:*

(i) $E(T^*) = \pi(T^*)$;

(ii) $E_a(T^*) = \pi_a(T^*)$;

(iii) $E(T^*) = \pi_a(T^*)$.

Consequently, property (gR), property (gw), property (gS), generalized a-Weyl's theorem and generalized Weyl;s theorem are equivalent for T^*.

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{SBF_{-}^{+}}(T)$. We prove first the equality $\sigma_{SBF_{-}^{+}}(T^*) = \sigma_{BW}(T^*)$. If $\lambda \notin \sigma_{SBF_{-}^{+}}(T)$ then $T - \lambda$ is a lower semi-B-Fredholm operator and $ind(T - \lambda) \geq 0$. As T has SVEP, then it follows from Theorem 2.5 of [12] that $T - \lambda$ is a B-Weyl operator and so $\lambda \notin \sigma_{BW}(T)$. As $\sigma_{BW}(T) = \sigma_{BW}(T^*)$. Then $\lambda \notin \sigma_{BW}(T^*)$. So $\sigma_{BW}(T^*) \subseteq \sigma_{SBF_{-}^{+}}(T)$. As $\sigma_{SBF_{-}^{+}}(T) = \sigma_{SBF_{+}^{-}}(T^*)$, then $\sigma_{BW}(T^*) \subseteq \sigma_{SBF_{+}^{-}}(T^*)$. Since the other inclusion is always verified, it then follows that $\sigma_{BW}(T^*) = \sigma_{SBF_{+}^{-}}(T^*)$. Now we show that $\sigma_{BW}(T^*) = \sigma_D(T^*)$. Since we have always $\sigma_{SBF_{-}^{+}}(T) \subseteq \sigma_{SF_{-}^{+}}(T)$, then T has SVEP at every $\lambda \in \sigma_{SF_{-}^{+}}(T)$. Hence T^* satisfies generalized Browder's theorem. So $\sigma_D(T^*) = \sigma_{BW}(T^*)$. Finally, we have $\sigma_{BW}(T^*) = \sigma_{SBF_{+}^{-}}(T^*) = \sigma_D(T^*)$ and $\sigma(T^*) = \sigma_a(T^*)$,from which we obtain $E(T^*) = E_a(T^*)$ and $\pi(T^*) = \pi_a(T^*)$. The SVEP at every $\lambda \in \sigma_{SBF_{-}^{+}}(T)$ ensure by Corollary 2.7 of [15] that generalized a-Browder's theorem holds for T^*, and hence, by Corollary 2.1 of [5] and Theorem 2.4, property (gR), property (gw), property (gS), generalized a-Weyl's theorem and generalized Weyl's theorem are equivalent for T^*. □

An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent $(T - \lambda)^{-1}$, or equivalently $0 < a(T - \lambda) = d(T - \lambda) < \infty$. An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be *a-polaroid* if every isolated point of $\sigma_a(T)$ is a pole of the resolvent $(T-\lambda)^{-1}$, or equivalently $0 < a(T-\lambda) = d(T-\lambda) < \infty$. Clearly,

$$T\ a\text{-polaroid} \Rightarrow T\ \text{polaroid}. \tag{1}$$

and the opposite implication is not generally true.

Theorem 2.18. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is polaroid. Then the following assertions hold:*

(i) T *satisfies property* (S).

(ii) T *satisfies property* (gS).

Proof. (i) Note first that if T is polaroid then $\pi^0(T) = E^0(T)$. In fact, if $\lambda \in E^0(T)$ then λ is isolated in $\sigma(T)$ and hence $0 < a(T - \lambda) = d(T - \lambda) < \infty$. Moreover, $\alpha(T - \lambda) < \infty$, so by Theorem 3.4 of [1] it follows that $\beta(T - \lambda)$ is also finite, thus $\lambda \in \pi^0(T)$. This shows that $E^0(T) \subseteq \pi^0(T)$, since the other inclusion is

always verified, we have $E^0(T) = \pi^0(T)$. Therefore, T satisfies property (S).
(ii) Note first that if T is polaroid then $\pi(T) = E(T)$. In fact, if $\lambda \in E(T)$ then λ is isolated in $\sigma(T)$ and hence $0 < a(T-\lambda) = d(T-\lambda) < \infty$. Hence $\lambda \in \pi(T)$. This shows that $E(T) \subseteq \pi(T)$, since the other inclusion is always verified, we have $E(T) = \pi(T)$. Therefore, T satisfies property (gS). $\square$

Recall from [16] that $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy property (ab) if $\Delta(T) = \pi_a^0(T)$, T is said to satisfy property (aw) if $\Delta(T) = E_a^0(T)$, T is said to satisfy property (gab) if $\Delta^g(T) = \pi_a(T)$, and T is said to satisfy property (gaw) if $\Delta^g(T) = E_a(T)$.

Theorem 2.19. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is a-polaroid and T satisfies Browder's theorem. Then*

(i) Property (S), property (ab), property (aw) and Weyl's theorem are equivalent for T.

(ii) Property (gS), property (gab), property (gaw) and generalized Weyl's theorem are equivalent for T.

Proof. (i) If T is a-polaroid, it follows from proof of Theorem 2.21 of [2] that $\pi^0(T) = E_a^0(T)$. Since $\pi^0(T) \subseteq \pi_a^0(T) \subseteq E_a^0(T)$ by Lemma 2.1 of [2], we have

$$\pi^0(T) = E^0(T) = E_a^0(T) = \pi_a^0(T).$$

Now, if T satisfies Browder's theorem, we then have $\sigma_w(T) = \sigma_b(T)$ and so $\Delta_b(T) = \Delta(T)$. Therefore,

$$E_a^0(T) = \Delta(T) = \pi_a^0(T) = \Delta_b(T) = E^0(T).$$

That is, property (S), property (ab), property (aw) and Weyl's theorem are equivalent for T.
(ii) Note first that if T is a-polaroid then $\pi(T) = E_a(T)$. In fact, if $\lambda \in E_a(T)$ then λ is isolated in $\sigma_a(T)$ and hence $0 < a(T-\lambda) = d(T-\lambda) < \infty$. Hence $\lambda \in \pi(T)$. This shows that $E_a(T) \subseteq \pi(T)$, since we have always $\pi(T) \subseteq E(T) \subseteq E_a(T)$ and $\pi(T) \subseteq \pi_a(T)$. Therefore,

$$E_a(T) = \pi(T) = E(T) = \pi_a(T).$$

Now, since Browder's theorem is equivalent to generalized Browder's theorem by Theorem 2.1 of [5], then T satisfies generalized Browder's theorem. Hence $\sigma_{BW}(T) = \sigma_D(T)$ and so, $\Delta^g(T) = \Delta_b^g(T)$. Therefore,

$$E_a(T) = \Delta^g(T) = \pi_a(T) = \Delta_b^g(T) = E(T).$$

That is, property (gS), property (gab), property (gaw) and generalized Weyl's theorem are equivalent for T. $\square$

Let $H_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non-constant on each of the components of its domain. Define, by the classical calculus, $f(T)$ for every $f \in H_{nc}(\sigma(T))$.

Theorem 2.20. *Suppose that $T \in \mathscr{L}(\mathscr{X})$ is polaroid. Then*

(i) $f(T)$ satisfies property (S) for all $f \in H_{nc}(\sigma(T))$.

(ii) $f(T)$ satisfies property (gS) for all $f \in H_{nc}(\sigma(T))$.

Proof. It follows from Lemma 3.11 of [3] that $f(T)$ is polaroid. Hence the result follows now from Theorem 2.18. □

3. Conclusion

In this last part, we give a summary of the known Weyl type theorems as in [11], including the properties introduced in [4, 6, 14, 24, 28], and in this paper. We use the abbreviations $gaW, aW, gW, W, (gw), (w), (Bw), (Bgw), (R), (S)$ and (gS) to signify that an operator $T \in \mathscr{L}(\mathscr{X})$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), property (Bw), property (Bgw), property (R), property (S) and property (gS). Similarly, the abbreviations gaB, aB, gB, B, $(gb), (b), (Bb)$ and (Bgb) have analogous meaning with respect to Browder's theorem .

The following table summarizes the meaning of various theorems and properties.

gaW	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	gaB	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi_a(T)$
gW	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$	gB	$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$
aW	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	aB	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_a^0(T)$
W	$\sigma(T) \setminus \sigma_W(T) = E^0(T)$	B	$\sigma(T) \setminus \sigma_W(T) = \pi^0(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$
(w)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$	(b)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi^0(T)$
(Bw)	$\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$	(Bb)	$\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$
(Bgw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$	(Bgb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^0(T)$
(gR)	$\sigma_a(T) \setminus \sigma_{LD}(T) = E(T)$	(R)	$\sigma_a(T) \setminus \sigma_{ub}(T) = E^0(T)$
(gS)	$\sigma(T) \setminus \sigma_D(T) = E(T)$	(S)	$\sigma(T) \setminus \sigma_b(T) = E^0(T)$

In the following diagram, which extends the similar diagram presented in [28], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw), property (gb), property (Bw), property (Bgw), property (Bb), property (Bgb), property (R), property (gR), property (S) and property (gS). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$
\begin{tikzcd}
gW \arrow[d, "2.4"] & gb \arrow[r, "{[14]}"] & gB \arrow[r, leftrightarrow, "{[5]}"] & B & aB \arrow[l, "{[19]}"'] \arrow[r, "{[19]}"] & B & W \arrow[l, "{[8]}"'] \\
gS \arrow[d, "2.2"] & gw \arrow[l, "2.6"'] \arrow[r, "{[6]}"] \arrow[u, "{[14]}"] \arrow[d, "{[6]}"] & gW \arrow[u, "{[11]}"] \arrow[d, "{[11]}"] & gaW \arrow[l, "{[11]}"'] \arrow[r, "{[11]}"] \arrow[d, "{[11]}"] & gaB \arrow[d, leftrightarrow, "{[5]}"] & Bgw \arrow[l, "{[28]}"'] \arrow[r, "{[28]}"] \arrow[d, "{[28]}"] & Bgb \arrow[d, "{[28]}"] \\
S & w \arrow[l, "2.6"'] \arrow[r, "{[14]}"] & W & aW \arrow[l, "{[25]}"] \arrow[r, "{[11]}"'] & aB & gb \arrow[l, "{[14]}"] \arrow[r, "{[14]}"'] \arrow[d, "{[14]}"] & gaB \arrow[d, leftrightarrow, "{[5]}"] \\
W \arrow[u, "2.4"] & Bgw \arrow[l, "{[28]}"'] \arrow[r, "{[28]}"'] \arrow[u, "{[28]}"] \arrow[d, "2.12"] & Bw \arrow[r, "{[28]}"'] \arrow[u, "{[28]}"] & Bb & Bgb \arrow[l, "{[28]}"] \arrow[r, "{[28]}"'] & b \arrow[r, "{[14]}"'] & aB \\
S & R \arrow[l, "2.10"'] & gR \arrow[l, "2.9"] \arrow[r, "2.10"'] & gS & & &
\end{tikzcd}
$$

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