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On Local Property of Absolute Summability of Factored Fourier Series

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Abstract. We establish two general theorems on the local properties of the absolute summability of factored Fourier series by applying a recently defined absolute summability, $|A, \alpha_n|_k$ summability, and the class $\mathcal{S}\left(\alpha_n, \phi_n\right)$, which generalize some well known results and can be applied to improve many classical absolute summability methods.

1. Introduction

Let $A := (a_{nk})$ be a lower triangular matrix and $\{s_n\}$ the partial sums of $\sum a_n$. Let $\{\alpha_n\}$ be a nonnegative sequence, then the series $\sum a_n$ is said to be summable $|A, \alpha_n|_k$, $k \geq 1$, if (see [19])

$$\sum_{n=1}^{\infty} \alpha_n |A_n - A_{n-1}|^k < \infty,$$

where

$$A_n := \sum_{v=1}^{n} a_{nv} s_v.$$

In particular, if $\alpha_n = n^{k-1}$, then $|A, \alpha_n|_k$ −summability reduces to the $|A|_k$-summability (see [17]). Let A be the Cesàro matrices $C := (c_{nv})$ of order α, that is,

$$c_{nv} := \frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}, \ v = 0, 1, \cdots, n,$$

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where

$$A_n^{\alpha} := \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)},\ n = 0, 1, \cdots .$$

When $\alpha_n = n^{\delta k+k-1}, k \geq 1, \delta \geq 0$, $|A, \alpha_n|_k$ −summability is usually called $|C, \alpha; \delta|_k$-summability. Therefore, a series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$, $\alpha > -1$, if (see [9])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \left|\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}\right|^k < \infty,$$

where

$$\sigma_n^{\alpha} := \sum_{j=0}^{n} \frac{A_{n-j}^{\alpha-1}}{A_n^{\alpha}} s_j.$$

For any positive sequence $\{p_n\}$ such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$, the corresponding Riesz matrix R has the entries

$$r_{nv} := \frac{p_v}{P_n},\quad v = 0, 1, \cdots, n,\ \ n = 0, 1, 2, \cdots .$$

Taking $\alpha_n = \left(\frac{P_n}{p_n}\right)^{\delta k+k-1}$ and $\alpha_n = n^{\delta k+k-1}$, we get two special absolute summability, $\left|\overline{N}, p_n; \delta\right|_k$ summability and $\left|R, p_n; \delta\right|_k$ summability, of $|R, \alpha_n|_k$ summability, respectively. In particular, if $np_n \asymp P_n$, then $\left|\overline{N}, p_n; \delta\right|_k$ summability and $\left|R, p_n; \delta\right|_k$ summability are equivalent. See [2] and [3] for more details on $\left|\overline{N}, p_n; \delta\right|_k$ summability and $\left|R, p_n; \delta\right|_k$ summability.

One can find more examples of $|A, \alpha_n|_k$ −summability for different weight sequences $\{\alpha_n\}$ and different summability matrices A discussed in many papers, see [2], [3], [7], [10], and [16] for examples.

Let f be a function with period 2π, integrable (L) over $(-\pi, \pi)$. Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} C_n(t).$$

It is well known that (see [18]) the convergence of the Fourier series at $t = x$ is a local property of the generating function $f(t)$ (i.e., it depends only on the behavior of f in a arbitrarily small neighborhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function $f(t)$.

In 1939, Bosanquet and Kestelman (see [8]) showed that even the summability $|C, 1|$ of the Fourier series at a point is not a local property of f. Mohanty ([11]) subsequently observed that the summability $\left|R, \log n, 1\right|$ of the factored series

$$\sum C_n(t) / \log(n+1),$$

at any point is a local property of f, whereas the summability $|C,1|$ of this series is not. Several generalizations of Mohanty's result have been made by many authors, for examples, see, Bhatt ([1]), Bor ([3]-[5]), Borwein ([6]), Sarigöl ([14], [15]), etc.

For any lower triangular matrix A, associated it with two lower triangular matrices $\overline{A}$ and $\widehat{A}$ defined by

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nr},\ v = 0,1,2,\cdots,n \text{ and } n = 0,1,2,\cdots,$$

and

$$\widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v},\ v = 0,1,\cdots,n-1; n = 1,2,3,\cdots.\ \widehat{a}_{nn} = a_{nn} = \overline{a}_{nn}.$$

Sarigöl ([15]) proved the following theorem:

Theorem A. *Let A be a lower triangular matrix with nonnegative entries and $\{X_n\}$ a sequence of numbers, satisfying*

(i) $a_{n-1,v} \geq a_{nv}$ for $n \geq v+1$,

(ii) $\overline{a}_{n0} = 1,\ n = 0,1,\cdots,$

(iii) $\sum_{v=1}^{n-1} a_{vv}\widehat{a}_{n,v+1} = O\left(a_{nn}\right),$

(iv) $\Delta X_n = O\left(\frac{1}{n}\right),\ X_n = \left(na_{nn}\right)^{-1}, n = 1,2,\cdots, X_0 = 0.$

If for number sequences $\{\theta_n\}$ and $\{\lambda_n\}$ the following conditions:

(v) $\sum_{v=1}^{\infty} \left(\theta_v a_{vv}\right)^{k-1} X_v^{k-1}\frac{1}{v}\lambda_v^k < \infty,$

(vi) $\sum_{v=1}^{\infty} \left(\theta_v a_{vv}\right)^{k-1} X_v^{k}\Delta\lambda_v < \infty,$

(vii) $\sum_{n=v+1}^{\infty} \left(\theta_n a_{nn}\right)^{k-1} \left|\Delta\widehat{a}_{n,v+1}\right| = O\left(\left(\theta_v a_{vv}\right)^{k-1} a_{vv}\right)$

and

(viii) $\sum_{n=v+1}^{\infty} \left(\theta_n a_{nn}\right)^{k-1}\widehat{a}_{n,v+1} = O\left(\left(\theta_v a_{vv}\right)^{k-1}\right),$

hold, then the summability of $\left|A,\theta_n^{k-1}\right|_k,\ k \geq 1,$ of the *series* $\sum \lambda_n X_n C_n(t)$ *at any point is a local property of f, where $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.*

Theorem A generalized some well known results on the local property of summability of factored Fourier series. Although, there are some matrices satisfying the conditions in Theorem A, a Cesàro's matrix may not satisfy all the conditions (i)-(iii). In fact, (ii) and (iii) do not hold for any $\alpha > 1$ or $\alpha < 1$. Furthermore, Rhaly's generalized Cesàro matrices and the $p-$Cesàro matrices do not satisfy the conditions of Theorem A neither (see Section 3 for the definitions of Rhaly's generalized Cesàro matrices and the $p-$Cesàro matrices).

In the present paper, we establish a new factor theorem which generalizes Theorem A, and can be applied to many well known matrices, including the ones mentioned above. We need the following class of matrices, $\mathcal{S}\left(\alpha_n,\phi_n\right)$, which is recently introduced by Yu and Zhou ([20]):

Definition 1.1. *Let $\{\alpha_n\}$, $\left\{\phi_n\right\}$ be sequences of positive numbers. We say that a lower triangular matrix $A := (a_{nk}) \in \mathcal{S}\left(\alpha_n,\phi_n\right)$, if it satisfies the following conditions*

$$\sum_{i=0}^{n-1} \left|\Delta_i\widehat{a}_{ni}\right| = O\left(\phi_n\right); \tag{T1}$$

$$\left|\widehat{a}_{ni}\right| = O\left(\phi_n\right),\ i = 0, 1, \cdots, n; \tag{T2}$$

$$\sum_{n=i+1}^{\infty} \alpha_n \phi_n^{k-1} \left|\Delta_i \widehat{a}_{ni}\right| = O\left(\alpha_i \phi_i^k\right); \tag{T3}$$

$$\sum_{n=i+1}^{\infty} \alpha_n \phi_n^{k-1} \left|\widehat{a}_{n,i+1}\right| = O\left(\alpha_i \phi_i^{k-1}\right). \tag{T4}$$

Our main results are the following:

Theorem 1.2. *Let* $\{\alpha_n\}$, *and* $\left\{\phi_n\right\}$ *be sequences of positive numbers. Let* $\{\lambda_n\} \in BV$ *be a sequence of complex numbers*[1)] *such that* $\lambda_{n+1} = O(|\lambda_n|)$ *for* $n = 1, 2, \cdots$, *and*

(A) $\sum_{n=0}^{\infty} \alpha_n \phi_n^k X_n^k \left|\lambda_n^k\right| < \infty$,

(B) $\sum_{n=0}^{\infty} \alpha_n \phi_n^{k-1} X_n^k |\Delta\lambda_n| < \infty$.

If $A \in \mathcal{S}\left(\alpha_n, \phi_n\right)$ *satisfies*

$$\sum_{v=0}^{n} \left|a_{vv} \widehat{a}_{n,v+1}\right| = O\left(\phi_n\right), \tag{1}$$

$$\Delta X_n = O\left(\phi_n\right),\ X_n = \frac{\phi_n}{a_{nn}}, \tag{2}$$

then the summability of $|A, \alpha_n|_k$ *for* $k \geq 1$, *of the series* $\sum C_n(t) \lambda_n X_n$ *at any point is a local property of* f.

Remark 1. The restrictions of $\{\lambda_n\}$ in Theorem A are relaxed in Theorem 1.2 to the simple conditions that $\{\lambda_n\} \in BV$ and $\lambda_{n+1} = O(|\lambda_n|)$, which obviously hold when $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.

Theorem 1.3. *The result of Theorem 1.2 also holds when (1) and (2) are replaced by*

$$\sum_{v=0}^{n} \left|\widehat{a}_{n,v+1} \phi_v\right| = O\left(\phi_n\right), \tag{3}$$

$$\Delta X_n = O\left(n^{-1}\right),\ X_n = \frac{1}{n\phi_n}, n = 1, 2, \cdots, X_0 = 0, \tag{4}$$

respectively.

Remark 2. If the matrix A satisfies the condition $\overline{a}_{n0} = 1,\ n = 0, 1, \cdots$, then the indexes of the summations in (A), (B), (1) and (3) only need to run from 1 instead of 0, which can be observed in the proofs of the theorems.

Remark 3. Let $\phi_n := a_{nn}, \alpha_n = \theta_n^{k-1}$. If the matrix A satisfies the conditions in Theorem A, then we can easily have that $A \in \mathcal{S}\left(\alpha_n, \phi_n\right)$. That is, Theorem A can be regarded as a corollary of Theorem 1.3.

[1)] We say a sequence of complex numbers $\{\lambda_n\} \in BV$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.

We prove the theorems in Section 2. In Section 3, we show that some well known matrices such as Cesàro's matrices, Rhaly's generalized Cesàro matrices, the $p-$Cesàro matrices, and Riesz's matrices are in $\mathcal{S}\left(\alpha_n, \phi_n\right)$ for some certain sequences $\{\alpha_n\}$ and $\left\{\phi_n\right\}$, and then derive some new theorems on the local property of some factored Fourier series, as applications of the above theorems.

2. Proofs of the Main Results

We prove Theorem 1.2 in this section. The proof of Theorem 1.3 is similar.

The behavior of the Fourier series, as far as convergence is concerned, at a particular value of x, depends on the behavior of the function in the immediate neighborhood of this point only. Therefore, in order to prove the theorem, it is sufficient to prove that if $\{s_n\}$ is bounded, then under the conditions of Theorem 1, $\sum a_n\lambda_n X_n$ is summable $|A, \alpha_n|_k$, $k \geq 1$. Let T_n be the $n-$th term of the $A-$transform of $\sum_{i=0}^{n} \lambda_i a_i X_i$. Then

$$T_n = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i\lambda_i X_i = \sum_{i=0}^{n} a_i\lambda_i X_i \sum_{v=i}^{n} a_{nv} = \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i X_i.$$

Thus,

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i X_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i \lambda_i X_i \\
&= \sum_{i=0}^{n} \widehat{a}_{ni} a_i \lambda_i X_i = \sum_{i=0}^{n} \widehat{a}_{ni} \lambda_i X_i \left(s_i - s_{i-1}\right) \\
&= \sum_{i=0}^{n-1} \left(\widehat{a}_{ni}\lambda_i X_i - \widehat{a}_{n,i+1}\lambda_{i+1}X_{i+1}\right) s_i + a_{nn}\lambda_n s_n X_n \\
&= \sum_{i=0}^{n-1} \widehat{a}_{n,i+1} \Delta\lambda_i X_i s_i + \sum_{i=0}^{n-1} \widehat{a}_{n,i+1}\lambda_{i+1} \Delta X_i s_i + \sum_{i=0}^{n-1} \left(\Delta_i \widehat{a}_{ni}\right) \lambda_i X_i s_i \\
&\quad + a_{nn}\lambda_n X_n s_n \\
&=: T_{n1} + T_{n2} + T_{n3} + T_{n4}.
\end{aligned}$$

Therefore, it is sufficient to prove that

$$\sum_{n=1}^{\infty} \alpha_n \left|T_{ni}\right|^k < \infty, \text{ for } i = 1, 2, 3, 4. \tag{5}$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{m+1} \alpha_n \left|T_{n1}\right|^k &= O(1) \sum_{n=1}^{m+1} \alpha_n \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1}\right| \left|X_i\right| \left|\Delta\lambda_i\right| \right)^k \\
&= O(1) \sum_{n=1}^{m+1} \alpha_n \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1}\right| \left|X_i^k\right| \left|\Delta\lambda_i\right| \right) \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1}\right| \left|\Delta\lambda_i\right| \right)^{k-1}.
\end{aligned}$$

Since $\{\lambda_n\} \in BV$, we have

$$\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1}\right| |\Delta\lambda_i| = O\left(\phi_n\right),$$

by (T2). Hence

$$\begin{aligned}
\sum_{n=1}^{m+1} \alpha_n |T_{n1}|^k &= O(1) \sum_{n=1}^{m+1} \alpha_n \phi_n^{k-1} \sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1}\right| \left|X_i^k\right| |\Delta\lambda_i| \\
&= O(1) \sum_{i=0}^{m} \left|X_i^k\right| |\Delta\lambda_i| \sum_{n=i+1}^{m+1} \alpha_n \phi_n^{k-1} \left|\widehat{a}_{n,i+1}\right| \\
&= O(1) \sum_{i=0}^{m} \alpha_i \phi_i^{k-1} \left|X_i^k\right| |\Delta\lambda_i| = O(1),
\end{aligned} \tag{6}$$

by (T3), and (B) of Theorem 1.2.

It follows from (2) that $\Delta X_i = O(a_{ii} X_i)$. Then by Hölder's inequality, (1) and condition (A) of the Theorem 1.2, we have

$$\begin{aligned}
\sum_{n=1}^{m+1} \alpha_n |T_{n2}|^k &= O(1) \sum_{n=1}^{m+1} \alpha_n \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1} \lambda_{i+1} \Delta X_i\right|\right)^k \\
&= O(1) \sum_{n=1}^{m+1} \alpha_n \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1} \lambda_i a_{ii} X_i\right|\right)^k \\
&= O(1) \sum_{n=1}^{m+1} \alpha_n \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1} a_{ii}\right| \left|\lambda_i^k\right| \left|X_i^k\right|\right) \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1} a_{ii}\right|\right)^{k-1} \\
&= O(1) \sum_{n=1}^{m+1} \alpha_n \phi_n^{k-1} \left(\sum_{i=0}^{n-1} \left|\widehat{a}_{n,i+1} a_{ii}\right| \left|\lambda_i^k\right| \left|X_i^k\right|\right) \\
&= O(1) \sum_{i=0}^{m} \left|\lambda_i^k\right| \left|X_i^k\right| |a_{ii}| \sum_{n=i+1}^{m+1} \alpha_n \phi_n^{k-1} \left|\widehat{a}_{n,i+1}\right| \\
&= O(1) \sum_{i=0}^{m} \alpha_n \phi_n^{k-1} \left|\lambda_i^k\right| \left|X_i^k\right| |a_{ii}| \\
&= O(1) \sum_{i=0}^{m} \alpha_n \phi_n^{k} \left|\lambda_i^k\right| \left|X_i^k\right| = O(1),
\end{aligned}$$

where we also used the fact that $\widehat{a}_{nn} = a_{nn} = O\left(\phi_n\right)$, which follows from (T2).

By (T1), (T3) and condition (A), we have

$$\begin{aligned}
\sum_{n=1}^{m+1}\alpha_n\left|T_{n3}\right|^k &= O(1)\sum_{n=1}^{m+1}\alpha_n\left(\sum_{i=0}^{n-1}\left|\Delta\widehat{a}_{n,i+1}\lambda_i X_i\right|\right)^k\\
&= O(1)\sum_{n=1}^{m+1}\alpha_n\left(\sum_{i=0}^{n-1}\left|\Delta\widehat{a}_{n,i+1}\right|\left|\lambda_i^k\right|\left|X_i^k\right|\right)\left(\sum_{i=0}^{n-1}\left|\Delta\widehat{a}_{n,i+1}\right|\right)^{k-1}\\
&= O(1)\sum_{n=1}^{m+1}\alpha_n\phi_n^{k-1}\sum_{i=0}^{n-1}\left|\Delta\widehat{a}_{n,i+1}\right|\left|\lambda_i^k\right|\left|X_i^k\right|\\
&= O(1)\sum_{i=0}^{m}\left|\lambda_i^k\right|\left|X_i^k\right|\sum_{n=i+1}^{m+1}\alpha_n\phi_n^{k-1}\left|\Delta\widehat{a}_{n,i+1}\right|\\
&= O(1)\sum_{i=0}^{m}\alpha_i\phi_i^k\left|\lambda_i^k\right|\left|X_i^k\right| = O(1). \qquad (7)
\end{aligned}$$

By using $a_{nn} = O\left(\phi_n\right)$ again, we have

$$\begin{aligned}
\sum_{n=1}^{m+1}\alpha_n\left|T_{n4}\right|^k &= O(1)\sum_{n=1}^{m+1}\alpha_n\left|a_{nn}\lambda_n X_n\right|^k\\
&= O(1)\sum_{n=1}^{m+1}\alpha_n\phi_n^k\left|\lambda_n^k\right|\left|X_n^k\right|\\
&= O(1). \qquad (8)
\end{aligned}$$

Combining (6)-(8), we have (5). This proves Theorem 1.2.

3. Applications of the Theorems

3.1. Cesàro's Matrices

We will use the following formula often in the proofs (see [21]) for proof, for example):

$$A_n^\alpha = \frac{n^\alpha}{\Gamma(\alpha+1)}\left(1+O\left(\frac{1}{n}\right)\right). \qquad (9)$$

In this subsection, we set

$$\phi_0 := 1,\ \phi_n := \begin{cases} n^{-1}, & \alpha > 1 \\ \frac{1}{A_n^\alpha} = c_{nn}, & 0 < \alpha \le 1, \end{cases} \quad n = 1, 2, \cdots.$$

By (9), we see that

$$\phi_n \sim \begin{cases} n^{-1}, & \alpha > 1 \\ n^{-\alpha}, & 0 < \alpha \le 1, \end{cases} \quad n = 1, 2, \cdots.$$

Recall that a nonnegative sequence $\{a_n\}$ is said to be almost decreasing, if there is a positive constant K such that

$$a_n \ge K a_m$$

holds for all $n \le m$, and it is said to be quasi-β-power decreasing, if $\left\{n^{\beta} a_n\right\}$ is almost decreasing.

It should be noted that every decreasing sequence is an almost decreasing sequence, and every almost decreasing sequence is a quasi-β-power decreasing sequence for any non-positive index β, but the converse is not true.

Lemma 3.1. *([20])Let $\alpha > 0$, and let $\{\alpha_n\}$ be a sequence of positive numbers. If $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε-power decreasing for some $\varepsilon > 0$, then $C \in \mathcal{S}\left(\alpha_n, \phi_n\right)$.*

A direct calculation leads to

$$\begin{aligned}
\widehat{c}_{ni} &= \frac{1}{A_n^{\alpha}} \sum_{j=i}^{n} A_{n-j}^{\alpha-1} - \frac{1}{A_{n-1}^{\alpha}} \sum_{j=i}^{n-1} A_{n-1-j}^{\alpha-1} \\
&= \frac{A_{n-i}^{\alpha}}{A_n^{\alpha}} - \frac{A_{n-1-i}^{\alpha}}{A_{n-1}^{\alpha}} = \frac{i A_{n-i}^{\alpha-1}}{n A_n^{\alpha}}.
\end{aligned}$$

Thus, for $0 < \alpha \le 1$,

$$\begin{aligned}
\sum_{v=0}^{n} \left| c_{vv} \widehat{c}_{n,v+1} \right| &= O\left(\frac{1}{n^{1+\alpha}}\right) \sum_{v=1}^{n} \frac{(v+1) A_{n-v-1}^{\alpha-1}}{A_v^{\alpha}} \\
&= O\left(\frac{1}{n^2}\right) \sum_{v=1}^{n/2} v^{1-\alpha} + O\left(\frac{1}{n^{2\alpha}}\right) \sum_{v=1}^{n/2} v^{\alpha-1} \\
&= O\left(\phi_n\right). \qquad (10)
\end{aligned}$$

Similarly, for $\alpha > 1$,

$$\sum_{v=1}^{n} \frac{1}{v} \left| \widehat{c}_{n,v+1} \right| = O\left(\phi_n\right). \qquad (11)$$

Now set

$$X_n \equiv 1 = \begin{cases} \frac{\phi_n}{c_{nn}}, & 0 < \alpha \le 1, \\ (n\phi_n)^{-1}, & \alpha > 1. \end{cases}$$

Then X_n satisfies (2) and (4) for $0 < \alpha \le 1$ and $\alpha > 1$ respectively. Now, applying Lemma 3.1, (10), (11), Theorem 1.2 and Theorem 1.3, we have the following

Theorem 3.2. *Let $\alpha > 0$, $\{\alpha_n\}$ be sequences of positive numbers. Let $\{\lambda_n\} \in BV$ be a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and*

(a) $\sum_{n=1}^{\infty} \alpha_n \phi_n^k \left|\lambda_n^k\right| < \infty$,

(b) $\sum_{n=1}^{\infty} \alpha_n \phi_n^{k-1} |\Delta\lambda_n| < \infty$.

If $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε-power decreasing, then the summability of $|C, \alpha_n|_k$ for $k \ge 1$, of the series $\sum C_n(t) \lambda_n$ at any point is a local property of f.

As examples, we give two corollaries of Theorem 3.2.

Corollary 3.3. *Let $\{\lambda_n\} \in BV$ be a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and*

(c) $\sum_{n=1}^{\infty} n^{\delta k-1} \log^{\gamma} n \left|\lambda_n^k\right| < \infty$,

(d) $\sum_{n=1}^{\infty} n^{\delta k} \log^{\gamma} n \left|\Delta\lambda_n\right| < \infty$,

then the summability of $\left|C, n^{\delta k+k-1} \log^{\gamma} n\right|_k$, for $\alpha \geq 1$, $\gamma \in R$, $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$, of the series $\sum C_n(t)\lambda_n$ at any point is a local property of f.

Proof. Let $\alpha_n = n^{\delta k+k-1} log^{\gamma} n$, $n = 1, 2, \cdots$, $\alpha_0 = 1$. Since $\alpha \geq 1$, then $\phi_n = n^{-1}$. It is then obvious that (c) implies (a), and (d) implies (b). From the condition that $0 \leq \delta < \frac{1}{k}$, we see that there exists an $\varepsilon > 0$ such that $\delta k - 1 + \varepsilon < 0$, and thus $\left\{n^{\delta k-1+\varepsilon} \log^{\gamma} n\right\}$ is quasi decreasing for $\gamma \in \mathbb{R}$. In other words, $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε-power decreasing. Therefore, by Theorem 3.2, we have Corollary 3.3. □

Corollary 3.4. *Let $\{\lambda_n\} \in BV$ be a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and*

(c′) $\sum_{n=1}^{\infty} n^{\delta k+(1-\alpha)k-1} \log^{\gamma} n \left|\lambda_n^k\right| < \infty$,

(d′) $\sum_{n=1}^{\infty} n^{\delta k+(1-\alpha)(k-1)} \log^{\gamma} n X_n \left|\Delta\lambda_n\right| < \infty$,

then the summability of $\left|C, n^{\delta k+k-1} \log^{\gamma} n\right|_k$, for $0 < \alpha < 1$, $\gamma \in R$, $k \geq 1$ and $0 \leq \delta < \frac{2-\alpha+(1-\alpha)k}{k}$, of the series $\sum C_n(t)\lambda_n$ at any point is a local property of f.

Proof. Note that $\phi_n = n^{-\alpha}$ for $0 < \alpha < 1$. Then the proof of Corollary 3.4 is similar to that of Corollary 3.3. □

3.2. *Rhaly's Generalized Cesàro Matrices*

Let D be the Rhaly generalized Cesàro matrix (see [12]), that is, D has entries of the form $d_{nk} = t^{n-k}/(n+1)$, $k = 0, 1, \cdots, n$, $n = 1, 2, \cdots$. When $t = 1$, the Rhaly generalized Cesàro matrix reduces to the Cesàro matrix of order 1. We shall restrict our attention to $0 < t < 1$. In this case, D does not satisfy condition (ii) of Theorem A. It is routine to deduce that

$$\widehat{d}_{nv} = \sum_{r=v}^{n} \frac{t^{n-r}}{n+1} - \sum_{r=v}^{n-1} \frac{t^{n-1-r}}{n} = \frac{1}{1-t}\left(\frac{1-t^{n-v+1}}{n+1} - \frac{1-t^{n-v}}{n}\right). \tag{12}$$

Set $\phi_0 = 1$, $\phi_n = n^{-1}, n = 1, 2, \cdots$. By (12), we have

$$\begin{aligned}\widehat{d}_{nv} &= \frac{1}{1-t}\left(\frac{1-t^{n-v+1}}{n+1} - \frac{1-t^{n-v}}{n}\right)\\ &= -\frac{1-t^{n-v} - nt^{n-v}(1-t)}{(1-t)\,n\,(n+1)}\\ &= O\left(\frac{1}{n(n+1)} + \frac{nt^{n-v}}{n(n+1)}\right).\end{aligned}$$

Thus

$$\sum_{v=0}^{n}\left|d_{vv}\widehat{d}_{n,v+1}\right| = O\left(\frac{1}{n^2}\right)\sum_{v=1}^{n}\frac{1}{v} + O\left(\frac{1}{n}\right)\sum_{v=1}^{n} t^{n-v} = O\left(\phi_n\right). \tag{13}$$

Lemma 3.5. *([20])Let $0 < t < 1$, and $\{\alpha_n\}$ be a sequences of positive numbers. If $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε−power decreasing for some $\varepsilon > 0$, then $D \in \mathcal{S}\left(\alpha_n, \phi_n\right)$.*

Now set

$$X_n = \frac{\phi_n}{d_{nn}} = \frac{n+1}{n}.$$

Then $X_n = O(1)$ and $\Delta X_n = O\left(\phi_n\right)$. Therefore, by Lemma 3.5, (13) and Theorem 1.2, we have

Theorem 3.6. *Let $0 < t < 1$ and let $\{\alpha_n\}$ be a sequence of positive numbers. Assume that $\{\lambda_n\} \in BV$ is a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and (A), (B) in Theorem 1.2 hold. If $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε-power decreasing for some $\varepsilon > 0$, then the summability of $|D, \alpha_n|_k$ for $k \geq 1$, of the series $\sum C_n(t) \frac{n+1}{n} \lambda_n$ at any point is a local property of f.*

Obviously, we can also have a corollary of Theorem 3.6 that is similar to Corollary 3.3. We omit the details here.

3.3. $p-$Cesàro Matrices

Let E be the $p-$Cesàro matrix (see [13]), that is, the entries of E has the form $e_{ni} = 1/(n+1)^p$, $i = 0, 1, \cdots, n$, $n = 1, 2, \cdots$. When $p = 1$, the $p-$Cesàro matrix reduces to the Cesàro matrix of order 1 again. Also, E does not satisfy condition (ii) of Theorem A. We restrict our attention to the case when $1 < p \leq 2$.

Set $\phi_0 = 1$, $\phi_n = n^{-p}, n = 1, 2, \cdots$. Then

$$\widehat{e_{ni}} = \bar{e}_{ni} - \bar{e}_{n-1,i} = \frac{n-i+1}{(n+1)^p} - \frac{(n-i)}{n^p}, \tag{14}$$

and

$$\Delta_i \widehat{e_{ni}} = \widehat{e_{n,i+1}} - \widehat{e_{ni}} = e_{ni} - e_{n-1,i} = \frac{1}{(n+1)^p} - \frac{1}{n^p}. \tag{15}$$

By (14), we have

$$\widehat{e_{ni}} = (n-i)\left(\frac{1}{(n+1)^p} - \frac{1}{n^p}\right) + \frac{1}{(n+1)^p} = O\left(\phi_n\right). \tag{16}$$

Now set $X_n = \frac{\phi_n}{e_n n}$. Then direct calculations yield that

$$\Delta X_n = O\left(n^{-2}\right) = O(\phi_n),\ 1 < p \leq 2,$$

and

$$\sum_{v=1}^{n} \left| e_{vv} \widehat{e}_{n,v+1} \right| = O\left(\phi_n\right).$$

Lemma 3.7. *([20])Let $p > 1$ and $\{\alpha_n\}$ be a sequence of positive numbers. If $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε-power decreasing for some $\varepsilon > 0$ such that $p - 2 + \varepsilon > 0$, then $D \in \mathcal{S}\left(\alpha_n, \phi_n\right)$.*

Therefore, we have

Theorem 3.8. *Let $1 < p \leq 2$ and let $\{\alpha_n\}$ be a sequence of positive numbers. Assume that $\{\lambda_n\} \in BV$ is a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and (A), (B) in Theorem 1.2 hold. If $\left\{\alpha_n \phi_n^{k-1} n^{-1}\right\}$ is quasi-ε-power decreasing for some $\varepsilon > 0$ such that $p - 2 + \varepsilon > 0$, then the summability of $|E, \alpha_n|_k$ for $k \geq 1$, of the series $\sum C_n(t) X_n \lambda_n$ at any point is a local property of f.*

3.4. Riesz's matrices

We firstly establish a general result, then apply it to the Riesz's matrices.

Lemma 3.9. *([20]) Let A be a lower triangular matrix with nonnegative entries, and $\{\alpha_n\}$ be a sequence of positive numbers. If*

(I) $\bar{a}_{n0} = 1,\ n = 0, 1, \cdots,$
(II) $a_{n-1,v} \geq a_{nv}$ for $n \geq v + 1$,
(III) $na_{nn} = O(1)$,
(IV) $\sum_{n=v+1}^{\infty} \alpha_n n^{-k+1} \left|\Delta_v \widehat{a}_{nv}\right| = O\left(\alpha_v a_{vv} v^{-k+1}\right)$,
(V) $\sum_{n=v+1}^{\infty} \alpha_n n^{-k+1} \widehat{a}_{n,v+1} = O\left(\alpha_v v^{-k+1}\right)$,

then $A \in \mathcal{S}\left(\alpha_n, n^{-1}\right)$.

Now, by setting $\phi_0 = 1, X_0 = 0, \phi_n := n^{-1}, X_n = \left(n\phi_n\right)^{-1}, n = 1, 2, \cdots$, and applying Theorem 2, we have

Theorem 3.10. *Let $\{\alpha_n\}$ be a sequence of positive numbers, and let A be a lower triangular matrix with nonnegative entries satisfying conditions (I)-(V) of Lemma 3.9. Assume that $\{\lambda_n\} \in BV$ is a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and (A), (B) in Theorem 1.2 hold. If (3) and (4) hold, then the summability of $|A, \alpha_n|_k$ for $k \geq 1$, of the series $\sum C_n(t) X_n \lambda_n$ at any point is a local property of f.*

We now show that under some necessary conditions, Riesz matrix R satisfies all the conditions in Lemma 3.9. For any positive sequence $\{p_n\}$ such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$, the corresponding Riesz matrix R has the entries $r_{nv} := \frac{p_v}{P_n}$, $v = 0, 1, \cdots, n;\ n = 0, 1, 2, \cdots$. Now obviously, we have $\bar{r}_{n0} = 1$ and $r_{n-1,v} \geq r_{nv}$ for $n \geq v + 1$. Direct calculations yield that (set $P_{-1} = 0$)

$$\widehat{r}_{nv} = \frac{P_{v-1} p_n}{P_n P_{n-1}}, \tag{17}$$

and

$$\left|\Delta_v \widehat{r}_{nv}\right| = \frac{p_n p_v}{P_n P_{n-1}}. \tag{18}$$

So if $np_n = O(P_n)$ and

$$\sum_{n=v+1}^{m+1} \alpha_n n^{-k+1} \frac{p_n}{P_n P_{n-1}} = O\left(\alpha_v v^{-k+1} P_v^{-1}\right), \tag{19}$$

then R satisfies all conditions in Lemma 3.9.

Thus, we have (note that $X_n := \left(n\phi_n\right)^{-1}$)

Theorem 3.11. *Let $\{p_n\}$ be a positive sequence satisfying $P_n \to \infty$, $np_n = O(P_n)$ and (19). Assume that $\{\lambda_n\} \in BV$ is a sequence of complex numbers such that $\lambda_{n+1} = O(|\lambda_n|)$ for $n = 1, 2, \cdots$, and (A), (B) in Theorem 1.2 hold. If (3) and (4) hold, then the summability of $|R, \alpha_n|_k$ for $k \geq 1$, of the series $\sum C_n(t) X_n \lambda_n$ at any point is a local property of f.*

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