Filomat 28:8 (2014), 1687–1697
DOI 10.2298/FIL1408687S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Predictor Homotopy Analysis Method (Pham) for Nano Boundary Layer Flows with Nonlinear Navier Boundary Condition: Existence of Four Solutions

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Abstract. In the present work, the classical laminar boundary layer equation of the flow away from the origin past a wedge with the no-slip boundary condition replaced by a nonlinear Navier boundary condition is considered. This boundary condition contains an arbitrary index parameter, denoted by $n > 0$, which appears in the differential equation to be solved. Predictor homotopy analysis method (PHAM) is applied to this problem and more, it is proved corresponding to the value $n = \frac{1}{3}$, there exist four solutions. Furthermore, these solutions are approximated by analytical series solution using PHAM for further physical interpretations.

1. Introduction

In the theoretical investigations of the boundary layer equations, it has more usually been applied the no-slip boundary conditions at the fluid-solid interface, which is a fundamental notion in fluid mechanics [7, 16, 32]. Also, it often assumes that the fluid velocity component is zero relative to the solid boundary. This is not true for fluid flows at the micro or nano scale and the no-slip boundary condition does not apply but a certain degree of tangential velocity slip must be replaced [12]. Navier [26] proposed a boundary condition which states the component of the fluid velocity tangential to the boundary walls is proportional to the tangential stress. Subsequently, by some researchers [11, 23, 30, 36], the linear Navier boundary condition has been extended to a nonlinear form

$$|u| = l\left(\left|\frac{\partial u}{\partial y}\right|\right)^{n}, \tag{1}$$

2010 *Mathematics Subject Classification*. 35Q30; 65N99; 76D05.
Keywords. Nano boundary layer flow, nonlinear Navier boundary condition, predictor homotopy analysis method, existence of for solution
Received: 26 July 2013; Accepted: 03 February 2014
Communicated by Professor Jelena Manojlovic
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where $l > 0$ is the constant slip length and $n > 0$ is an arbitrary power parameter. Consider the steady two-dimensional boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

subject to the boundary conditions

$$|u| = l\left(\left|\frac{\partial u}{\partial y}\right|\right)^n \quad \text{and} \quad v = 0 \quad \text{at} \quad y = 0, \tag{4}$$

$$u \equiv U(x) = ax^m \ \text{ as } \ y \to +\infty, \tag{5}$$

where x and y are the dimensionless Cartesian coordinates measured along the plate and normal to it. u and v are the velocity components along the x and y axes. U is a given external inviscid velocity field. The parameters a, m and n are constants. The case $a > 0$ is of main interest when describing flows away from the origin, and $a < 0$ when the external stream flows towards the origin (for more details see [10, 29]). Matthews and Hill [22, 23] introduced the following similarity transformation, by using Lie symmetries analysis

$$\eta = x^{-\frac{n-1}{3n-2}}y, \tag{6}$$

and the stream function $f(\eta)$ defined by

$$\frac{u}{U} = \frac{1}{a}f', \qquad x^{\frac{2n-1}{3n-2}}\frac{v}{U} = \frac{1}{a}\left(\frac{n-1}{3n-2}\eta f' - \frac{2n-1}{3n-2}\eta f\right), \tag{7}$$

with $m = \frac{n}{3n-2}$ and $n \neq \frac{3}{2}$. By these transformations, Eqs. (2) and (3) can be written as

$$f''' + \frac{2n-1}{3n-2}ff'' - \frac{n}{3n-2}\left(f'^2 - a^2\right) = 0, \tag{8}$$

where the prime denotes differentiation with respect to η. The boundary conditions depend on the value of n. For $n \neq \frac{1}{2}$, the boundary conditions are

$$f = 0 \quad \text{and} \quad |f'| = l\left(|f''|\right)^n \quad \text{at} \quad \eta = 0, \tag{9}$$

$$f' \to a \ \text{ as } \ \eta \to +\infty. \tag{10}$$

A further simplification shows the parameter a could be removed from the governing equation and the second boundary condition (10) if η and l are multiplied by $\sqrt{|a|}$ and $|a|^{\frac{3n-2}{2}}$, respectively; and f is multiplied by $\frac{1}{\sqrt{a}}$ for $a > 0$. Therefore, the differential equation of the flow away from the origin past a wedge is converted to

$$f''' + \frac{2n-1}{3n-2}ff'' - \frac{n}{3n-2}\left(f'^2 - 1\right) = 0, \tag{11}$$

with the boundary condition

$$f(0) = 0, \quad |f'(0)| = l\left(|f''(0)|\right)^n, \quad f'(+\infty) = 1. \tag{12}$$

Homotopy analysis method (HAM) has been developed by Liao [17–19] to obtain series solutions of controllable convergence to various nonlinear problems. This technique has successfully been applied in the latter decade to several nonlinear problems such as the viscous flows of non-Newtonian fluids [13–15, 27, 28], the KdV-type equations [1, 33], finance problems [42], Riemann problems related to nonlinear shallow water equations [39], projectile motion [41], Glauert-jet flow [8], nonlinear water waves [35], groundwater flows [34], Burgers-Huxley equation [24], time-dependent Emden-Fowler type equations [6], differential difference equation [38], Laplace equation with Dirichlet and Neumann boundary conditions [25], thermal-hydraulic networks [9], and so on. Also, Liao applied the HAM in erudite way to obtain new branches of solutions of nonlinear problems [20, 21, 31, 40]. The HAM logically contains some other non-perturbation techniques, such as Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method.

The predictor homotopy analysis method, which is the reconstruction of the homotopy analysis method by adding rule of multiplicity of solutions and so-called prescribed parameter, has been recently proposed by E. Shivanian and S. Abbasbandy [2–4] to predict multiplicity of solutions of boundary value problem. It takes the advantage of the convergence-controller parameter in erudite way in order to achieve this important goal that is anticipating existence of multiple solutions besides obtaining all branches of solutions analytically. Consequently, this method might obtain new unfamiliar class of solutions of nonlinear problems which is of fundamental interest for practical using in science and engineering. This method has been successfully applied to predict and obtain multiple solutions of a class of nonlinear reactive transport model and some fractional differential equations [5, 37].

In the next sections, we consider the boundary value problem (11) and (12) when $n = \frac{1}{3}$. In this case, Matthews and Hill [23] have shown that the solution is not unique. Using PHAM we prove corresponding to the mentioned value of n, there exist exactly four solutions for some values of slip length l for further physical interpretations.

2. Application of the Predictor Homotopy Analysis Method

Assuming $n = \frac{1}{3}$, the problem (11) and (12) is converted to the following one

$$f''' + \frac{1}{3} f f'' + \frac{1}{3}\left(f'^2 - 1\right) = 0, \tag{13}$$

with the boundary condition

$$f(0) = 0, \quad |f'(0)| = l\sqrt[3]{|f''(0)|}, \quad f'(+\infty) = 1. \tag{14}$$

As it is usual in the frame of PHAM, we prescribe the parameter α and use the initial condition $f'(0) = \alpha$. Now, the boundary conditions (14) are replaced by

$$f(0) = 0, \quad f'(0) = \alpha, \quad f'(+\infty) = 1. \tag{15}$$

It will be used the following equation as forcing condition which is played important role in determining the multiplicity of solutions

$$|f'(0)| = l\sqrt[3]{|f''(0)|}. \tag{16}$$

2.1. Zero-order Deformation Equation

Based on the boundary conditions (15), it is straightforward to use the set of base functions

$$\{\eta^m e^{-n\eta} | m, n = 0, 1, 2, ...\}. \tag{17}$$

Let denote $f_0(\eta) = (\alpha - 1) + \eta - (\alpha - 1)e^{-\eta}$ an initial approximation guess of the exact solution $f(\eta)$ which satisfies the boundary conditions (15) automatically. Also, as that is well-known in the frame of PHAM, assume $\hbar \neq 0$ denote convergence-controller parameter, $H(\eta) = 1$ an auxiliary function, and $\mathfrak{L}$ an auxiliary linear operator. Now using $p \in [0, 1]$ as an embedding parameter, we construct the zero-order deformation equation and the corresponding boundary conditions as follows:

$$(1 - p)\mathfrak{L}\left[\varphi(\eta, \alpha; p) - f_0(\eta, \alpha)\right] = pH(\eta)\mathcal{N}\left[\varphi(\eta, \alpha; p)\right], \tag{18}$$

$$\varphi(0, \alpha; p) = 0, \quad \frac{\partial\varphi}{\partial\eta}(0, \alpha; p) = \alpha, \quad \frac{\partial\varphi}{\partial\eta}(+\infty, \alpha; p) = 1, \tag{19}$$

where $\varphi(\eta, \alpha; p)$ is an unknown function to be determined. Also, the auxiliary linear and nonlinear operators $\mathfrak{L}$ and $\mathcal{N}$ are defined as

$$\mathfrak{L}\left[\varphi(\eta, \alpha; p)\right] = \frac{\partial^3\varphi}{\partial\eta^3} + \frac{\partial^2\varphi}{\partial\eta^2}, \tag{20}$$

$$\mathcal{N}\left[\varphi(\eta, \alpha; p)\right] = \frac{\partial^3\varphi}{\partial\eta^3} + \frac{1}{3}\varphi\frac{\partial^2\varphi}{\partial\eta^2} + \frac{1}{3}\left[\left(\frac{\partial\varphi}{\partial\eta}\right)^2 - 1\right]. \tag{21}$$

When $p = 0$, the zero order deformation equation (18) becomes

$$\mathfrak{L}[\varphi(\eta, \alpha; 0) - f_0(\eta, \alpha)] = 0, \tag{22}$$

which gives $\varphi(\eta, \alpha; 0) = f_0(\eta, \alpha)$. When $p = 1$, the Eq. (18) leads to

$$\mathcal{N}[\varphi(\eta, \alpha; 1)] = 0, \tag{23}$$

which is exactly the same as the original Eq. (13) provided that $\varphi(\eta, \alpha; 1) = f(\eta, \alpha)$. We now expand the function $\varphi(\eta, \alpha; p)$ in a Taylor series with respect to the embedding parameter p. This Taylor expansion can be written in the form

$$\varphi(\eta, \alpha; p) = f_0(\eta, \alpha) + \sum_{m=1}^{+\infty} f_m(\eta, \alpha)p^m, \tag{24}$$

where

$$f_m(\eta, \alpha) = \frac{1}{m!}\frac{\partial^m\varphi(\eta, \alpha; p)}{\partial p^m}\Big|_{p=0}, \quad m = 0, 1, 2, ... \tag{25}$$

As it is well known in during the frame of PHAM, when the linear operator $\mathfrak{L}$, the initial approximation $f_0(\eta,\alpha)$, the auxiliary parameter $\hbar \neq 0$, and the auxiliary function $H(\eta) \neq 0$ are chosen properly, the series (24) converges for $p = 1$, and thus

$$f(\eta,\alpha) = f_0(\eta,\alpha) + \sum_{m=1}^{+\infty} f_m(\eta,\alpha) = \sum_{m,n=0}^{+\infty} a_{m,n}\eta^m e^{-n\eta}. \tag{26}$$

2.2. High-order Deformation Equation

Assume that the linear operator $\mathfrak{L}$, the initial approximation $f_0(\eta,\alpha)$, and the auxiliary function $H(\eta) \neq 0$ have been chosen properly (it is worth mentioning here that $\hbar \neq 0$ so-called convergence-controller parameter will be determined later), the unknown functions $f_m(\eta,\alpha)$ in Eq. (26) can be determined with the aid of the high-order deformation equations as follows. At first we define the vector $\overrightarrow{f}_n = \{f_0(\eta), f_1(\eta),, f_n(\eta)\}$ then, differentiating the zero-order deformation equation (18) m times with respect to the embedding parameter p, dividing it by $m!$, setting subsequently $p = 0$ and taking into account the boundary conditions (19), one obtains the mth-order deformation equation

$$\mathfrak{L}[f_m(\eta,\alpha) - \chi_m f_{m-1}(\eta,\alpha)] = \hbar H(\eta)\mathcal{R}_m(\vec{f}_{m-1},\eta,\alpha), \tag{27}$$

subject to the boundary conditions

$$f_m(0,\alpha) = 0, \qquad f'_m(0,\alpha) = 0, \qquad f'_m(+\infty,\alpha) = 0, \tag{28}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \tag{29}$$

and

$$\begin{aligned} \mathcal{R}_m(\vec{f}_{m-1},\eta,\alpha) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\varphi(\eta,\alpha;p)]}{\partial p^{m-1}}\Big|_{p=0} \\ &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\sum_{n=0}^{+\infty} u_n(\eta,\alpha)p^n]}{\partial p^{m-1}}\Big|_{p=0} \\ &= f'''_{m-1} + \frac{1}{3}\sum_{j=0}^{m-1} f_j f''_{m-1-j} + \frac{1}{3}\left[\sum_{j=0}^{m-1} f'_j f'_{m-1-j} - (1-\chi_m)\right]. \end{aligned} \tag{30}$$

The high-order deformation equation (27) obviously, is just the linear ordinary differential equation with boundary condition (28) and, can be easily solved by using some symbolic software programs such as Mathematica or Maple. In this way, starting by $f_0(\eta,\alpha)$, we obtain the functions $f_m(\eta,\alpha)$ for $m = 1,2,3,...$ from Eq. (27) and (28) successively. Accordingly, the M-th order approximate solution of the problem (13) and (15) is given by

$$f(\eta,\alpha) \approx F_M(\eta,\alpha,\hbar) = f_0(\eta,\alpha) + \sum_{m=1}^{M} f_m(\eta,\alpha) = \sum_{m,n=0}^{M'} a_{m,n}\eta^m e^{-n\eta}. \tag{31}$$

2.3. Prediction of the Multiplicity of Solutions

It is noteworthy to indicate that up to this stage, the linear operator$\mathfrak{L}$, the initial approximation $f_0(\eta, \alpha)$, and the auxiliary function $H(\eta) \neq 0$ have been chosen properly so that the series solutions (31) would be convergence. However, there are still two unknown parameters in the series (31) namely α (prescribed parameter) and $\hbar$ (convergence-controller parameter) which should be determined. It is essential that existence of unique or multiple solutions in terms of the basic functions (17) for the original boundary value problem (13)-(14) depends on the fact whether the forcing condition (16) ($|f'(0)| = l\sqrt[3]{|f''(0)|}$), admits unique or multiple values for the formally introduced parameter α in the boundary conditions (15). This stage is called *rule of multiplicity of solutions* that is a criterion in order to know how many solutions the boundary value problem (13)-(14) admits. The so-called *rule of multiplicity of solutions* is applied as follows:
Consider the Mth order approximate solution (31) and set into the forcing condition (16), so the following equation is derived

$$|\alpha|^3 = l^3|F_M(0, \alpha, \hbar))|. \tag{32}$$

The above equation has two unknown parameters namely α, and $\hbar$ which controls the convergence of the PHAM series (31). It is a basic feature of PHAM that the series solution (31) and its derivatives converge at $\eta = 0$ only in that range of $\hbar$, where the parameter α does not change with the variation of $\hbar$. This means that in the plot of α as function of $\hbar$ according to Eq. (32) in an implicitly way, in the convergence range of the series $f'(0)$ some horizontal plateau occur or equivalently in the plot of the residual of (32) via α for some different $\hbar$, multiple crosses with horizontal axis occur. The number of such horizontal plateaus where $\alpha(\hbar)$ becomes constant or equivalently number of crosses with horizontal axis, gives key role about the multiplicity of the solutions of problem (it is called this point as *rule of multiplicity of solutions*).

3. Results and Discussions

As it has been mentioned before, Eq (32) is played important role in determining the multiplicity of solutions. We have used this equation to plot α as function of $\hbar$ in Figure 1. As it is seen in this Figure, two horizontal plateaus occur which yields $f'(0)$ admits two values. Equivalently, we have shown the residual of equation (32), i.e.

$$\mathrm{RES}(\alpha, \hbar) = |\alpha|^3 - l^3|F_M(0, \alpha, \hbar))|, \tag{33}$$

in Figure 2 for different values of $\hbar$, namely $\hbar = -0.7, -0.8, -0.9$. It is clear from the Figure that there exist two crosses with horizontal axis which confirms again $\alpha = f'(0)$ admits two values. Our computations on series solution (31) with more and more terms shows these two values for α are

$$\alpha = f'(0) = 1.027595 \quad \alpha = f'(0) = 1.801706. \tag{34}$$

On the other hand, the enforcing condition (16) indicates that for each value of α there are corresponding two values for $f''(0)$ as indicated in Figure 3. Therefore, there exist exactly four solutions for the original boundary value problem (13)-(14) with the following properties:

$$l = 2, \quad f(0) = 0, \quad f'(0) = 1.027595, \quad f''(0) = 0.135636,$$
$$l = 2, \quad f(0) = 0, \quad f'(0) = 1.027595, \quad f''(0) = -0.135636,$$
$$l = 2, \quad f(0) = 0, \quad f'(0) = 1.801706, \quad f''(0) = 0.731075,$$
$$l = 2, \quad f(0) = 0, \quad f'(0) = 1.801706, \quad f''(0) = -0.731075.$$

We have done the same actions for when $l = 1.8$ and $l = 2.2$. Also, the approximate solutions for these three slip length parameters are shown in Figures 4, 5 and 6.

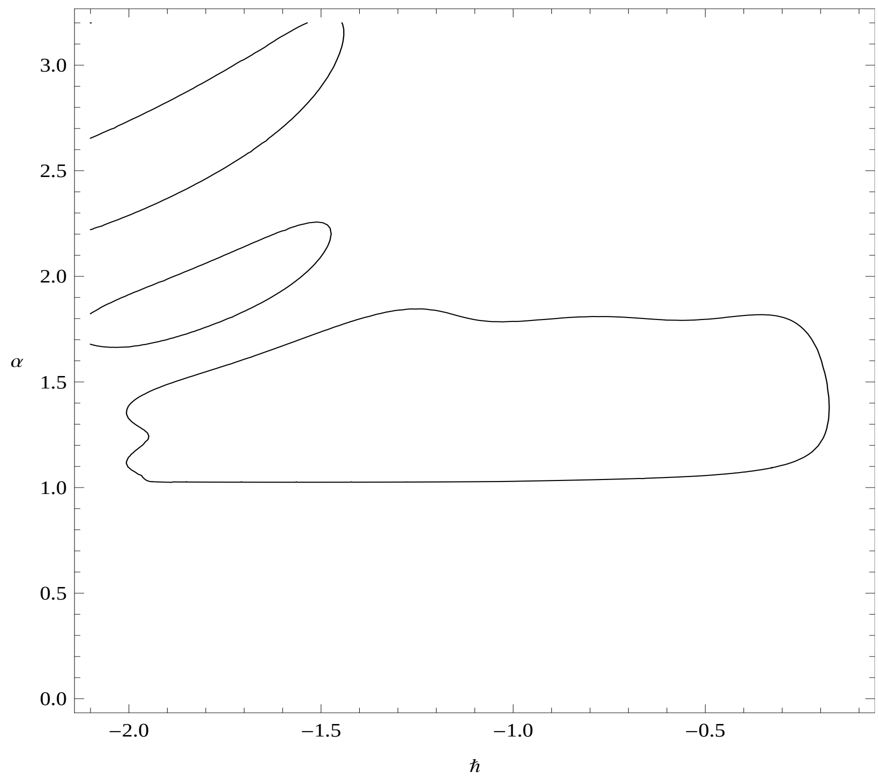

Fig. 1: Diagram of $\alpha = f'(0)$ as function of $\hbar$ in according to Eq. (32) with $M = 18$ and $l = 2$.

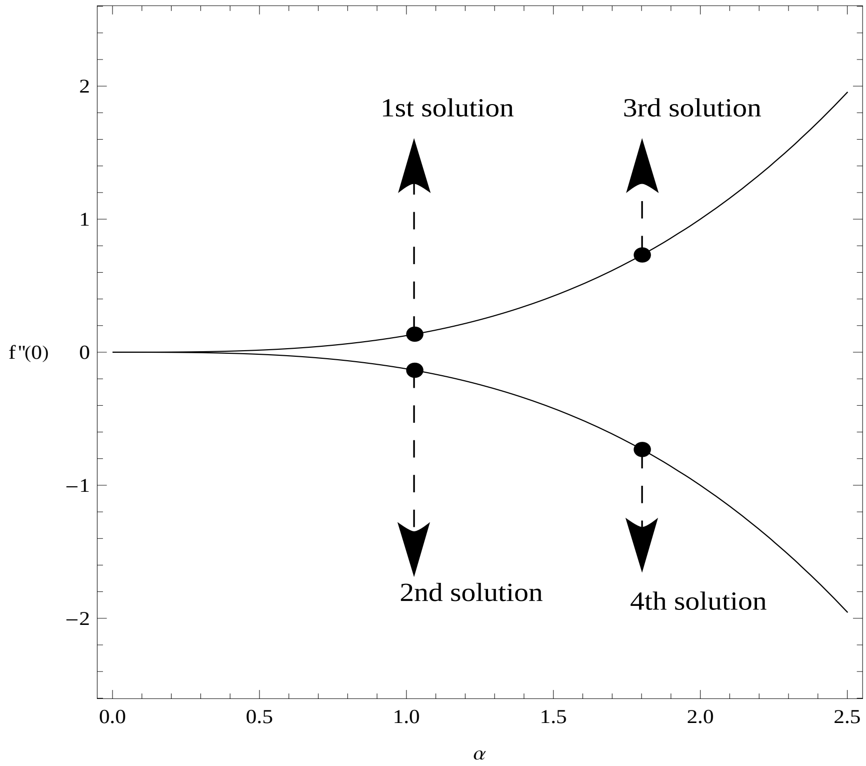

Fig. 2: Diagram of RES($\alpha, \hbar$) as function of α for $\hbar = -0.7, -0.8, -0.9$ with $l = 2$.

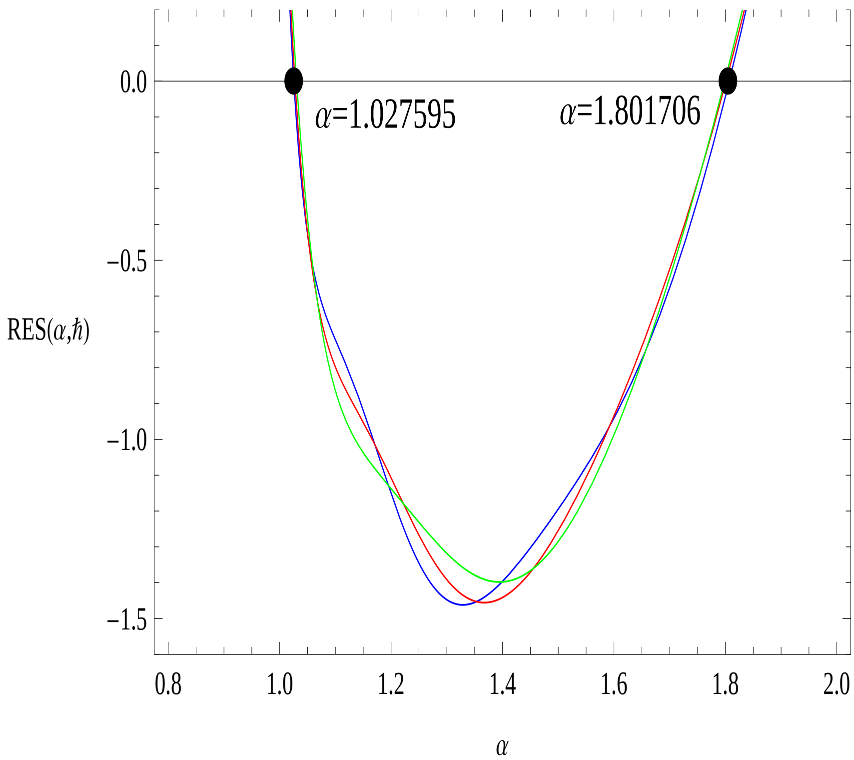

Fig. 3: Diagram of $f''(0)$ via $\alpha = f'(0)$ in according to Eq. (16) when $l = 2$.

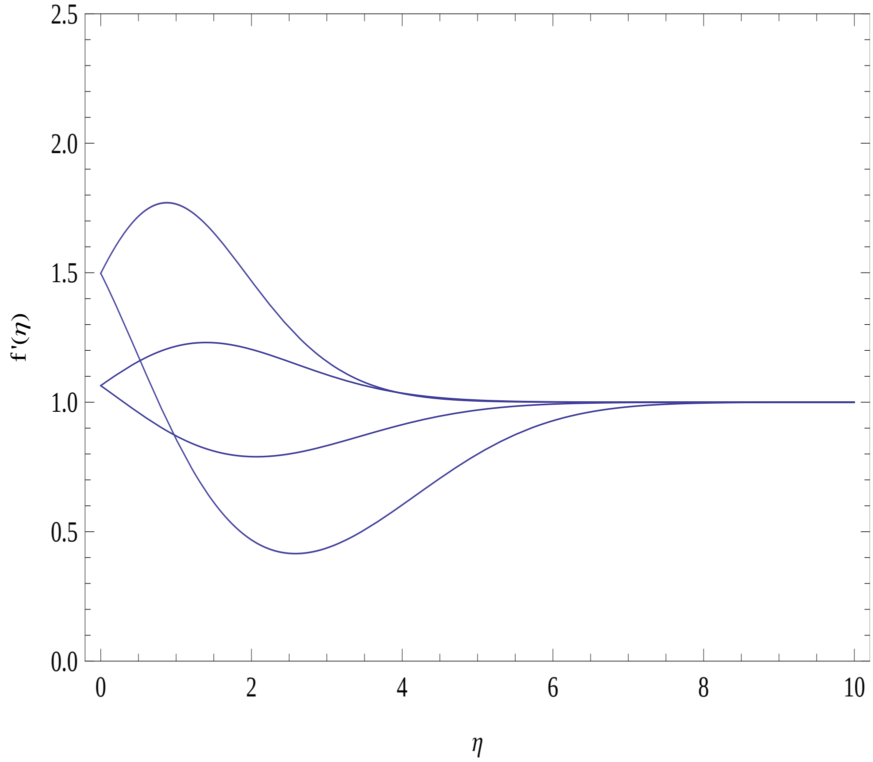

Fig. 4: Diagram of $f'(\eta)$: Existence of four solutions for the problem (13)-(14) when $l = 1.8$.

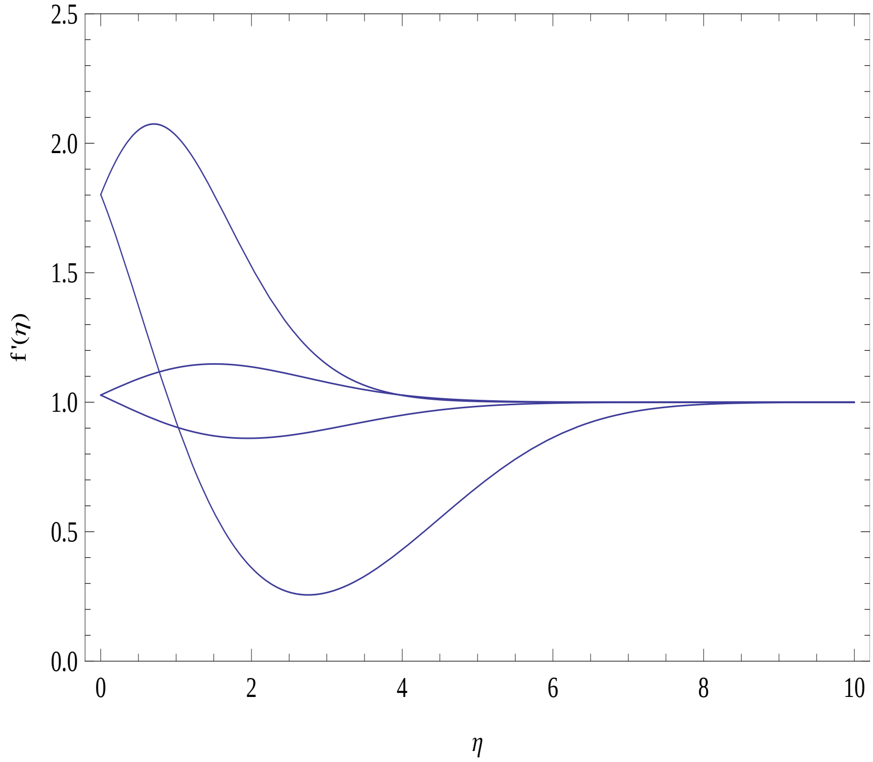

Fig. 5: Diagram of $f'(\eta)$: Existence of four solutions for the problem (13)-(14) when $l = 2$.

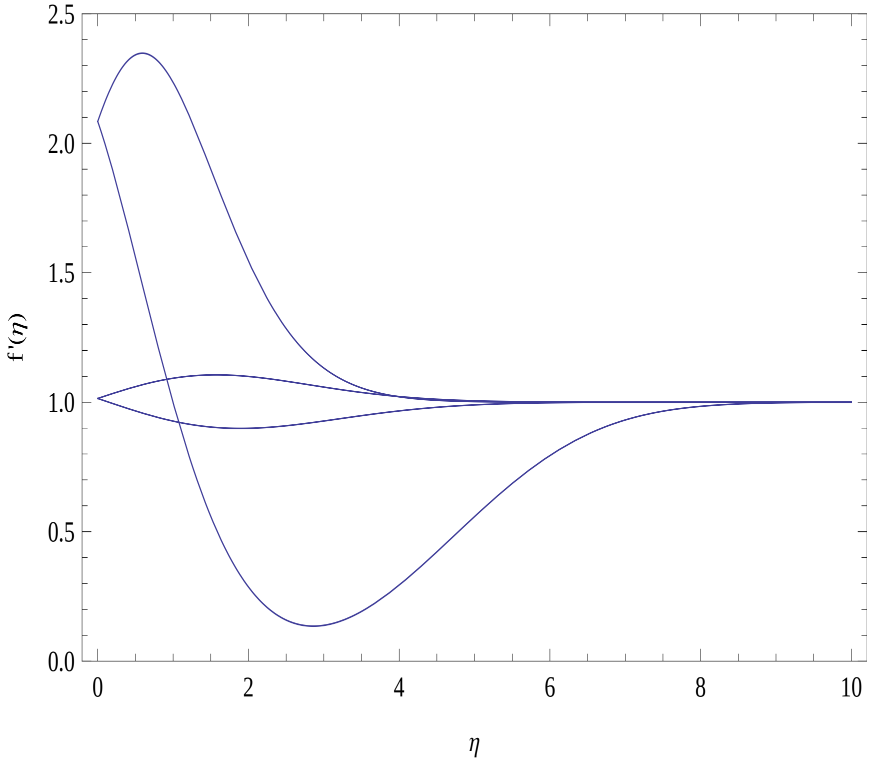

Fig. 6: Diagram of $f'(\eta)$: Existence of four solutions for the problem (13)-(14) when $l = 2.2$.

4. Conclusions

In this study the classical laminar boundary layer equations is considered with the nonlinear Navier boundary condition Eq. (1) which contains the arbitrary power index n. For $n = \frac{1}{3}$, we have proved by the predictor homotopy analysis method (PHAM), the problem could be admit four solutions for some values of slip length parameter. Also, approximate solutions have been shown graphically in Figures 4, 5 and 6 for further physical interpretations.

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