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Some Cesaro-Type Summability Spaces of Order α and Lacunary Statistical Convergence of Order α

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Abstract. In the paper [32], we have defined the concepts of lacunary statistical convergence of order α and strong $N_\theta(p)-$summability of order α for sequences of complex (or real) numbers. In this paper we continue to examine others relations between lacunary statistical convergence of order α and strong $N_\theta(p)-$summability of order α.

1. Introduction

The idea of statistical convergence was given by Zygmund [34] in the first edition of his monograph puplished in Warsaw in 1935. The consept of statistical convergence was introduced by Steinhaus [33] and Fast [10] and later reintroduced by Schoenberg [31] independently. Over the years and under different names statistical convergence was discussed in the theory of Fourier analysis, ergodik theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Bhardwaj and Bala [2], Çolak ([3],[4]), Connor [5], Et et al. [9], Fridy [12], Fridy and Orhan ([13], [14]), Güngör et al. ([17],[18]), Işık [19], Mursaleen et al. ([23],[24],[25]), Rath and Tripathy [28], Salat [30] and many others.

The $\alpha-$ *density* of a subset E of $\mathbb{N}$ was defined by Çolak [3]. Let α be a real number such that $0 < \alpha \leq 1$. The $\alpha-$ *density* of a subset E of $\mathbb{N}$ is defined by

$$\delta_\alpha(E) = \lim_n \frac{1}{n^\alpha} |\{k \leq n : k \in E\}| \text{ provided the limit exists,}$$

where $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n.

If $x = (x_k)$ is a sequence such that x_k satisfies property $p(k)$ for all k except a set of $\alpha-$density zero, then we say that x_k satisfies $p(k)$ for "almost all k according to α" and we abbreviate this by "$a.a.k(\alpha)$".

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The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [15] and after then statistical convergence of order α and strong $p-$Cesàro summability of order α studied by Çolak [3].

Let $x = (x_k) \in w$ and $0 < \alpha \leq 1$ be given. The sequence (x_k) is said to be statistically convergent of order α if there is a complex number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0$$

i.e. for $a.a.k\,(\alpha)$ $|x_k - L| < \varepsilon$ for every $\varepsilon > 0$, in which case we say that x is statistically convergent of order α, to L. In this case we write $S^\alpha - \lim x_k = L$ [3].

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Throught this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Subsequently lacunary sequences have been studied in ([6],[11],[13],[14],[21]).

2. Main Results

In this section we give the main results of the paper. In Theorem 2.10 we give the inclusion relations between the sets of $S_\theta^\alpha-$statistically convergent sequences for different $\alpha^{'}$s and different $\theta^{'}$s. In Theorem 2.12 we give the relationships between strong $N_\theta^\alpha(p)-$summability and strong $N_\theta^\beta(p)-$summability for different $\theta^{'}$s. In Theorem 2.14 we give the relationship between strong $N_\theta^\beta(p)-$summability and $S_\theta^\alpha-$statistical convergence for different $\theta^{'}$s.

Definition 2.1 [32] Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in w$ is said to be $S_\theta^\alpha-$statistically convergent (or lacunary statistically convergent sequence of order α) if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| = 0,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denote the αth power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = \left(h_1^\alpha, h_2^\alpha, ..., h_r^\alpha, ...\right)$. In this case we write $S_\theta^\alpha - \lim x_k = L$. The set of all $S_\theta^\alpha-$statistically convergent sequences will be denoted by S_θ^α. For $\theta = (2^r)$ we shall write S^α instead of S_θ^α and in the special case $\alpha = 1$ and $\theta = (2^r)$ we shall write S instead of S_θ^α.

The lacunary statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general. For this $x = (x_k)$ be defined as follows:

$$x_k = \begin{cases} 1, k = 2r \\ 0, k \neq 2r \end{cases} \quad r = 1, 2, 3, ...$$

then both

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - 1| \geq \varepsilon\}\right| \leq \lim_{r\to\infty} \frac{k_r - k_{r-1}}{2h_r^\alpha} = \lim_{r\to\infty} \frac{h_r}{2h_r^\alpha} = 0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r : |x_k-0|\geq\varepsilon\}\right|\leq\lim_{r\to\infty}\frac{k_r-k_{r-1}}{2h_r^\alpha}=\lim_{r\to\infty}\frac{h_r}{2h_r^\alpha}=0$$

for $\alpha>1$, such that $x=(x_k)$ lacunary statistically convergence of order α, both to 1 and 0, i.e., $S_\theta^\alpha-\lim x_k=1$ and $S_\theta^\alpha-\lim x_k=0$. But this is impossible.

Definition 2.2 [32] Let $\theta=(k_r)$ be a lacunary sequence, $\alpha\in(0,1]$ be any real number and p be a positive real number. A sequence x is said to be strongly $N_\theta^\alpha(p)-$summable (or strongly $N_\theta(p)-$summable of order α) if there is a real number L such that

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}|x_k-L|^p=0.$$

In this case we write $N_\theta^\alpha(p)-\lim x_k=L$. The set of all strongly $N_\theta(p)-$summable sequences of order α will be denoted by $N_\theta^\alpha(p)$. In the special case $\alpha=1$ we shall write $N_\theta(p)$ instead of $N_\theta^\alpha(p)$ and also in the special case $\theta=(2^r)$ we shall write w_p^α instead of $N_\theta^\alpha(p)$. If $L=0$, then we shall write $w_{p,0}^\alpha$ instead of w_p^α. The set of all strongly $N_\theta(p)-$summable sequences of order α, to 0 will be denoted by $N_{\theta,0}^\alpha(p)$.

Definition 2.3 Let $0<\alpha\leq1$ and $\theta=(k_r)$ be a lacunary sequence. The sequence x is said to be an $S_\theta^\alpha-$Cauchy sequence if there is a subsequence $\left\{x_{k'(r)}\right\}$ of x such that $k'(r)\in I_r$ for each r, $\lim_r x_{k'(r)}=L$, and for every $\varepsilon>0$

$$\lim_r\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\left|x_k-x_{k'(r)}\right|\geq\varepsilon\right\}\right|=0.$$

Theorem 2.4 [32] Let $0<\alpha\leq1$ and $x=(x_k)$, $y=(y_k)$ be sequences of real numbers, then

(i) If $S_\theta^\alpha-\lim x_k=x_0$ and $c\in\mathbb{C}$, then $S_\theta^\alpha-\lim(cx_k)=cx_0$,

(ii) If $S_\theta^\alpha-\lim x_k=x_0$ and $S_\theta^\alpha-\lim y_k=y_0$, then $S_\theta^\alpha-\lim(x_k+y_k)=x_0+y_0$.

The proofs of the following two theorems are obtained by using techniques Fridy ([12],Theorem 1) and Fridy and Orhan ([14],Theorem 2) respectively, therefore we give them without proofs.

Theorem 2.5 Let $0<\alpha\leq1$, then the following statements are equivalent:

(i) x is a statistically convergent sequence of order α;

(ii) x is a statistically Cauchy sequence of order α;

(iii) x is a sequence for which there is a convergent sequence y such that $x_k=y_k$ $a.a.k(\alpha)$.

Theorem 2.6 Let $0<\alpha\leq1$ and $\theta=(k_r)$ be a lacunary sequence. The sequence x is $S_\theta^\alpha-$convergent if and only if x is an $S_\theta^\alpha-$Cauchy sequence.

Theorem 2.7 Let $0<\alpha\leq1$ and $\theta=(k_r)$ be a lacunary sequence. If $\liminf_r q_r>1$, then $w_p^\alpha\subset N_\theta^\alpha(p)$.

Proof. If $\liminf_r q_r>1$ there exists $\delta>0$ such that $1+\delta\leq q_r$ for all $r\geq1$. Then for $x\in w_{p,0}^\alpha$, we write

$$\begin{aligned}\tau_r^\alpha&=\frac{1}{h_r^\alpha}\sum_{k\in I_r}|x_i|^p=\frac{1}{h_r^\alpha}\sum_{i=1}^{k_r}|x_i|^p-\frac{1}{h_r^\alpha}\sum_{i=1}^{k_{r-1}}|x_i|^p\\&=\frac{k_r^\alpha}{h_r^\alpha}\left(\frac{1}{k_r^\alpha}\sum_{i=1}^{k_r}|x_i|^p\right)-\frac{k_{r-1}^\alpha}{h_r^\alpha}\left(\frac{1}{k_{r-1}^\alpha}\sum_{i=1}^{k_{r-1}}|x_i|^p\right).\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r^\alpha}{h_r^\alpha} \le \frac{(1+\delta)^\alpha}{\delta^\alpha} \text{ and } \frac{k_{r-1}^\alpha}{h_r^\alpha} \le \frac{1}{\delta^\alpha}.$$

Hence $x \in N_{\theta,0}^\alpha(p)$.

Theorem 2.8 Let $0 < \alpha \le 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then $N_\theta(p) \subset w_p^\alpha$.

Proof. Let $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then there exists a constant $M > 0$ such that $\frac{k_r}{k_{r-1}^\alpha} < M$ for all $r \ge 1$. Now let $x \in N_{\theta,0}(p)$ and $\varepsilon > 0$, then we can find $R > 0$ and $K > 0$ such that $\sup_{i \ge R} \tau_i < \varepsilon$ and $\tau_i < K$ for all $i = 1, 2, \dots$. Then if t is any integer with $k_{r-1} < t \le k_r$, where $r > R$, we can write

$$\begin{aligned} \frac{1}{t^\alpha}\sum_{i=1}^{t}|x_i|^p &\le \frac{1}{k_{r-1}^\alpha}\sum_{i=1}^{k_r}|x_i|^p = \frac{1}{k_{r-1}^\alpha}\left(\sum_{I_1}|x_i|^p + \sum_{I_2}|x_i|^p + \dots + \sum_{Ir}|x_i|^p\right) \\ &= \frac{k_1}{k_{r-1}^\alpha}\tau_1 + \frac{k_2 - k_1}{k_{r-1}^\alpha}\tau_2 + \dots + \frac{k_R - k_{R-1}}{k_{r-1}^\alpha}\tau_R + \frac{k_{R+1} - k_R}{k_{r-1}^\alpha}\tau_{R+1} + \dots + \frac{k_r - k_{r-1}}{k_{r-1}^\alpha}\tau_r \\ &\le \left(\sup_{i\ge 1}\tau_i\right)\frac{k_R}{k_{r-1}^\alpha} + \left(\sup_{i\ge R}\tau_i\right)\frac{k_r - k_R}{k_{r-1}^\alpha} < K\frac{k_R}{k_{r-1}^\alpha} + \varepsilon M \end{aligned}$$

Hence $x \in w_{p,0}^\alpha$.

Theorem 2.9 If $x \in w^\alpha \cap N_\theta^\alpha$ and $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$ then $N_\theta^\alpha - \lim x_k = w^\alpha - \lim x_k$.

Proof. Let $N_\theta^\alpha - \lim x_k = L$ and $w^\alpha - \lim x_k = L'$, and suppose that $L \ne L'$. Since $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$ by Theorem 2.8 we have $N_{\theta,0}(p) \subset w_{p,0}^\alpha$. Since $\left(x - L'\right) \in N_{\theta,0}(p)$, it follows that $\left(x - L'\right) \in w_{p,0}^\alpha$ and therefore $\frac{1}{t^\alpha}\sum_{i=1}^{t}\left|x_i - L'\right| \to 0$. Then we have

$$\frac{1}{t^\alpha}\sum_{i=1}^{t}\left|x_i - L'\right| + \frac{1}{t^\alpha}\sum_{i=1}^{t}|x_i - L| \ge \frac{1}{t^\alpha}\left|L - L'\right| > 0,$$

and hence $L = L'$.

Theorem 2.10 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be such that $0 < \alpha \le \beta \le 1$,

(i) If

$$\lim_{r\to\infty}\inf \frac{h_r^\alpha}{\ell_r^\beta} > 0 \tag{1}$$

then $S_{\theta'}^\beta \subseteq S_\theta^\alpha$,

(ii) If

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^\beta} = 1 \tag{2}$$

then $S_\theta^\alpha \subseteq S_{\theta'}^\beta$.

Proof. (*i*) Suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let (1) be satisfied. For given $\varepsilon > 0$ we have

$$\{k \in J_r : |x_k - L| \geq \varepsilon\} \supseteq \{k \in I_r : |x_k - L| \geq \varepsilon\}$$

and so

$$\frac{1}{\ell_r^\beta} |\{k \in J_r : |x_k - L| \geq \varepsilon\}| \geq \frac{h_r^\alpha}{\ell_r^\beta} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}|$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$ and $\ell_r = s_r - s_{r-1}$. Now taking the limit as $r \to \infty$ in the last inequality and using (1) we get $S_{\theta'}^\beta \subseteq S_\theta^\alpha$.

(*ii*) Let $x = (x_k) \in S_\theta^\alpha$ and (2) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^\beta} |\{k \in J_r : |x_k - L| \geq \varepsilon\}| &= \frac{1}{\ell_r^\beta} |\{s_{r-1} < k \leq k_{r-1} : |x_k - L| \geq \varepsilon\}| \\
&+ \frac{1}{\ell_r^\beta} |\{k_r < k \leq s_r : |x_k - L| \geq \varepsilon\}| \\
&+ \frac{1}{\ell_r^\beta} |\{k_{r-1} < k \leq k_r : |x_k - L| \geq \varepsilon\}| \\
&\leq \frac{k_{r-1} - s_{r-1}}{\ell_r^\beta} + \frac{s_r - k_r}{\ell_r^\beta} + \frac{1}{\ell_r^\beta} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \\
&= \frac{\ell_r - h_r}{\ell_r^\beta} + \frac{1}{\ell_r^\beta} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \\
&\leq \frac{\ell_r - h_r^\beta}{h_r^\beta} + \frac{1}{h_r^\beta} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \\
&\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right) + \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}|
\end{aligned}$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{\ell_r}{h_r^\beta} = 1$ by (2) the first term and since $x = (x_k) \in S_\theta^\alpha$ the second term of right hand side of above inequality tend to 0 as $r \to \infty$ $\left(\text{Note that } \left(\frac{\ell_r}{h_r^\beta} - 1\right) \geq 0\right)$. This implies that $S_\theta^\alpha \subseteq S_{\theta'}^\beta$.

From Theorem 2.10 we have the following results.

Corollary 2.11 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$.
If (1) holds then,

(*i*) $S_{\theta'}^\alpha \subseteq S_\theta^\alpha$ for each $\alpha \in (0, 1]$,
(*ii*) $S_{\theta'} \subseteq S_\theta^\alpha$ for each $\alpha \in (0, 1]$,
(*iii*) $S_{\theta'} \subseteq S_\theta$.

If (2) holds then,

(*i*) $S_\theta^\alpha \subseteq S_{\theta'}^\alpha$, for each $\alpha \in (0, 1]$,

(*ii*) $S_\theta^\alpha \subseteq S_{\theta'}$, for each $\alpha \in (0, 1]$, (*iii*) $S_\theta \subseteq S_{\theta'}$.

Theorem 2.12 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. Then we have

(*i*) If (1) holds then $N_{\theta'}^{\beta}(p) \subset N_{\theta}^{\alpha}(p)$,

(*ii*) If (2) holds and $x \in \ell_\infty$ then $N_{\theta}^{\alpha}(p) \subset N_{\theta'}^{\beta}(p)$.

Proof. (*i*) Omitted.

(*ii*) Let $x = (x_k) \in N_{\theta}^{\alpha}(p)$ and suppose that (2) holds. Since $x = (x_k) \in \ell_\infty$ then there exists some $M > 0$ such that $|x_k - L| \leq M$ for all k. Now, since $I_r \subseteq J_r$ and $h_r \leq \ell_r$ for all $r \in \mathbb{N}$, we may write

$$\begin{aligned} \frac{1}{\ell_r^{\beta}} \sum_{k \in J_r} |x_k - L|^p &= \frac{1}{\ell_r^{\beta}} \sum_{k \in J_r - I_r} |x_k - L|^p + \frac{1}{\ell_r^{\beta}} \sum_{k \in I_r} |x_k - L|^p \\ &\leq \left(\frac{\ell_r - h_r}{\ell_r^{\beta}} \right) M^p + \frac{1}{\ell_r^{\beta}} \sum_{k \in I_r} |x_k - L|^p \\ &\leq \left(\frac{\ell_r - h_r^{\beta}}{h_r^{\beta}} \right) M^p + \frac{1}{h_r^{\beta}} \sum_{k \in I_r} |x_k - L|^p \\ &\leq \left(\frac{\ell_r}{h_r^{\beta}} - 1 \right) M^p + \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} |x_k - L|^p \end{aligned}$$

for every $r \in \mathbb{N}$. Therefore $x = (x_k) \in N_{\theta'}^{\beta}(p)$.

From Theorem 2.12 we have the following results.

Corollary 2.13 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.

If (1) holds then,

(*i*) $N_{\theta'}^{\alpha}(p) \subseteq N_{\theta}^{\alpha}(p)$, for each $\alpha \in (0, 1]$,

(*ii*) $N_{\theta'}(p) \subseteq N_{\theta}^{\alpha}(p)$, for each $\alpha \in (0, 1]$,

(*iii*) $N_{\theta'}(p) \subseteq N_{\theta}(p)$,

If (2) holds then,

(*i*) $\ell_\infty \cap N_{\theta}^{\alpha}(p) \subset N_{\theta'}^{\alpha}(p)$, for each $\alpha \in (0, 1]$,

(*ii*) $\ell_\infty \cap N_{\theta}^{\alpha}(p) \subset N_{\theta'}(p)$ for each $\alpha \in (0, 1]$,

(*iii*) $\ell_\infty \cap N_{\theta}(p) \subset N_{\theta'}(p)$.

Theorem 2.14 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. Then

(*i*) Let (1) holds, if a sequence is strongly $N_{\theta'}^{\beta}(p)$−summable to L, then it is S_{θ}^{α}−statistically convergent to L,

(*ii*) Let (2) holds, if a bounded sequence is S_{θ}^{α}−statistically convergent to L then it is strongly $N_{\theta'}^{\beta}(p)$−summable to L.

Proof. (i) For any sequence $x = (x_k)$ and $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{k\in J_r}|x_k - L|^p &= \sum_{\substack{k\in J_r\\ |x_k-L|\geq\varepsilon}}|x_k - L|^p + \sum_{\substack{k\in J_r\\ |x_k-L|<\varepsilon}}|x_k - L|^p\\ &\geq \sum_{\substack{k\in I_r\\ |x_k-L|\geq\varepsilon}}|x_k - L|^p\\ &\geq |\{k\in I_r : |x_k - L|\geq\varepsilon\}|\,\varepsilon^p\end{aligned}$$

and so that

$$\begin{aligned}\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|x_k - L|^p &\geq \frac{1}{\ell_r^\beta}|\{k\in I_r : |x_k - L|\geq\varepsilon\}|\,\varepsilon^p\\ &\geq \frac{h_r^\alpha}{\ell_r^\beta}\frac{1}{h_r^\alpha}|\{k\in I_r : |x_k - L|\geq\varepsilon\}|\,\varepsilon^p.\end{aligned}$$

Since (1) holds it follows that if $x = (x_k)$ is strongly $N_{\theta'}^{\beta}(p)-$summable to L, then it is $S_\theta^\alpha-$statistically convergent to L.

(ii) Suppose that $S_\theta^\alpha - \lim x_k = L$ and $x = (x_k) \in \ell_\infty$. Then there exists some $M > 0$ such that $|x_k - L| \leq M$ for all k, then for every $\varepsilon > 0$ we may write

$$\begin{aligned}\frac{1}{\ell_r^\beta}\sum_{k\in J_r}|x_k - L|^p &= \frac{1}{\ell_r^\beta}\sum_{k\in J_r-I_r}|x_k - L|^p + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|x_k - L|^p\\ &\leq \left(\frac{\ell_r - h_r}{\ell_r^\beta}\right)M^p + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|x_k - L|^p\\ &\leq \left(\frac{\ell_r - h_r^\beta}{\ell_r^\beta}\right)M^p + \frac{1}{\ell_r^\beta}\sum_{k\in I_r}|x_k - L|^p\\ &\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right)M^p + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |x_k-L|\geq\varepsilon}}|x_k - L|^p + \frac{1}{h_r^\beta}\sum_{\substack{k\in I_r\\ |x_k-L|<\varepsilon}}|x_k - L|^p\\ &\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right)M^p + \frac{M^p}{h_r^\beta}|\{k\in I_r : |x_k - L|\geq\varepsilon\}| + \frac{h_r}{h_r^\beta}\varepsilon^p\\ &\leq \left(\frac{\ell_r}{h_r^\beta} - 1\right)M^p + \frac{M^p}{h_r^\alpha}|\{k\in I_r : |x_k - L|\geq\varepsilon\}| + \frac{\ell_r}{h_r^\beta}\varepsilon^p\end{aligned}$$

for all $r \in \mathbb{N}$. Using (2) we obtain that $N_{\theta'}^{\beta}(p) - \lim x_k = L$, whenever $S_\theta^\alpha - \lim x_k = L$.

From Theorem 2.14 we have the following results.

Corollary 2.15 Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, $0 < p < \infty$ and let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.

If (1) holds then,

(i) If a sequence is strongly $N_{\theta'}^{\alpha}(p)-$summable to L, then it is $S_\theta^\alpha-$statistically convergent to L,

(ii) If a sequence is strongly $N_{\theta'}(p)$ −summable to L, then it is S_{θ}^{α}−statistically convergent to L,

(iii) If a sequence is strongly $N_{\theta'}(p)$ −summable to L, then it is S_{θ}−statistically convergent to L.

If (2) holds then,

(i) If a bounded sequence $x = (x_k)$ is S_{θ}^{α}−statistically convergent to L then it is strongly $N_{\theta'}^{\alpha}(p)$ −summable to L,

(ii) If a bounded sequence $x = (x_k)$ is S_{θ}^{α}−statistically convergent to L then it is strongly $N_{\theta'}(p)$ −summable to L,

(iii) If a bounded sequence $x = (x_k)$ is S_{θ}−statistically convergent to L then it is strongly $N_{\theta'}(p)$ −summable to L.

3. Results Related to Modulus Function

In this section we give the inclusion relations between the sets of S_{θ}^{α}−statistically convergent sequences and strongly $w_{(p)}^{\alpha}[\theta, f]$ −summable sequences with respect to the modulus function f.

The notion of a modulus was introduced by Nakano [26]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. Maddox [22] and Ruckle [29] used a modulus function to construct some sequence spaces. Later on using a modulus different sequence spaces have been studied by Altin [1], Et ([7], [8]) ,Gaur and Mursaleen [16], Isık [20], Nuray and Savas [27] and many others.

Definition 3.1 Let f be a modulus function, $p = (p_k)$ be a sequence of strictly positive real numbers and $\alpha \in (0, 1]$ be any real number. We define the sequence space $w_{(p)}^{\alpha}[\theta, f]$ as follows:

$$w_{(p)}^{\alpha}[\theta, f] = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} [f(|x_k - L|)]^{p_k} = 0, \text{ for some } L \right\}.$$

In the special case $p_k = p$, for all $k \in \mathbb{N}$ and $f(x) = x$ we shall write $N_{\theta}^{\alpha}(p)$ instead of $w_{(p)}^{\alpha}[\theta, f]$. If $x \in w_{(p)}^{\alpha}[\theta, f]$, then we say that x is strongly $w_{(p)}^{\alpha}[\theta, f]$ −summable with respect to the modulus function f and write $w_{(p)}^{\alpha}[\theta, f] - \lim x_k = L$.

In the following theorems we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Theorem 3.2 Let $\alpha, \beta \in (0, 1]$ be real numbers such that $\alpha \leq \beta$, f be a modulus function and let $\theta = (k_r)$ be a lacunary sequence, then $w_{(p)}^{\alpha}[\theta, f] \subset S_{\theta}^{\beta}$.

Proof. Let $x \in w^{\alpha}_{(p)}[\theta, f]$ and let $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_r$, $|x_k - L| \geq \varepsilon$ and $k \in I_r$, $|x_k - L| < \varepsilon$ respectively. Since $h_r^{\alpha} \leq h_r^{\beta}$ for each r we may write

$$\begin{aligned}
\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} [f(|x_k - L|)]^{p_k} &= \frac{1}{h_r^{\alpha}} \left[\sum\nolimits_1 [f(|x_k - L|)]^{p_k} + \sum\nolimits_2 [f(|x_k - L|)]^{p_k} \right] \\
&\geq \frac{1}{h_r^{\beta}} \left[\sum\nolimits_1 [f(|x_k - L|)]^{p_k} + \sum\nolimits_2 [f(|x_k - L|)]^{p_k} \right] \\
&\geq \frac{1}{h_r^{\beta}} \sum\nolimits_1 [f(\varepsilon)]^{p_k} \\
&\geq \frac{1}{h_r^{\beta}} \sum\nolimits_1 \min([f(\varepsilon)]^h, [f(\varepsilon)]^H) \\
&\geq \frac{1}{h_r^{\beta}} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \ \min([f(\varepsilon)]^h, [f(\varepsilon)]^H).
\end{aligned}$$

Since $x \in w^{\alpha}_{(p)}[\theta, f]$, the left hand side of the above inequality tends to zero as $r \to \infty$. Therefore the right hand side tends to zero as $r \to \infty$ and hence $x \in S^{\beta}_{\theta}$.

Theorem 3.3 If the modulus f is bounded and $\lim_{r\to\infty} \frac{h_r}{h_r^{\alpha}} = 1$ then $S^{\alpha}_{\theta} \subset w^{\alpha}_{(p)}[\theta, f]$.

Proof. Let $x \in S^{\alpha}_{\theta}$ and suppose that f is bounded and $\varepsilon > 0$ be given. Since f is bounded there exists an integer K such that $f(x) \leq K$, for all $x \geq 0$. Then for each $r \in \mathbb{N}$ we may write

$$\begin{aligned}
\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} [f(|x_k - L|)]^{p_k} &= \frac{1}{h_r^{\alpha}} \left(\sum\nolimits_1 [f(|x_k - L|)]^{p_k} + \sum\nolimits_2 [f(|x_k - L|)]^{p_k} \right) \\
&\leq \frac{1}{h_r^{\alpha}} \sum\nolimits_1 \max\left(K^h, K^H\right) + \frac{1}{h_r^{\alpha}} \sum\nolimits_2 [f(\varepsilon)]^{p_k} \\
&\leq \max\left(K^h, K^H\right) \frac{1}{h_r^{\alpha}} \left| \{k \in I_r : f(|x_k - L|) \geq \varepsilon\} \right| \\
&+ \frac{h_r}{h_r^{\alpha}} \max\left(f(\varepsilon)^h, \ f(\varepsilon)^H \right).
\end{aligned}$$

Hence $x \in w^{\alpha}_{(p)}[\theta, f]$.

Theorem 3.4 If $\lim p_k > 0$ and $x = (x_k)$ is strongly $w^{\alpha}_{(p)}[\theta, f]-$summable to L with respect to the modulus function f, then $w^{\alpha}_{(p)}[\theta, f] - \lim x_k = L$ uniquely.

Proof. Let $\lim p_k = s > 0$. Suppose that $w^{\alpha}_{(p)}[\theta, f] - \lim x_k = L$, and $w^{\alpha}_{(p)}[\theta, f] - \lim x_k = L_1$. Then

$$\lim_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} [f(|x_k - L|)]^{p_k} = 0,$$

and

$$\lim_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} [f(|x_k - L_1|)]^{p_k} = 0.$$

Definition of f, we have

$$\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} [f(|L - L_1|)]^{p_k} \leq \frac{D}{h_r^{\alpha}} \sum_{k \in I_r} [f(|x_k - L|)]^{p_k} + \frac{D}{h_r^{\alpha}} \sum_{k \in I_r} [f(|x_k - L_1|)]^{p_k},$$

where $\sup_k p_k = H$ and $D = \max\left(1, 2^{H-1}\right)$. Hence

$$\lim_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[f\left(|L - L_1|\right) \right]^{p_k} = 0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $L - L_1 = 0$. Thus the limit is unique.

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