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# $(m,n)$-Jordan Derivations

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**Abstract.** A subspace lattice $\mathcal{L}$ on $H$ is called *commutative subspace lattice* if all projections in $\mathcal{L}$ commute pairwise. It is denoted by $CSL$. If $\mathcal{L}$ is a $CSL$, then $alg\mathcal{L}$ is called a $CSL$ algebra. Under the assumption $m+n \neq 0$ where $m,n$ are fixed integers, if $\delta$ is a mapping from $\mathcal{L}$ into itself satisfying the condition $(m+n)\delta(A^2) = 2m\delta(A)A + 2nA\delta(A)$ for all $A \in \mathcal{A}$, we call $\delta$ an $(m,n)$ *Jordan derivation*. We show that if $\delta$ is a norm continuous linear $(m,n)$ mapping from $\mathcal{A}$ into it self then $\delta$ is a $(m,n)$-Jordan derivation.

## 1. Introduction.

**Definition 1.1.** *Let $X$ be a ring (or an algebra ) with the unit $I$. An additive (or linear) map $\delta$ from $X$ into it self is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in X$.*

**Definition 1.2.** *An additive (or linear) map $\delta$ from a ring (or an algebra) $X$ into itself is called a Jordan derivation if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in X$.*

**Definition 1.3.** *Let $H$ be a separable complex Hilbert space and let $B(H)$ be the set of all bounded linear maps from $H$ into itself. By a subspace lattice on $H$, we mean a collection $\mathcal{L}$ of subspaces of $H$ with $0$ and $H$ in $\mathcal{L}$ such that every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belonging to $\mathcal{L}$. For a subspace lattice $\mathcal{L}$ of $H$, $alg\mathcal{L}$ denotes the algebra of all operators on $H$ that leave members of $\mathcal{L}$ invariant. It is also disregard the distinction between a subspace and the orthogonal projection onto it. A Hilbert space subspace lattice $\mathcal{L}$ is called a commutative subspace lattice if it consists of mutually commuting projections. If $\mathcal{L}$ is a commutative subspace lattice then $alg\mathcal{L}$ is called a CSL-algebra.*

In [2], Vukman defined a new type of Jordan derivation, named $(m,n)$-Jordan derivation as follows: let $m \geq 1, n \geq 1$ be some fixed integers with $m \neq n$, and let $\mathcal{A}$ be an algebra. Suppose there exists a nonzero additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying the relation $(m+n)\delta(x^2) = 2m\delta(x)x + 2nx\delta(x)$ for all $x \in \mathcal{A}$ is called $(m,n)$-Jordan derivation.

## 2. $(m,n)$-Jordan Derivations on $CSL$-Algebras.

In this paper we will study $(m,n)$-Jordan derivation on $CSL$-algebras. Assume that $m+n \neq 0$. We proceed with the following lemmata.

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**Lemma 2.1.** *Let $\mathcal{A}$ be a unital algebra. If $\delta$ is an $(m, n)$-Jordan derivation from $\mathcal{A}$ into it self, then for each idempotent $P \in \mathcal{A}$, $(m + n)\delta(P) = 2m\delta(P)P + 2nP\delta(P)$.*

*Proof.* It is obvious from $I = I.I$ that $(m+n)\delta(I) = (m+n)\delta(I.I) = 2m\delta(I)I+2nI\delta(I) = 2m\delta(I)+2n\delta(I) = 2(m+n)\delta(I)$. Thus $(m + n)\delta(I) = 0$. Since we know that $m + n \neq 0$, therefore we have $\delta(I) = 0$. For any idempotent $P \in \mathcal{A}, P(I - P) = 0$. Then we have

$$\begin{aligned}(m + n)\delta(P(I - P) + (I - P)P) &= 2m\delta(P)(I - P) + 2m\delta(I - P)P + 2nP\delta(I - P) + 2n(I - P)\delta(P)\\ &= 2m\delta(P) + 2n\delta(P) - 4m\delta(P)P - 4nP\delta(P)\\ &\Rightarrow (m + n)\delta(P) = 2m\delta(P)P + 2nP\delta(P).\end{aligned}$$

□

**Lemma 2.2.** *Let $\mathcal{A}$ and $\delta$ be as in Lemma 2.1. Then for each idempotent $P \in \mathcal{A}$ and every element $A \in \mathcal{A}$, we have*
*(i)$(m + n)\delta(PA + AP) = 2m\delta(P)A + 2m\delta(A)P + 2nP\delta(A) + 2nA\delta(P)$*
*(ii)$(m + n)\delta(PAP) = m\delta(P)PA + m\delta(P)AP + mP\delta(A)P + mA\delta(P)P - m\delta(P)A + nP\delta(P)A + nP\delta(A)P + 2nPA\delta(P) + nAP\delta(P) - nA\delta(P)$.*

*Proof.* (i) For any idempotent $P \in \mathcal{A}, P(I - P)PA = (I - P)PA = 0$. Thus we have

$$\begin{aligned}&(m + n)\delta(P(I - P)A + (I - P)AP)\\ &\quad= 2m\delta((I - P)A)P + 2m\delta(P)(I - P)A + 2n(I - P)A\delta(A) + 2nP\delta((I - P)A)\\ &\quad= 2m\delta(A)P - 2m\delta(PA)P + 2m\delta(P)A - 2m\delta(P)PA + 2nA\delta(P) - 2nPA\delta(P) + 2nP\delta(A) - 2nP\delta(PA),\end{aligned} \tag{1}$$

and

$$\begin{aligned}&(m + n)\delta((I - P)PA + PA(I - P))\\ &\quad= 2m\delta(I - P)PA + 2m\delta(PA)(I - P) + 2n(I - P)\delta(PA) + 2n(PA)\delta(I - P)\\ &\quad= 2m\delta(A)P - 2m\delta(PA)P + 2m\delta(PA) - 2m\delta(P)PA + 2nA\delta(P) - 2nPA\delta(P) + 2nP\delta(A) - 2nP\delta(PA).\end{aligned} \tag{2}$$

Combining the equations above then they give

$$2m\delta(PA) + 2n\delta(PA) = 2m\delta(A)P + 2m\delta(P)A + 2nA\delta(P) + 2nP\delta(A). \tag{3}$$

Since AP(I-P)=A(I-P)P=0, with the similar proof of above equations above.

$$2m\delta(AP) + 2n\delta(AP) = 2m\delta(A)P + 2m\delta(P)A + 2nA\delta(P) + 2nP\delta(A) \tag{4}$$

Combining (3) and (4) we have

$$(m + n)\delta(AP + PA) = 2m\delta(A)P + 2m\delta(P)A + 2nA\delta(P) + 2nP\delta(A).$$

Replacing $A$ by $PA + AP$ in (i), we have

$$\begin{aligned}&(m + n)\delta(P(PA + AP) + (PA + AP)P)\\ &\quad= 2m\delta(P)(PA + AP) + 2m\delta(PA + AP)P + 2nP\delta(PA + AP) + 2n(PA + AP)\delta(P)\end{aligned}$$

$$\begin{aligned}&\Rightarrow(m + n)\delta(PA + AP) + 2(m + n)\delta(PAP)\\ &\quad= 2m\delta(P)(PA + AP) + 2m\delta(PA + AP)P + 2nP\delta(PA + AP) + 2n(PA + AP)\delta(P).\end{aligned}$$

Then it implies

$$\begin{aligned}
&2m\delta(P)A + 2m\delta(A)P + 2nP\delta(A) + 2nA\delta(P) + 2(m+n)\delta(PAP)\\
&= 2m\delta(P)(PA + AP) + 2m(\delta(P)A + P\delta(A) + \delta(A)P + A\delta(P)P) + 2nP(\delta(P)A\\
&\quad + P\delta(A) + \delta(A)P + A\delta(P)) + 2n(PA + AP)\delta(P)\\
&= 2m\delta(P)(PA + AP) + 2m\delta(P)AP + 2mP\delta(A)P + 2m\delta(A)P + 2mA\delta(P)P\\
&\quad + 2nP\delta(P)A + 2nP\delta(A) + 2nP\delta(A)P + 2nPA\delta(P) + 2nPA\delta(P) + 2nAP\delta(P)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 2m\delta(P)A + 2nA\delta(P) + 2(m+n)\delta(PAP)\\
&\quad = 2m\delta(P)PA + 4m\delta(P)AP + 2mP\delta(A)P + 2mA\delta(P)P + 2nP\delta(P)A + 2nP\delta(A)P + 4nPA\delta(P) + 2nAP\delta(P).
\end{aligned}$$

Hence we have

$$\begin{aligned}
(m+n)\delta(PAP) &= m\delta(P)PA + m\delta(P)AP + mP\delta(A)P + mA\delta(P)P - m\delta(P)A + nP\delta(P)A\\
&\quad + nP\delta(A)P + 2nPA\delta(P) + nAP\delta(P) - nA\delta(P)
\end{aligned}$$

which is the proof of (ii). □

**Corollary 2.3.** *Let $\mathcal{A}$ and $\delta$ be as in Lemma 2.1. Suppose that $\mathcal{B}$ is the subalgebra of $\mathcal{A}$ generated by all idempotents in $\mathcal{A}$. Then for any $T \in \mathcal{B}$ and any $A \in \mathcal{A}$, we have $(m+n)\delta(TA+AT) = 2m\delta(A)T + 2m\delta(T)A + 2nA\delta(T) + 2nT\delta(A)$.*

**Lemma 2.4.** *Let $\mathcal{L}$ be a CSL on H. If $\delta$ is a $(m,n)$-Jordan derivation from $alg\mathcal{L}$ into itself, then for all $S, T \in alg\mathcal{L}$ and $P \in \mathcal{L}$, we have*
*(i)$(m+n)\delta(SPT(I-P)) = 2m\delta(S)(PT(I-P)) + 2nS\delta(PT(I-P))$*
*(ii)$(m+n)\delta(PS(I-P)T) = 2m\delta(PS(I-P))T + PS(I-P)\delta(T)$.*

*Proof.* (i)Let $P$ be in $\mathcal{L}$. Since $(m+n)\delta(P) = 2m\delta(P)P + 2nP\delta(P)$, we see that $P\delta(P)P = (I-P)\delta(P)(I-P) = 0$. So $\delta(P) = P\delta(P)(I-P)$. Thus by Lemma 2.2, for every $T \in alg\mathcal{L}$,

$$\begin{aligned}
(m+n)\delta(PT(I-P)) &= (m+n)\delta(PPT(I-P) + PT(I-P)P)\\
&= 2m\delta(P)(PT(I-P) + 2m\delta(PT(I-P))P + 2nP\delta(PT(I-P)) + 2nPT(I-P)\delta(P)\\
&= 2m\delta(PT(I-P)P) + 2nP\delta(PT(I-P)).
\end{aligned}$$

This implies $\delta(PT(I-P)) = P\delta(PT(I-P))(I-P)$ for every $T \in alg\mathcal{L}$. By Lemma 2.2 (ii), we have $(I-P)\delta(PTP) = \delta((I-P)T(I-P)P = 0$ for every $T \in alg\mathcal{L}$. Since $PT(I-P) = P - (P - PT(I-P))$ and $PT(I-P)$ is an idempotent, by Corollary 2.3, for $S, T \in alg\mathcal{L}$,

$$\begin{aligned}
(m+n)\delta(SPT(I-P) &= (m+n)(\delta(PSPPT(I-P) + PT(I-P)PSP))\\
&= 2m\delta(PSP)(PT(I-P)) + 2m\delta(PT(I-P))PSP\\
&\quad + 2nPSP\delta(PT(I-P)) + 2nPT(I-P)\delta(PSP)\\
&= 2m\delta(PSP)(PT(I-P)) + 2nPSP\delta(PT(I-P))\\
&= 2m\delta(S)(PT(I-P)) + 2nS\delta(PT(I-P)).
\end{aligned}$$

With proof (i), the proof of (ii) is also true. □

By the lemmata above and the fact that a *CSL*-algebra contains all idempotent elements then we have the following result.

**Theorem 2.5.** *Let $\mathcal{L}$ be a CSL-algebra on H. If $\delta$ is a norm continuous linear $(m,n)$ mapping from $\mathcal{A}$ into it self then $\delta$ is a $(m,n)$-Jordan derivation.*

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