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Stochastic Lotka-Volterra Systems under Regime Switching with Jumps

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Abstract. A stochastic Lotka-Volterra model with Markovian switching driven by jumps is proposed and investigated. In the model, the white noise, color noise and jumping noise are taken into account at the same time. This model is more feasible and applicable. Firstly, sufficient conditions for stochastic permanence and extinction are presented. Then the moment average in time and the asymptotic pathwise properties are estimated. Our results show that these properties have close relations with the jumps and the stationary probability distribution of the Markov chain. Finally, several numerical simulations are provided to illustrate the effectiveness of the results.

1. Introduction

The deterministic autonomous Lotka-Volterra system can be described by

$$\mathrm{d}x_i(t) = x_i(t)\Big[b_i + \sum_{j=1}^{n} a_{ij}x_j(t)\Big]\mathrm{d}t, \quad i = 1, \cdots, n, \tag{1}$$

or in the matrix form

$$\mathrm{d}x(t) = diag(x_1(t), \cdots, x_n(t))\Big[b + Ax(t)\Big]\mathrm{d}t,$$

where $x_i(t)$ denotes the population size of the ith species at time t and

$$x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n, \quad b = (b_1, \cdots, b_n)^T \in \mathbb{R}^n_+, \quad A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}.$$

Due to the importance in theory and practice, model (1) has received great attention and many good results have been reported, here we only mention [7, 12, 14, 18] and references therein.

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However, population systems in the real world are inevitably subject to environmental noise and there are various types of environmental noise, such as white and color noise (see e.g. [6, 9, 28, 30]). If the effects of environmental noise are taken into account, the system will change significantly. Firstly, we consider the color noise, also called telegraph noise [24, 32]. The color noise can be regarded as a switching between two or more regimes of environment, which differ by factors such as rain falls or nutrition [6, 30]. Jeffries [13] has pointed that the growth rates and the carrying capacities are often subject to environmental noise, as we know that the growth rates of some species in the rainy season are different from those in the dry season. Since the switching among the different environments is memoryless and the waiting time for the next switch has an exponential distribution, we can make use of a right-continuous Markov chain $r(t)$ with finite state space $\mathbb{S} = \{1, \cdots, m\}$ to model the regime switching. Incorporating the color noise into the model(1), it changes into

$$\mathrm{d}x_i(t) = x_i(t)\Big[b_i(r(t)) + \sum_{j=1}^{n} a_{ij}(r(t))x_j(t)\Big]\mathrm{d}t, \quad i = 1, \cdots, n. \tag{2}$$

This system can be explained as follows: if the initial state $r(0) = \varsigma$, then Eq.(2) obeys

$$\mathrm{d}x_i(t) = x_i(t)\Big[b_i(\varsigma) + \sum_{j=1}^{n} a_{ij}(\varsigma)x_j(t)\Big]\mathrm{d}t, \quad i = 1, \cdots, n$$

till time τ_1 when the Markov chain switches to k from ς; then the system obeys

$$\mathrm{d}x_i(t) = x_i(t)\Big[b_i(k) + \sum_{j=1}^{n} a_{ij}(k)x_j(t)\Big]\mathrm{d}t, \quad i = 1, \cdots, n$$

until the next switching. The system will continue to switch as long as the Markov chain switches. Takeuchi et al. [32] considered a two-dimensional predator-prey system with regime switching. They obtained an important result which reveals the significant effect of the environmental noise on the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative (see e.g. [8]).

Now let us turn into another type of environment noise, the white noise. Suppose that $a_{ij}(r(t))$ is affected by the white noise [24, 28] with

$$a_{ij}(r(t)) \to a_{ij}(r(t)) + \sigma_{ij}(r(t))\dot{B}(t),$$

then the stochastic Lotka-Volterra system under Markovian switching can be described by the Itô equation

$$\mathrm{d}x_i(t) = x_i(t)\Big[b_i(r(t)) + \sum_{j=1}^{n} a_{ij}(r(t))x_j(t)\Big]\mathrm{d}t + x_i(t)\sum_{j=1}^{n} \sigma_{ij}(r(t))x_j(t)\mathrm{d}B(t), \quad i = 1, \cdots, n. \tag{3}$$

Many scholars studied this stochastic model and considered the effects of the noise. For example, Luo and Mao [24] presented sufficient conditions for the existence of global positive solutions and showed that the solution is stochastically ultimate bounded. Liu et al. [20] studied a stochastic delay Gilpin-Ayala competition system under regime switching, considered the stochastically ultimate boundedness and the asymptotic moment estimation of the solution. Many other authors also studied the population systems with white noise, see [3, 10, 11, 21, 25, 26, 28] and references therein. Here we do not illustrate them one by one.

In addition, the population systems may suffer from sudden environmental shocks, for example, red tides, earthquakes, floods, hurricanes, epidemics and so on, see [4, 5]. These events are so strong that they break the continuity of the solution, so models with white noise and color noise cannot explain these phenomena, however the model with jumps can do [4, 5] and it provides a feasible and more realistic model. Stochastic differential equations (SDEs) with jumps have received considerable attention in recent years, and there are many references about the knowledge of jumps. Among them Situ [31] and Applebaum [2] are good references.

Inspired by the above discussions, in this paper we propose the following stochastic Lotka-Volterra model under regime switching with jumps:

$$\mathrm{d}x(t) = diag(x_1(t^-), \cdots, x_n(t^-))\Big[\Big(b(r(t)) + A(r(t))x(t^-)\Big)\mathrm{d}t + \sigma(r(t))x(t^-)\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma(r(t), z)N(\mathrm{d}t, \mathrm{d}z)\Big] \tag{4}$$

and the ith component of $x(t)$ is expressed by

$$\mathrm{d}x_i(t) = x_i(t^-)\Big[b_i(r(t)) + \sum_{j=1}^{n} a_{ij}(r(t))x_j(t^-)\Big]\mathrm{d}t + x_i(t^-)\sum_{j=1}^{n}\sigma_{ij}(r(t))x_j(t^-)\mathrm{d}B(t) + x_i(t^-)\int_{\mathbb{Z}}\gamma_i(r(t), z)N(\mathrm{d}t, \mathrm{d}z).$$

In the model, $x(t^-)$ denotes the left limit of $x(t)$, N is a Poisson random measure generated by a Poisson point process with characteristic measure λ on a measurable subset $\mathbb{Z}$ of $(0, \infty)$ with $\lambda(\mathbb{Z}) < \infty$, $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \lambda(\mathrm{d}z)\mathrm{d}t$ is the corresponding martingale measure. For the biological background, we assume $\gamma_i(r, z) > -1$, $r \in \mathbb{S}$, $z \in \mathbb{Z}$, $1 \le i \le n$ are bounded functions, see Remark 4 in [35].

It is worth to point out that there are few papers ([4, 5, 23]) to consider the population systems with jumps. Bao et al. [4] do pioneering work in this area. They proposed the stochastic Lotka-Volterra competitive population dynamics with jumps, talked about the existence and uniqueness, boundedness, tightness, Lyapunov exponents and extinction of the positive solutions. Further, Bao and Yuan [5] developed the general Lotka-Volterra population dynamics with jumps, they showed that both jump process and Brownian motion can suppress the explosion of the solution. Recently, Liu and Wang [23] considered a predator-prey system with jumps, and established sufficient conditions for stability in mean and extinction of the system.

Here we propose a new type of models with white noise, color noise and jumps. So far as our knowledge is concerned, this kind of models has not been reported, not to mention the properties of the solution. Therefore our paper is valuable and meaningful. Our main aim is to investigate the asymptotic properties of the solutions and reveal the effects of the three noise on the population system.

In this paper, we mainly consider model (4). In Sections 2 and 3, we present sufficient conditions for stochastic permanence and extinction respectively. The moment average in time and the asymptotic pathwise estimate of the solution are considered in Sections 4 and 5 respectively. Our results show that the properties of the solution have close relations with the jump and the stationary distribution of the Markov chain. In Section 6, we provide several figures to illustrate our main results. We conclude the paper with conclusions and further remarks in Section 7.

2. Stochastic Permanence

In this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e $\mathcal{F}_t$ is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)$ denote the standard Brownian motion defined on this probability space. $N((0, t] \times \mathbb{Z})$ is a Poisson random measure, and $r(t)$ is a right-continuous Markov chain with values in a finite state space $\mathbb{S} = \{1, 2, \cdots, m\}$ with the generator $Q = (q_{ij})_{m\times m}$ given by

$$\mathbb{P} = \{r(t + \triangle t) = j | r(t) = i\} = \begin{cases} q_{ij}\triangle t + o(\triangle t), & \text{if } j \ne i; \\ 1 + q_{ii}\triangle t + o(\triangle t), & \text{if } j = i, \end{cases}$$

where $\triangle t > 0$, $q_{ij} \ge 0$ is transition rate from i to j if $i \ne j$ while $\sum_{j=1}^{m} q_{ij} = 0$. We assume $B(t)$, $N((0, t] \times \mathbb{Z})$ and $r(t)$ are independent. Further assume that Markov chain $r(t)$ is irreducible which means that the system can switch from any regime to any other regime. It is known that (see [1]) the irreducibility implies that the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_m) \in R^{1\times m}$ which can be determined by solving the following linear equation

$$\pi Q = 0 \tag{5}$$

subject to

$$\sum_{i=1}^{m}\pi_i = 1 \ \text{ and } \ \pi_{\rm i} > 0, \ \ \forall {\rm i} \in \mathbb{S}.$$

Let $\mathbb{R}^n_+$ denote the positive cone in $\mathbb{R}^n$, namely, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0, \forall 1 \leq i \leq n\}$. If A is a vector or matrix, $A \gg 0$ means all elements of A are positive and the transpose of A is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\rm trace(A^TA)}$ whilst its operator norm is expressed by $\|A\| = \sup\{|Ax| : |x| = 1\}$. For a symmetric $n \times n$ matrix A, define

$$\lambda^+_{\max}(A) = \sup_{x\in\mathbb{R}^n_+,|x|=1} x^TAx.$$

Let us emphasis that $\lambda^+_{\max}(A)$ is different from the largest eigenvalue $\lambda_{\max}(A)$ but does have some similar properties as $\lambda_{\max}(A)$ has. It is obvious that $\lambda^+_{\max}(A) \leq \lambda_{\max}(A)$ and $x^TAx \leq \lambda^+_{\max}(A)|x|^2$ for all $x \in \mathbb{R}^n_+$. For further properties of $\lambda^+_{\max}(A)$, please see [29].

For convenience, in the following, for any constant sequence $c_{ij}(k)$, $1 \leq i, j \leq n, 1 \leq k \leq m$, we adopt the following notations:

$$\check{c} = \max_{1\leq i,j\leq n,k\in\mathbb{S}} c_{ij}(k), \ \ \check{c}(k) = \max_{1\leq i,j\leq n} c_{ij}(k),$$

$$\hat{c} = \min_{1\leq i,j\leq n,k\in\mathbb{S}} c_{ij}(k), \ \ \hat{c}(k) = \min_{1\leq i,j\leq n} c_{ij}(k).$$

Since $x_i(t)$ in system (4) represents the population size of the ith species at time t, it should be nonnegative. For further study, we firstly establish some conditions under which the solution of SDE (4) has a unique global positive solution.

Assumption 2.1. *For each $r \in \mathbb{S}$, $\sigma_{ii}(r) > 0$ for $1 \leq i \leq n$ whilst $\sigma_{ij}(r) \geq 0$ if $i \neq j$.*

Assumption 2.2. *There exists a positive constance K such that*

$$\int_{\mathbb{Z}}\Big[\ln(1+\gamma(r,z))\Big]^2\lambda(dz) \leq K, \ \ \forall r \in \mathbb{S}.$$

Theorem 2.3. *Under Assumptions 2.1 and 2.2, for any initial value $x(0) = \bar{x} \in \mathbb{R}^n_+$ and $r(0) \in \mathbb{S}$, Eq.(4) admits a unique global solution $x(t) \in \mathbb{R}^n_+$ for any $t \geq 0$ almost surely.*

Proof. The proof is motivated by Luo and Mao [24], we will illustrate it in Appendix A. □

Theorem 2.3 guarantees that the solution of (4) will remain in the positive cone $\mathbb{R}^n_+$. But from the biological point of view, the positivity and non-explosion of the solution are often not good enough. However permanence and extinction are two of important and interesting properties which mean that the population system will survive or die out in the future, respectively. In this section, we will give the definitions of stochastically ultimate boundedness and stochastic permanence firstly, then present sufficient conditions for them.

Definition 2.4. *[16] The solution $x(t)$ of Eq.(4) is called stochastically ultimate bounded, if for any initial value $\overline{x} \in \mathbb{R}^n_+$, $\forall\epsilon \in (0, 1)$, $\exists H = H_\epsilon > 0$, the solution $x(t)$ of Eq.(4) satisfies*

$$\limsup_{t\to+\infty} \mathbb{P}[|x(t)| > H] < \epsilon.$$

Definition 2.5. *[22] The solution $x(t)$ of Eq.(4) is said to be stochastically permanent, if for any $\varepsilon \in (0, 1)$, there is a pair of positive constants $H_1 = H_1(\varepsilon)$ and $H_2 = H_2(\varepsilon)$ such that*

$$\liminf_{t\to+\infty} \mathbb{P}[|x(t)| \leq H_1] \geq 1-\varepsilon, \ \ \liminf_{t\to+\infty} \mathbb{P}[|x(t)| \geq H_2] \geq 1-\varepsilon,$$

where $x(t)$ is an arbitrary solution of the equation with initial value $\overline{x} \in \mathbb{R}^n_+$.

From the definitions we can see that stochastic permanence implies stochastically ultimate boundedness, so we begin with the easier one.

Lemma 2.6. *Under Assumptions 2.1 and 2.2, for any $p \in (0,1)$, there exists a positive constant $K = K(p)$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}|x(t)|^p \le K.$$

where $x(t)$ is the solution of Eq.(4) with the initial data $\bar{x} \in \mathbb{R}^n_+$.

Proof. The method of the proof is classical and we adopt the ideas of [24]. Define

$$V(x) = \sum_{i=1}^n x_i^p, \quad x \in \mathbb{R}^n_+ \tag{6}$$

and a sequence of stopping time

$$\sigma_k = \inf\{t \ge 0, |x(t)| > k\}. \tag{7}$$

Clearly $\sigma_k \uparrow \infty$ a.s. as $k \to \infty$. Applying the generalized Itô's formula to $V(x)$ leads to

$$\begin{aligned} \mathrm{d}V(x(t)) &= \sum_{i=1}^n p x_i^p \Big[b_i + \sum_{j=1}^n a_{ij}x_j + \frac{1}{2}(p-1)(\sum_{j=1}^n \sigma_{ij}x_j)^2\Big]\mathrm{d}t + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i)^p - 1]\lambda(\mathrm{d}z)\mathrm{d}t \\ &\quad + \sum_{i=1}^n p x_i^p (\sum_{j=1}^n \sigma_{ij}x_j)\mathrm{d}B(t) + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i)^p - 1]\widetilde{N}(\mathrm{d}t,\mathrm{d}z) \\ &= LV(x(t),r(t))\mathrm{d}t + \sum_{i=1}^n p x_i^p (\sum_{j=1}^n \sigma_{ij}x_j)\mathrm{d}B(t) + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i)^p - 1]\widetilde{N}(\mathrm{d}t,\mathrm{d}z). \end{aligned} \tag{8}$$

Here

$$\begin{aligned} LV(x,r) &= \sum_{i=1}^n p x_i^p \Big[b_i(r) + \sum_{j=1}^n a_{ij}(r)x_j + \frac{1}{2}(p-1)(\sum_{j=1}^n \sigma_{ij}(r)x_j)^2\Big] \\ &\quad + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i(r,z))^p - 1]\lambda(\mathrm{d}z). \end{aligned} \tag{9}$$

By Assumption 2.1,

$$\begin{aligned} LV(x,r) &\le \sum_{i=1}^n p x_i^p \Big[b_i(r) + \sum_{j=1}^n a_{ij}(r)x_j\Big] - \frac{p(1-p)}{2}\sum_{i=1}^n x_i^{p+2}\sigma_{ii}^2(r) \\ &\quad + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i(r,z))^p - 1]\lambda(\mathrm{d}z) \\ &:= F(x,r) - V(x), \end{aligned}$$

where

$$\begin{aligned} F(x,r) &= V(x) + \sum_{i=1}^n p x_i^p \Big[b_i(r) + \sum_{j=1}^n a_{ij}(r)x_j\Big] - \frac{p(1-p)}{2}\sum_{i=1}^n \sigma_{ii}^2(r)x_i^{p+2} \\ &\quad + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i(r,z))^p - 1]\lambda(\mathrm{d}z). \end{aligned}$$

Note that $F(x,r)$ is bounded in $\mathbb{R}^n_+ \times \mathbb{S}$, namely

$$\sup_{(x,r)\in\mathbb{R}^n_+\times\mathbb{S}} F(x,r) = K_1(p) < \infty.$$

Therefore we have $LV(x,r) \le K_1(p) - V(x)$. Applying Itô's formula to $[e^t V(x(t))]$ deduces that

$$\begin{aligned} d[e^t V(x(t))] &= e^t[V(x(t)) + LV(x(t), r(t))]dt + e^t\left(\sum_{i=1}^n p x_i^p(t) \sum_{j=1}^n \sigma_{ij}(r(t))x_j(t)\right)dB(t) \\ &\quad + e^t \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} [(1+\gamma_i(r(t),z))^p - 1]\widetilde{N}(dt, dz) \\ &\le K_1(p)e^t dt + e^t\left(\sum_{i=1}^n p x_i^p(t) \sum_{j=1}^n \sigma_{ij}(r(t))x_j(t)\right)dB(t) \\ &\quad + e^t \sum_{i=1}^n x_i(t)^p \int_{\mathbb{Z}} [(1+\gamma_i(r(t),z))^p - 1]\widetilde{N}(dt, dz). \end{aligned} \tag{10}$$

Integrating (10) from 0 to $t \wedge \sigma_k$, then taking expectations, we reach that

$$\mathbb{E}[e^{t\wedge\sigma_k} V(x(t\wedge\sigma_k))] \le V(\bar{x}) + K_1(p)\mathbb{E}\int_0^{t\wedge\sigma_k} e^s ds.$$

Letting $k \to \infty$ follows that

$$\mathbb{E}[V(x(t))] \le e^{-t} V(\bar{x}) + K_1(p).$$

Clearly,

$$|x|^2 \le n \max_{1\le i\le n} x_i^2.$$

So

$$|x|^p \le n^{\frac{p}{2}} \max_{1\le i\le n} x_i^p \le n^{\frac{p}{2}} V(x).$$

Therefore, we have

$$\mathbb{E}|x(t)|^p \le n^{\frac{p}{2}}(e^{-t}V(\bar{x}) + K_1(p)).$$

This implies

$$\limsup_{t\to+\infty} \mathbb{E}|x(t)|^p \le n^{\frac{p}{2}} K_1(p) := K(p),$$

which is our desired assertion. This completes the proof. □

Remark 2.7. *If the jump-diffusion coefficient $\gamma(r,z) \equiv 0$, then model (4) becomes the model (2.2) in [24], and the result of Lemma 2.6 changes into Lemma 3.2 of [24].*

As an application of Lemma 2.6 together with Chebyshev's inequality, we get the following result.

Theorem 2.8. *Under Assumptions 2.1 and 2.2, Eq.(4) is stochastically ultimate boundedness.*

In order to get the stochastic permanence, we need more conditions.

Assumption 2.9. *For some $u \in \mathbb{S}$, $q_{iu} > 0$, $\forall i \neq u$.*

Assumption 2.10. $\sum_{k=1}^{m} \pi_k \hat{\beta}(k) > 0$, *where* $\hat{\beta}(k) = 2\hat{b}(k) + \int_{\mathbb{Z}} \frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2} \lambda(\mathrm{d}z)$, $\hat{\gamma}(k,z) = \min_{1\le i\le n} \gamma_i(k,z)$ *and* $|\gamma(\check{k},z)| = \max_{1\le i\le n} |\gamma_i(k,z)|$

Lemma 2.11. *Let Assumptions 2.9 and 2.10 hold, then there exists a constant $0 < \theta < 1$ such that the matrix*

$$A(\theta) := \mathrm{diag}(\xi_1(\theta), \xi_2(\theta), \cdots, \xi_{\mathrm{m}}(\theta)) - \mathrm{Q} \tag{11}$$

is a nonsingular $M-$matrix, where $\xi_k(\theta) = \theta\hat{\beta}(k)$.

Proof. The proof is presented in Appendix B. □

Theorem 2.12. *Let Assumptions 2.1, 2.2, 2.9 and 2.10 hold, then SDE (4) is stochastically permanent.*

Proof. By Chebyshev's inequality and Lemma 2.6, we can get

$$\liminf_{t\to+\infty} \mathbb{P}[|x(t)| \le H_1] \ge 1 - \varepsilon.$$

In the following, we will prove the second part $\liminf_{t\to+\infty} \mathbb{P}[|x(t)| \ge H_2] \ge 1 - \varepsilon$. Define

$$U(x) = \sum_{i=1}^{n} x_i, \quad x \in \mathbb{R}_+^n. \tag{12}$$

Applying Itô's formula with jumps [31] leads to

$$\mathrm{d}U(x) = \sum_{i=1}^{n} x_i\Big(b_i + \sum_{j=1}^{n} a_{ij}x_j\Big)\mathrm{d}t + \sum_{i=1}^{n} x_i\Big(\sum_{j=1}^{n} \sigma_{ij}x_j\Big)\mathrm{d}B(t) + \int_{\mathbb{Z}} \Big(\sum_{i=1}^{n} x_i\gamma_i\Big)N(\mathrm{d}t, \mathrm{d}z),$$

where we drop t from $x(t)$, and $r(t)$ from $b(r(t))$ and etc. Define $V_1(x) = \frac{1}{U^2(x)}$, Itô's formula yields that

$$\begin{aligned}\mathrm{d}V_1(x) \quad = \quad & -\frac{2}{U^3}\sum_{i=1}^{n} x_i\Big(b_i + \sum_{j=1}^{n} a_{ij}x_j\Big)\mathrm{d}t - \frac{2}{U^3}\Big(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} \sigma_{ij}x_j\Big)\mathrm{d}B(t) + \frac{3}{U^4}\Big(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} \sigma_{ij}x_j\Big)^2\mathrm{d}t \\ & + \int_{\mathbb{Z}} \Big[\frac{1}{(U + \sum_{i=1}^{n} x_i\gamma_i)^2} - \frac{1}{U^2}\Big]N(\mathrm{d}t, \mathrm{d}z).\end{aligned}$$

For θ given in Lemma 2.11, by Theorem 2.10 [29], there exists a vector $\vec{p} = (p_1, p_2, \cdots, p_m)^T \gg 0$ such that $A(\theta)\vec{p} \gg 0$, which is equivalent to

$$p_k\theta\Big(2\hat{b}(k) + \int_{\mathbb{Z}} \frac{2\hat{\gamma}(k,z)}{(1 + |\gamma(\check{k},z)|)^2}\lambda(\mathrm{d}z)\Big) - \sum_{j=1}^{m} q_{kj}p_j > 0, \quad for \ \ 1 \le k \le m. \tag{13}$$

Define function $V_2 : \mathbb{R}_+^n \times \mathbb{S} \to \mathbb{R}_+$ by

$$V_2(x,k) = p_k(1 + V_1)^{\theta}.$$

Making use of Itô's formula follows that

$$\begin{aligned}\mathrm{d}V_2(x,k) \quad = \quad & LV_2(x,k)\mathrm{d}t + \theta p_k(1 + V_1)^{\theta-1} \cdot \frac{-2}{U^3}\Big(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} \sigma_{ij}x_j\Big)\mathrm{d}B(t) \\ & + \int_{\mathbb{Z}} p_k\Big[\Big(1 + V_1 + \frac{1}{(U + \sum_{i=1}^{n} x_i\gamma_i)^2} - \frac{1}{U^2}\Big)^{\theta} - (1 + V_1)^{\theta}\Big]\widetilde{N}(\mathrm{d}t, \mathrm{d}z),\end{aligned}$$

where

$$LV_2(x,k) = \theta p_k(1+V_1)^{\theta-2}\Big\{\frac{-2}{U^3}(1+V_1)\sum_{i=1}^{n}x_i\Big(b_i+\sum_{j=1}^{n}a_{ij}x_j\Big)+(1+V_1)\frac{3}{U^4}\Big(\sum_{i=1}^{n}x_i\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2$$
$$+\frac{2(\theta-1)}{U^6}\Big(\sum_{i=1}^{n}x_i\sum_{j=1}^{n}\sigma_{ij}x_j\Big)^2\Big\}+\sum_{j=1}^{m}q_{kj}p_j(1+V_1)^{\theta}$$
$$+\int_{\mathbb{Z}}p_k\Big[\Big(1+V_1+\frac{1}{(U+\sum_{i=1}^{n}x_i\gamma_i)^2}-\frac{1}{U^2}\Big)^{\theta}-(1+V_1)^{\theta}\Big]\lambda(\mathrm{d}z).$$

We compute that

$$\Big[\Big(1+V_1+\frac{1}{(U+\sum_{i=1}^{n}x_i\gamma_i)^2}-\frac{1}{U^2}\Big)^{\theta}-(1+V_1)^{\theta}\Big]$$
$$=(1+V_1)^{\theta}\Big\{\Big(1+\frac{-2U\sum_{i=1}^{n}x_i\gamma_i-(\sum_{i=1}^{n}x_i\gamma_i)^2}{(1+V_1)U^2(U+\sum_{i=1}^{n}x_i\gamma_i)^2}\Big)^{\theta}-1\Big\}$$
$$\leq(1+V_1)^{\theta}\theta\cdot-\frac{2U\sum_{i=1}^{n}x_i\gamma_i+(\sum_{i=1}^{n}x_i\gamma_i)^2}{(1+V_1)U^2(U+\sum_{i=1}^{n}x_i\gamma_i)^2}$$
$$\leq-\theta(1+V_1)^{\theta-1}\frac{1}{U^2}\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}$$
$$=(1+V_1)^{\theta-2}\Big[-\theta\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}-\theta\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}\Big].$$

Here in the first inequality, we use the fundamental inequality $x^r\leq 1+r(x-1), x\geq 0, 1\geq r\geq 0$. Further, we have

$$LV_2 \leq (1+V_1)^{\theta-2}\Big\{-V_1^2\Big(2\theta p_k\hat{b}(k)-\sum_{j=1}^{m}q_{kj}p_j+\int_{\mathbb{Z}}\theta p_k\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}\lambda(dz)\Big)-2\theta p_k\hat{a}(u)V_1^{1.5}$$
$$+V_1\Big(-2\theta p_k\hat{b}(k)+(2\theta+1)\theta p_k(\check{\sigma})^2(k)+2\sum_{j=1}^{m}q_{kj}p_j-\int_{\mathbb{Z}}\theta p_k\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}\lambda(\mathrm{d}z)\Big)$$
$$-2\theta p_k\hat{a}(k)V_1^{0.5}+3\theta p_k\check{\sigma}^2(k)+\sum_{j=1}^{m}q_{kj}p_j\Big\}. \tag{14}$$

Now, by (13) we can choose a sufficiently small η to satisfy

$$p_k\theta\Big(2\hat{b}(k)+\int_{\mathbb{Z}}\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}\lambda(\mathrm{d}z)\Big)-\sum_{j=1}^{m}q_{kj}p_j-\eta p_k>0,\ \textit{for all}\ 1\leq k\leq m. \tag{15}$$

Using Itô's formula again, we obtain

$$\mathbb{E}[e^{\eta t}V_2(x(t),r(t))]=V_2(\bar{x},r(0))+\mathbb{E}\int_0^t e^{\eta s}[LV_2(x(s),r(s))+\eta V_2(x(s))]\mathrm{d}s. \tag{16}$$

By (14) we follow that

$$
\begin{aligned}
LV_2+\eta V_2 \quad \le \quad & (1+V_1)^{\theta-2}\Big\{-V_1^2\Big(2\theta p_k\hat{b}(k)-\sum_{j=1}^{m}q_{kj}p_j+\int_{\mathbb{Z}}\theta p_k\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}\lambda(\mathrm{d}z)-\eta p_k\Big)\\
& -2\theta p_k\hat{a}(u)V_1^{1.5}+V_1\Big(-2\theta p_k\hat{b}(k)+(2\theta+1)\theta p_k(\check{\sigma})^2(k)+2\sum_{j=1}^{m}q_{kj}p_j\\
& -\int_{\mathbb{Z}}\theta p_k\frac{2\hat{\gamma}(k,z)}{(1+|\gamma(\check{k},z)|)^2}\lambda(\mathrm{d}z)+2\eta p_k\Big)-2\theta p_k\hat{a}(k)V_1^{0.5}+3\theta p_k\hat{\sigma}^2(k)+\sum_{j=1}^{m}q_{kj}p_j+\eta p_k\Big\}.
\end{aligned}
$$

According to (15), $LV_2+\eta V_2$ is bounded, namely, there exists a constant M such that $LV_2+\eta V_2\le M$. Therefore (16) changes into

$$\mathbb{E}[V_2(x,k)]\le e^{-\eta t}V_2(\bar{x},r(0))+M/\eta.$$

Further we have

$$\limsup_{t\to+\infty}\mathbb{E}[V_1^{\theta}(x(t))]\le\limsup_{t\to+\infty}\mathbb{E}[(1+V_1(x(t)))^{\theta}]\le M/\eta\hat{p}_k.$$

Note that, for $x\in\mathbb{R}_+^n$

$$(\sum_{i=1}^{n}x_i(t))^{\theta}\le(n\max_{1\le i\le n}x_i(t))^{\theta}=n^{\theta}(\max_{1\le i\le n}x_i^2(t))^{0.5\theta}\le n^{\theta}|x(t)|^{\theta}.$$

So we conclude that $\limsup_{t\to+\infty}\mathbb{E}[|x(t)|^{-2\theta}]\le n^{2\theta}M/\eta\hat{p}_k:=K$. For any given $\varepsilon>0$, let $H_2=(\varepsilon/K)^{\frac{1}{2\theta}}$, by Chebyshev's inequality, we see that

$$\mathbb{P}\{|x(t)|\le H_2\}=\mathbb{P}\{|x(t)|^{-2\theta}\ge H_2^{-2\theta}\}\le\frac{E(|x(t)|^{-2\theta})}{H_2^{-2\theta}}.$$

So, $\limsup_{t\to+\infty}\mathbb{P}\{|x(t)|\le H_2\}\le\varepsilon$. Therefore $\liminf_{t\to+\infty}\mathbb{P}\{|x(t)|\ge H_2\}\ge 1-\varepsilon$ is obtained. □

3. Extinction

In the previous section, we have concluded that under some conditions, the solution has good properties such as non-explosion, the ultimate boundedness and stochastic permanence. In other words, we show that under certain conditions the three noise will not spoil this nice properties. In this section, we will see that the effect of the jumping noise on system (4).

Assumption 3.1. *Assume that there exist positive numbers $c_1,c_2,\cdots,c_n$ such that*

$$-\lambda:=\max_{r\in\mathbb{S}}\Big\{\lambda_{\max}^{+}\Big(\bar{C}A(r)+A^T(r)\bar{C}\Big)\Big\}\le 0$$

where $\bar{C}=\mathrm{diag}(c_1,c_2,\cdots,c_n)$.

Lemma 3.2. *[19] Suppose that $M(t), t\ge 0$, is a local martingale with $M(0)=0$. Then*

$$\lim_{t\to+\infty}\rho_M(t)<\infty\ \Rightarrow\lim_{t\to+\infty}\frac{M(t)}{t}=0\ \ a.s.,$$

where

$$\rho_M(t)=\int_0^t\frac{\mathrm{d}\langle M\rangle(s)}{(1+s)^2},\ \ t\ge 0$$

and $\langle M\rangle(t)$ is Meyer's angle bracket process (see e.g. [15])

Theorem 3.3. *Under Assumptions 2.1, 2.2 and 3.1, the solution $x(t)$ of Eq.(4) obeys*

$$\limsup_{t\to+\infty}\frac{\ln|x(t)|}{t}\leq\sum_{r=1}^{m}\pi_r\check{\alpha}(r)\ a.s.,$$

where $\check{\alpha}(r)=\check{b}(r)+\int_{\mathbb{Z}}\ln(1+\check{\gamma}(r,z))\lambda(\mathrm{d}z)$.

Particularly, if $\sum_{r=1}^{m}\pi_r\check{\alpha}(r)<0$, *then*

$$\lim_{t\to\infty}|x(t)|=0\quad a.s.$$

In other words, the species of (4) will go to extinction.

Proof. Define $V(x)=Cx=\sum_{i=1}^{n}c_ix_i$, $x\in\mathbb{R}_+^n$, where $C=(c_1,c_2,\cdots,c_n)$. Applying Itô's formula with jumps [31] leads to

$$\begin{aligned}\mathrm{d}V(x(t)) &= x^T(t)\bar{C}\Big[b(r(t))+A(r(t))x(t)\Big]\mathrm{d}t+x^T(t)\bar{C}\sigma(r(t))x(t)\mathrm{d}B(t)\\ &\quad+\int_{\mathbb{Z}}\Big[x^T(t)\bar{C}\gamma(r(t),z)\Big]N(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

Making use of Itô's formula again to $\ln V(x(t))$ yields that

$$\begin{aligned}\mathrm{d}\ln V(x(t)) &= \frac{1}{V}x^T\bar{C}\Big[b(r(t))+A(r(t))x\Big]\mathrm{d}t+\frac{1}{V}x^T\bar{C}\sigma(r(t))x\mathrm{d}B(t)\\ &\quad-\frac{1}{2V^2}|x^T\bar{C}\sigma(r(t))x|^2\mathrm{d}t+\int_{\mathbb{Z}}\Big[\ln\Big(V+x^T\bar{C}\gamma(r(t),z)\Big)-\ln V\Big]N(\mathrm{d}t,\mathrm{d}z).\end{aligned}\tag{17}$$

Here for convenience and simplicity, we omit $x(t)$ in $V(x(t))$ and t in $x(t)$. Note that

$$\frac{x^T\bar{C}b(r(t))}{V}\leq\check{b}(r(t)),$$

$$\frac{x^T\bar{C}A(r(t))x}{V}=\frac{x^T[\bar{C}A(r(t))+A^T(r(t))\bar{C}]x}{2V}\leq-\frac{\lambda|x|^2}{2V}\leq-\frac{\lambda}{2|C|}|x|\leq0$$

and

$$\int_{\mathbb{Z}}\Big[\ln\Big(V+x^T\bar{C}\gamma(r(t),z)\Big)-\ln V\Big]\lambda(\mathrm{d}z)\leq\int_{\mathbb{Z}}\ln\Big(1+\check{\gamma}(r(t),z)\Big)\lambda(\mathrm{d}z).$$

Substituting the above three inequalities into (17), we obtain

$$\begin{aligned}\mathrm{d}\ln V(x(t)) &\leq \Big[\check{b}(r(t))+\int_{\mathbb{Z}}\ln\Big(1+\check{\gamma}(r(t),z)\Big)\lambda(\mathrm{d}z)\Big]\mathrm{d}t+\frac{1}{V}x^T\bar{C}\sigma(r(t))x\mathrm{d}B(t)\\ &\quad-\frac{1}{2V^2}|x^T\bar{C}\sigma(r(t))x|^2\mathrm{d}t+\int_{\mathbb{Z}}\Big[\ln\Big(V+x^T\bar{C}\gamma(r(t),z)\Big)-\ln V\Big]\widetilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

This implies

$$\begin{aligned}\ln V(x(t)) &\leq \ln V(\bar{x})+\int_0^t\Big[\check{b}(r(s))+\int_{\mathbb{Z}}\ln\Big(1+\check{\gamma}(r(s),z)\Big)\lambda(dz)\Big]\mathrm{d}s+M_1(t)\\ &\quad-\int_0^t\frac{1}{2V^2(x(s))}|x^T(s)\bar{C}\sigma(r(s))x(s)|^2\mathrm{d}s+Q_1(t).\end{aligned}\tag{18}$$

Where $M_1(t) = \int_0^t \frac{1}{V(x(s))} x^T(s)\bar{C}\sigma(r(s))x(s)\mathrm{d}B(s)$, $Q_1(t) = \int_0^t \int_{\mathbb{Z}} \Big[\ln\Big(V(x(s))+x^T(s)\bar{C}\gamma(r(s),z)\Big) - \ln V(x(s))\Big]\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$. The quadratic variations of $M_1(t)$ and $Q_1(t)$ are

$$\langle M_1\rangle(t) = \int_0^t \frac{|x^T(s)\bar{C}\sigma(r(s))x(s)|^2}{V^2(x(s))}\mathrm{d}s$$

and

$$\langle Q_1\rangle(t) = \int_0^t \int_{\mathbb{Z}} \Big[\ln\Big(V(x(s)) + x^T(s)\bar{C}\gamma(r(s),z)\Big) - \ln V(x(s))\Big]^2 \lambda(\mathrm{d}z)\mathrm{d}s$$

respectively. By virtue of exponential martingale inequality ([29] Theorem 2.14), for any positive constants T, α and β, we have

$$\mathbb{P}\Big(\sup_{0\le t\le T} \Big[M_1(t) - \frac{\alpha}{2}\langle M_1\rangle(t)\Big] > \beta\Big) \le e^{-\alpha\beta}.$$

Setting $T = n, \alpha = 1, \beta = 2\ln n$ leads to

$$\mathbb{P}\Big(\sup_{0\le t\le n} \Big[M_1(t) - \frac{1}{2}\langle M_1\rangle(t)\Big] > 2\ln n\Big) \le \frac{1}{n^2}.$$

According to $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ and Borel-Cantelli lemma [25], we reach that for almost all $\omega \in \Omega$, there exists a random integer $n_0 = n_0(\omega)$ such that for $n \ge n_0$,

$$\sup_{0\le t\le n} \Big[M_1(t) - \frac{1}{2}\langle M_1\rangle(t)\Big] \le 2\ln n.$$

This implies

$$M_1(t) \le 2\ln n + \frac{1}{2}\langle M_1\rangle(t) = 2\ln n + \frac{1}{2}\int_0^t \frac{|x^T(s)\bar{C}\sigma(r(s))x(s)|^2}{V^2(x(s))}\mathrm{d}s. \tag{19}$$

for all $0 \le t \le n, n \ge n_0$. By (19), (18) changes into

$$\ln V(x(t)) \le \ln V(\bar{x}) + \int_0^t \Big[\check{b}(r(s)) + \int_{\mathbb{Z}} \ln\Big(1 + \check{\gamma}(r(s),z)\Big)\lambda(\mathrm{d}z)\Big]\mathrm{d}s + 2\ln n + Q_1(t). \tag{20}$$

In addition, by Assumption 2.2

$$\begin{aligned}\langle Q_1\rangle(t) &= \int_0^t \int_{\mathbb{Z}} \Big[\ln\Big(V(x(s)) + x^T(s)\bar{C}\gamma(r(s),z)\Big) - \ln V(x(s))\Big]^2 \lambda(\mathrm{d}z)\mathrm{d}s \\ &\le \int_0^t \int_{\mathbb{Z}} \Big[\ln\Big(1 + \check{\gamma}(r(s),z)\Big)\Big]^2 \lambda(\mathrm{d}z)\mathrm{d}s \le Kt.\end{aligned}$$

Using Lemma 3.2, we get

$$\lim_{t\to+\infty} \frac{Q_1(t)}{t} = 0 \ \ a.s. \tag{21}$$

Dividing both sides of (20) by t, for $n-1 \le t \le n, n \ge n_0$, we have

$$\frac{\ln V(x(t))}{t} \le \frac{\ln V(\bar{x})}{t} + \frac{1}{t}\int_0^t \Big[\check{b}(r(s)) + \int_{\mathbb{Z}} \ln\Big(1 + \check{\gamma}(r(s),z)\Big)\lambda(\mathrm{d}z)\Big]ds + \frac{2\ln n}{n-1} + \frac{Q_1(t)}{t}.$$

Making use of Eq.(21) and the ergodic property of the Markov chain, we find

$$\limsup_{t\to+\infty} \frac{\ln V(x(t))}{t} \le \sum_{r=1}^{m} \pi_r \check{\alpha}(r) \ \ a.s.$$

which implies the required assertion. This completes the proof. □

Remark 3.4. *Theorem 3.3 reveals an important fact that the color noise and the jump noise can make the population extinction and from the result we can see that our result generalizes the existing conclusions.*

4. Moment Average in Time

Now we are in the position to talk about the moment average in time of the solution of Eq.(4), and we give the upper bound of the moment average in time which relates with the stationary probability distribution of the Markov chain.

Theorem 4.1. *Under Assumptions 2.1 and 2.2, for any constants $p \in (0,1)$, $\alpha \in (0,2)$ and for any initial value $\bar{x} \in \mathbb{R}_+^n$, $r(0) \in \mathbb{S}$, the solution $x(t)$ of Eq.(4) has the property*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}|x(t)|^{p+\alpha}\mathrm{d}t \le \sum_{r=1}^m \pi_r K_r,$$

where $K_r = \sup_{x\in R_+^n} F(x,r) < \infty$ and

$$F(x,r) = |x|^{p+\alpha} + \sum_{i=1}^n p x_i^p[b_i(r) + \sum_{j=1}^n a_{ij}(r)x_j] - \frac{p(1-p)}{2}\sum_{i=1}^n \sigma_{ii}^2(r)x_i^{p+2} + \sum_{i=1}^n x_i^p \int_{\mathbb{Z}} \Big[(1+\gamma_i(r,z))^p - 1\Big]\lambda(\mathrm{d}z).$$

Proof. Let $V(x)$ and σ_k be defined as (6) and (7), respectively. By Assumption 2.1,

$$LV(x,r) \le F(x,r) - |x|^{p+\alpha} \le K_r - |x|^{p+\alpha}. \tag{22}$$

Integrating both sides of (8) from 0 to $\sigma_k \wedge T$, using (22) and then taking expectations, we get

$$0 \le V(\bar{x}) + \mathbb{E}\int_0^{\sigma_k\wedge T} K_{r(t)}\mathrm{d}t - \mathbb{E}\int_0^{\sigma_k\wedge T} |x(t)|^{p+\alpha}\mathrm{d}t.$$

Letting $k \to \infty$ yields

$$\mathbb{E}\int_0^T |x(t)|^{p+\alpha}\mathrm{d}t \le V(\bar{x}) + \mathbb{E}\int_0^T K_{r(t)}\mathrm{d}t.$$

Dividing both sides by T, we have

$$\frac{1}{T}\mathbb{E}\int_0^T |x(t)|^{p+\alpha}\mathrm{d}t \le \frac{V(\bar{x})}{T} + \mathbb{E}\Big(\frac{1}{T}\int_0^T K_{r(t)}\mathrm{d}t\Big). \tag{23}$$

By view of the ergodic property of Markov chain, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_{r(t)}dt = \sum_{i=1}^m \pi_r K_r.$$

Letting $T \to \infty$ in (23) and using Fubini theorem lead to

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}|x(t)|^{p+\alpha}\mathrm{d}t \le \sum_{r=1}^m \pi_r K_r,$$

which is our aim. □

Remark 4.2. *In Theorem 4.1, if we choose $\gamma(r,z) = 0$, $p = 0.5$, $\alpha = 1.5$, then our result becomes Theorem 4.3 of [24]. In other words, we generalize the conclusion of [24].*

5. Asymptotic Pathwise Estimate

In the previous section we have considered how the solutions vary in $\mathbb{R}^n_+$ in probability or in moment. The next we will discuss pathwise properties of the solutions. First, we give a lemma which means that the total population of the ecosystem cannot grow too fast.

Lemma 5.1. *Let Assumptions 2.1 and 2.2 hold. Then for any initial value $\bar{x} \in \mathbb{R}^n_+$ and $r(0) \in \mathbb{S}$, the solution $x(t)$ of Eq.(4) has the property that*

$$\limsup_{t\to\infty} \frac{\ln|x(t)|}{\ln t} \leq 1 \ \ a.s.$$

Proof. Let U be defined as (12). Define $V(x) = \ln U(x) = \ln(\sum_{i=1}^n x_i)$. Using Itô's formula deduces that

$$\begin{aligned} \mathrm{d}V(x) \quad &= \quad \frac{1}{U}\sum_{i=1}^n x_i\Big(b_i + \sum_{j=1}^n a_{ij}x_j\Big)\mathrm{d}t + \frac{1}{U}\Big(\sum_{i=1}^n x_i \sum_{j=1}^n \sigma_{ij}x_j\Big)\mathrm{d}B(t) - \frac{1}{2U^2}\Big(\sum_{i=1}^n x_i \sum_{j=1}^n \sigma_{ij}x_j\Big)^2\mathrm{d}t \\ &\quad + \int_{\mathbb{Z}} \Big[\ln(U + \sum_{i=1}^n x_i\gamma_i) - \ln U\Big]N(\mathrm{d}t, \mathrm{d}z). \end{aligned}$$

Making use of Itô's formula again leads to

$$\begin{aligned} \mathrm{d}(e^t V(x)) \quad &= \quad e^t\Big[\ln\sum_{i=1}^n x_i + \frac{1}{U}\sum_{i=1}^n x_i\Big(b_i + \sum_{j=1}^n a_{ij}x_j\Big)\Big]\mathrm{d}t + \frac{e^t}{U}\Big(\sum_{i=1}^n x_i \sum_{j=1}^n \sigma_{ij}x_j\Big)\mathrm{d}B(t) \\ &\quad - \frac{e^t}{2U^2}\Big(\sum_{i=1}^n x_i \sum_{j=1}^n \sigma_{ij}x_j\Big)^2\mathrm{d}t + e^t\int_{\mathbb{Z}} \Big[\ln(U + \sum_{i=1}^n x_i\gamma_i) - \ln U\Big]N(\mathrm{d}t, \mathrm{d}z), \end{aligned}$$

which is equivalent to

$$\begin{aligned} e^t \ln\sum_{i=1}^n x_i(t) \quad &= \quad \int_0^t e^s\Big[\ln\sum_{i=1}^n x_i(s) + \frac{1}{U(x(s))}\sum_{i=1}^n x_i(s)\Big(b_i(r(s)) + \sum_{j=1}^n a_{ij}(r(s))x_j(s)\Big)\Big]\mathrm{d}s \\ &\quad + \ln\sum_{i=1}^n x_i(0) + M_2(t) - \int_0^t \frac{e^s}{2U^2(x(s))}\Big(\sum_{i=1}^n x_i(s)\sum_{j=1}^n \sigma_{ij}(r(s))x_j(s)\Big)^2\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{Z}} e^s\Big[\ln(U(x(s)) + \sum_{i=1}^n x_i(s)\gamma_i(r(s), z)) - \ln U(x(s))\Big]\lambda(\mathrm{d}z)\mathrm{d}s + Q_2(t), \end{aligned} \tag{24}$$

where

$$M_2(t) = \int_0^t \frac{e^s}{U(x(s))}\Big(\sum_{i=1}^n x_i(s)\sum_{j=1}^n \sigma_{ij}(r(s))x_j(s)\Big)\mathrm{d}B(s)$$

and

$$Q_2(t) = \int_0^t\int_{\mathbb{Z}} e^s\Big[\ln(U(x(s)) + \sum_{i=1}^n x_i(s)\gamma_i(r(s), z)) - \ln U(x(s))\Big]\widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

The quadratic variations of $M_2(t)$ and $Q_2(t)$ are

$$\langle M_2\rangle(t) = \int_0^t \frac{e^{2s}}{U^2(x(s))}\Big(\sum_{i=1}^n x_i(s)\sum_{j=1}^n \sigma_{ij}(r(s))x_j(s)\Big)^2\mathrm{d}s$$

and

$$\langle Q_2\rangle(t) = \int_0^t \int_{\mathbb{Z}} e^{2s}\Big[\ln(U(x(s)) + \sum_{i=1}^n x_i(s)\gamma_i(r(s),z)) - \ln U(x(s))\Big]^2 \lambda(\mathrm{d}z)\mathrm{d}s$$

respectively. By exponential martingale inequality with jumps ([2] Theorem 5.2.9), we see that

$$\mathbb{P}\Big\{\sup_{0\le t\le \tau k}\Big[M_2(t) - \frac{\varepsilon e^{-\tau k}}{2}\langle M_2\rangle(t) + Q_2(t) - \frac{1}{\varepsilon e^{-\tau k}}\int_0^t\int_{\mathbb{Z}}\Big((1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))})^{\varepsilon e^{s-\tau k}} - 1$$
$$-\varepsilon e^{s-\tau k}\ln(1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))})\Big)\lambda(\mathrm{d}z)\mathrm{d}s\Big] > \frac{\rho e^{\tau k}\ln k}{\varepsilon}\Big\} \le k^{-\rho},$$

where $0<\varepsilon<1$, $\rho>1$ and $\tau>0$. Using Borel-Cantelli lemma, we see that for almost all $\omega\in\Omega$, there exists a $k_0(\omega)$ such that for every $k\ge k_0(\omega)$,

$$\begin{aligned} M_2(t) + Q_2(t) \quad &\le \quad \frac{\varepsilon e^{-\tau k}}{2}\langle M_2\rangle(t) + \frac{1}{\varepsilon e^{-\tau k}}\int_0^t\int_{\mathbb{Z}}\Big((1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))})^{\varepsilon e^{s-\tau k}} - 1 \\ &\quad -\varepsilon e^{s-\tau k}\ln(1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))})\Big)\lambda(\mathrm{d}z)\mathrm{d}s + \frac{\rho e^{\tau k}\ln k}{\varepsilon},\ \ 0\le t\le \tau k. \end{aligned}$$

Substituting this inequality into (24) leads to

$$\begin{aligned} e^t\ln\sum_{i=1}^n x_i(t) \quad &\le \quad \int_0^t e^s\Big[\ln\sum_{i=1}^n x_i(s) + \frac{1}{U(x(s))}\sum_{i=1}^n x_i(s)\Big(b_i(r(s)) + \sum_{j=1}^n a_{ij}(r(s))x_j(s)\Big)\Big]\mathrm{d}s \\ &\quad +\ln\sum_{i=1}^n x_i(0) - \int_0^t \frac{e^s}{2U^2(x(s))}\Big(\sum_{i=1}^n x_i(s)\sum_{j=1}^n \sigma_{ij}(r(s))x_j(s)\Big)^2(1-\varepsilon e^{s-\tau k})\mathrm{d}s \\ &\quad +\int_0^t\int_{\mathbb{Z}} e^s\ln(1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))}\lambda(\mathrm{d}z)\mathrm{d}s + \frac{\rho e^{\tau k}\ln k}{\varepsilon} \\ &\quad +\frac{1}{\varepsilon e^{-\tau k}}\int_0^t\int_{\mathbb{Z}}\Big((1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))})^{\varepsilon e^{s-\tau k}} - 1 \\ &\quad -\varepsilon e^{s-\tau k}\ln(1+\frac{\sum_{i=1}^n x_i(s)\gamma_i(r(s),z)}{U(x(s))})\Big)\lambda(\mathrm{d}z)\mathrm{d}s \\ &:= \quad J_1(t,x) + \ln\sum_{i=1}^n x_i(0) - J_2(t,x) + J_3(t,x) + \frac{\rho e^{\tau k}\ln k}{\varepsilon} + J_4(t,x). \end{aligned} \tag{25}$$

Note that

$$J_1(t,x) \le \int_0^t e^s\Big[\ln\sum_{i=1}^n x_i(s) + \check{b} + \check{a}\sum_{i=1}^n x_i(s)\Big]\mathrm{d}s, \tag{26}$$

$$J_2(t,x) \ge \int_0^t \frac{e^s}{2}(\hat{\sigma})^2(\sum_{i=1}^n x_i(s))^2(1-\varepsilon e^{s-\tau k})\mathrm{d}s, \tag{27}$$

$$J_3(t,x) \le \int_0^t\int_{\mathbb{Z}} e^s\ln(1+\check{\gamma}(z))\lambda(\mathrm{d}z)\mathrm{d}s, \tag{28}$$

$$J_4(t,x) \le \int_0^t\int_{\mathbb{Z}} e^s\Big(\check{\gamma}(z) - \ln(1+\hat{\gamma}(z))\Big)\lambda(\mathrm{d}z)\mathrm{d}s. \tag{29}$$

Substituting (26)-(29) into (25), we reach that

$$e^t \ln(\sum_{i=1}^n x_i(t)) \le \ln(\sum_{i=1}^n x_i(0)) + \int_0^t e^s J(x(s))ds + \frac{\rho e^{\tau k}\ln k}{\varepsilon},$$

where

$$\begin{aligned} J(x(s)) &= -\frac{1}{2}(\hat{\sigma})^2(\sum_{i=1}^n x_i(s))^2(1-\varepsilon e^{s-\tau k}) + \ln\sum_{i=1}^n x_i(s) + \check{b} + \check{a}\sum_{i=1}^n x_i(s) \\ &\quad + \int_{\mathbb{Z}} \ln(1+\check{\gamma}(z))\lambda(\mathrm{d}z) + \int_{\mathbb{Z}}\Big(\check{\gamma}(z) - \ln(1+\hat{\gamma}(z))\Big)\lambda(\mathrm{d}z). \end{aligned}$$

Since the leading term of $J(x(s))$ is less than zero, it must be bounded, i.e. there is a positive constant K such that $J(x(s)) \le K < \infty$. So for $\tau(k-1) \le t \le \tau k$ and $k \ge k_0(\omega)$, we conclude that

$$\frac{\ln(\sum_{i=1}^n x_i(t))}{\ln t} \le \frac{\ln(\sum_{i=1}^n x_i(0))}{e^t \ln t} + \frac{K}{\ln t} + \frac{\rho e^{\tau k}\ln k}{\varepsilon e^{\tau(k-1)}\ln(\tau(k-1))}.$$

In addition, we have $\limsup_{t\to\infty}\frac{\ln|x(t)|}{\ln t} \le \limsup_{t\to\infty}\frac{\ln\sum_{i=1}^n x_i(t)}{\ln t} \le \frac{\rho e^{\tau}}{\varepsilon}$. Letting $\varepsilon \to 1$, $\rho \to 1$ and $\tau \to 0$ leads to our results. □

Using Lemma 5.1 and the fact $\lim_{t\to\infty}(\ln t)/t = 0$ we can get the following result.

Theorem 5.2. *Under Assumptions 2.1 and 2.2, the solutions of SDE (4) has the property*

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{t} \le 0 \ \ a.s.$$

6. Examples and Numerical Simulations

In this section, we will give several examples and numerical simulations to illustrate our results.

Let us consider the following two-species Lotka-Volterra competitive system

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\Big(b_1(r(t)) - a_{11}(r(t))x_1(t) - a_{12}(r(t))x_2(t)\Big)\mathrm{d}t + \Big(\sigma_{11}(r(t))x_1^2(t) + \sigma_{12}(r(t))x_1(t)x_2(t)\Big)\mathrm{d}B(t) \\ \qquad\qquad + x_1(t)\int_{\mathbb{Z}}\gamma_1(r(t),z)N(\mathrm{d}t,\mathrm{d}z), \\ \mathrm{d}x_2(t) = x_2(t)\Big(b_2(r(t)) - a_{21}(r(t))x_1(t) - a_{22}(r(t))x_2(t)\Big)\mathrm{d}t + \Big(\sigma_{21}(r(t))x_1(t)x_2(t) + \sigma_{22}(r(t))x_2^2(t)\Big)\mathrm{d}B(t) \\ \qquad\qquad + x_2(t)\int_{\mathbb{Z}}\gamma_2(r(t),z)N(\mathrm{d}t,\mathrm{d}z). \end{cases} \tag{30}$$

Where $r(t)$ is a right continuous Markov chain with values in the state space $\mathbb{S} = \{1,2\}$. Then system (30) can be regarded as the result of the following two systems switching from one to the other according to the movement of the Markovian chain:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\Big(b_1(1) - a_{11}(1)x_1(t) - a_{12}(1)x_2(t)\Big)\mathrm{d}t + \Big(\sigma_{11}(1)x_1^2(t) + \sigma_{12}(1)x_1(t)x_2(t)\Big)\mathrm{d}B(t) \\ \qquad\qquad + x_1(t)\int_{\mathbb{Z}}\gamma_1(1,z)N(\mathrm{d}t,\mathrm{d}z), \\ \mathrm{d}x_2(t) = x_2(t)\Big(b_2(1) - a_{21}(1)x_1(t) - a_{22}(1)x_2(t)\Big)\mathrm{d}t + \Big(\sigma_{21}(1)x_1(t)x_2(t) + \sigma_{22}(1)x_2^2(t)\Big)\mathrm{d}B(t) \\ \qquad\qquad + x_2(t)\int_{\mathbb{Z}}\gamma_2(1,z)N(\mathrm{d}t,\mathrm{d}z). \end{cases}$$

And

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\Big(b_1(2) - a_{11}(2)x_1(t) - a_{12}(2)x_2(t)\Big)\mathrm{d}t + \Big(\sigma_{11}(2)x_1^2(t) + \sigma_{12}(2)x_1(t)x_2(t)\Big)\mathrm{d}B(t) \\ \qquad\qquad + x_1(t)\int_{\mathbb{Z}}\gamma_1(2,z)N(\mathrm{d}t,\mathrm{d}z), \\ \mathrm{d}x_2(t) = x_2(t)\Big(b_2(2) - a_{21}(2)x_1(t) - a_{22}(2)x_2(t)\Big)\mathrm{d}t + \Big(\sigma_{21}(2)x_1(t)x_2(t) + \sigma_{22}(2)x_2^2(t)\Big)\mathrm{d}B(t) \\ \qquad\qquad + x_2(t)\int_{\mathbb{Z}}\gamma_2(2,z)N(\mathrm{d}t,\mathrm{d}z). \end{cases}$$

Here we choose the generator of the Markov chain $Q = \begin{pmatrix} -7 & 7 \\ 5 & -5 \end{pmatrix}$, by (5) the unique stationary distribution π of $r(t)$ is expressed by $\pi = (\pi_1, \pi_2) = (5/12, 7/12)$.

Example 6.1. *For model (30), let $\lambda(\mathbb{Z}) = 1$, the initial data $x_1(0) = x_2(0) = 0.6, r(0) = 2$ and the coefficients be*

$$
\begin{aligned}
&b_1(1) = 0.8, b_2(1) = 0.5, a_{11}(1) = 0.5, a_{12}(1) = 0.3, a_{21}(1) = 0.4, a_{22}(1) = 0.3,\\
&\sigma_{11}(1) = 0.5, \sigma_{12}(1) = 0, \sigma_{21}(1) = 0, \sigma_{22}(1) = 0.5, \gamma_1(1, z) \equiv -0.3, \gamma_2(1, z) \equiv -0.3.\\
&b_1(2) = 0.5, b_2(2) = 0.8, a_{11}(2) = 0.4, a_{12}(2) = 0.2, a_{21}(2) = 0.1, a_{22}(2) = 0.5,\\
&\sigma_{11}(2) = 0.1, \sigma_{12}(2) = 0, \sigma_{21}(2) = 0, \sigma_{22}(2) = 0.1, \gamma_1(2, z) \equiv -0.2, \gamma_2(2, z) \equiv -0.2.
\end{aligned}
$$

Then $\hat{\beta}(1) = 0.64, \hat{\beta}(2) = 0.72$. So we have

$$\sum_{i=1}^{2} \pi_i \hat{\beta}(i) = \frac{5}{12} \cdot 0.64 + \frac{7}{12} \cdot 0.72 > 0.$$

Then the conditions of Theorem 2.12 are all satisfied. Then by Theorem 2.12 the species $x(t)$ of model (30) is stochastic permanence. Figure 1 conforms this.

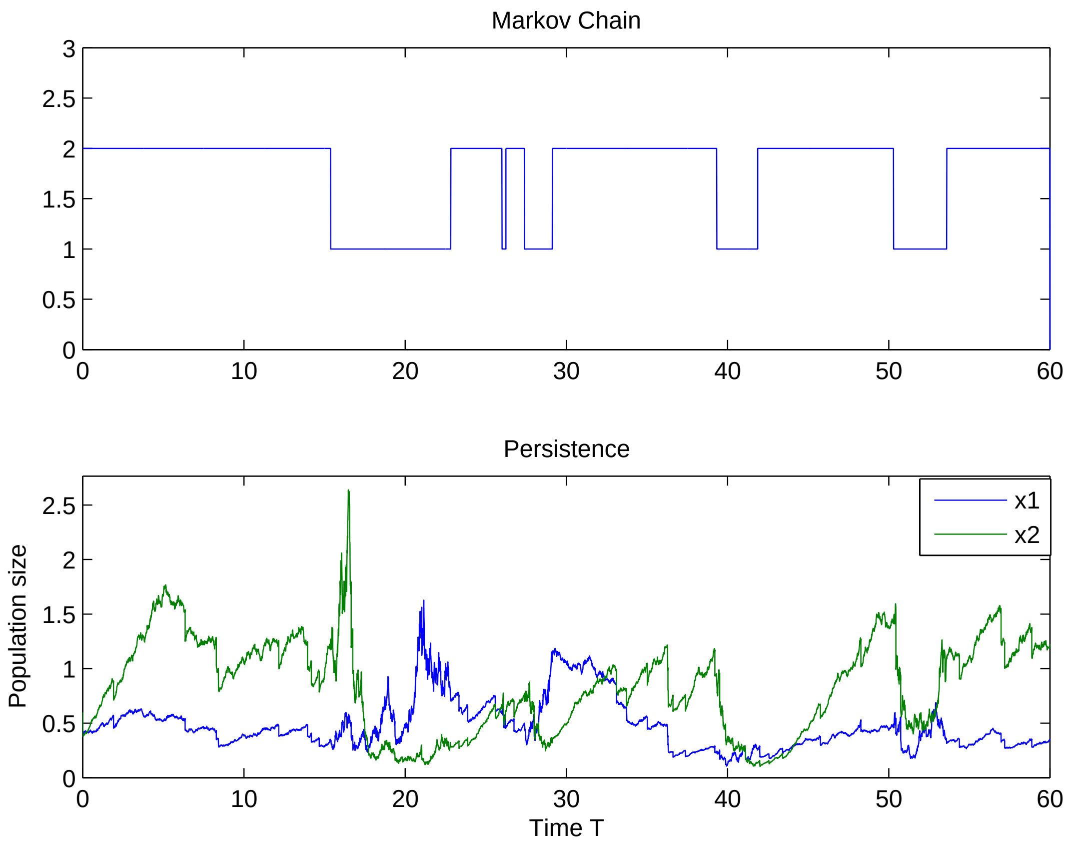

Figure 1: Numerical simulations of Example 6.1. The first figure is the numerical simulation of Markov chain, the second figure is the numerical simulation of system (30). From the figure, we can see that the species of (30) is stochastic persistence.

Example 6.2. *For model (30), let $\lambda(\mathbb{Z}) = 1$, the initial data $x_1(0) = x_2(0) = 0.6, r(0) = 2$ and the coefficients be*

$$b_1(1) = 0.4, b_2(1) = 0.5, a_{11}(1) = 0.5; a_{12}(1) = 0.3, a_{21}(1) = 0.4, a_{22}(1) = 0.3,$$
$$\sigma_{11}(1) = 0.5, \sigma_{12}(1) = 0, \sigma_{21}(1) = 0, \sigma_{22}(1) = 0.5, \gamma_1(1, z) \equiv -0.6, \gamma_2(1, z) \equiv -0.6.$$
$$b_1(2) = 0.4, b_2(2) = 0.3, a_{11}(2) = 0.4; a_{12}(2) = 0.2, a_{21}(2) = 0.1, a_{22}(2) = 0.5,$$
$$\sigma_{11}(2) = 0.1, \sigma_{12}(2) = 0, \sigma_{21}(2) = 0, \sigma_{22}(2) = 0.1, \gamma_1(2, z) \equiv -0.5, \gamma_2(2, z) \equiv -0.5.$$

By computation

$$\breve{\alpha}(1) = 0.5 + \int_{\mathbb{Z}} \ln(1 - 0.6)\lambda(dz) = -0.42, \ \ \breve{\alpha}(2) = 0.4 + \int_{\mathbb{Z}} \ln(1 - 0.5)\lambda(dz) = -0.29.$$

So $\sum_{i=1}^{2} \pi_i \breve{\alpha}(i) < 0$ is obtained. In view of Theorem 3.3 species $x(t)$ of (30) will be extinct. Figure 2 confirms this.

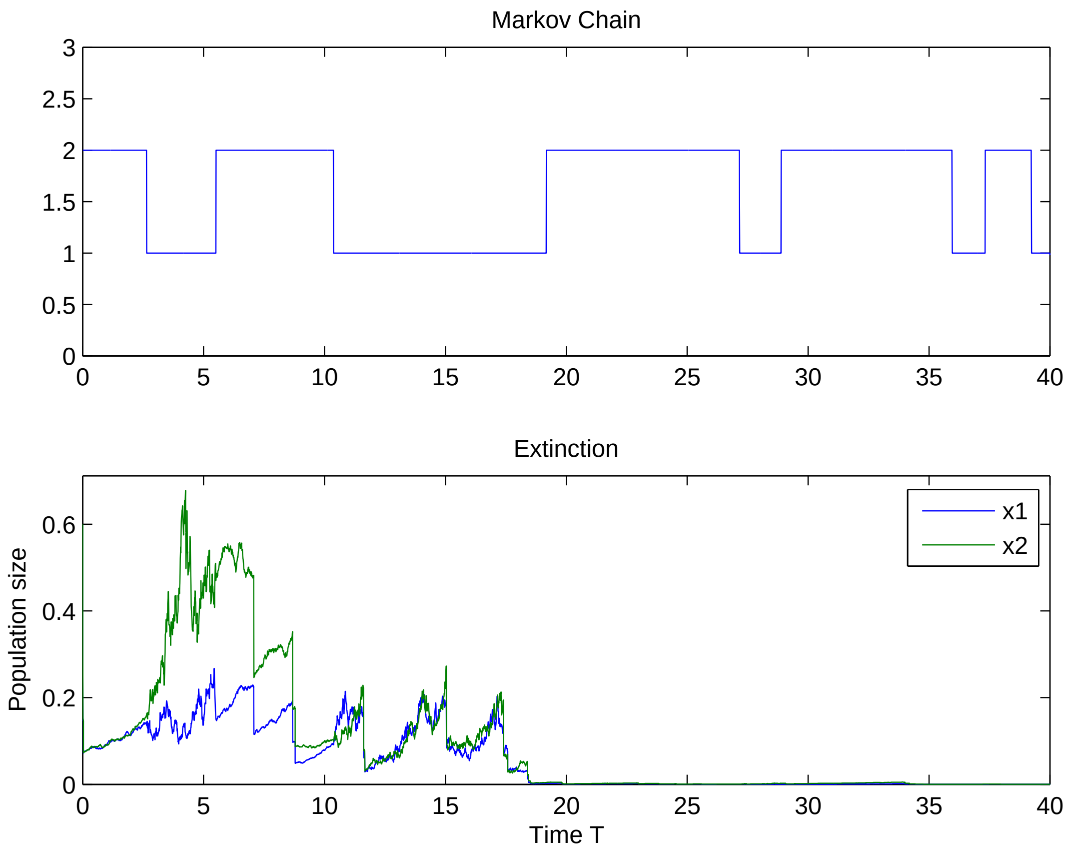

Figure 2: Numerical simulations of Example 6.2. The first figure is the numerical simulation of Markov chain, the second figure is the numerical simulation of system (30). From the figure, we can see that the species of (30) is extinct which reveals the fact that the jump noise make the population extinction.

7. Conclusions and Further Discussions

This paper is concerned with stochastic Lotka-Volterra systems under regime switching with jumps. This kind of model is more applicable. The asymptotic properties of positive solutions are examined. The effects of the color noise and jumping noise on the model are analyzed. Our key contributions are as follows.

(a) Our model is new. In the model, the white noise, color noise and jumping noise are introduced at the same time.

(b) By now, as our knowledge is concerned, the extinction and permanence of the model with three noise at the same time have not been reported. In this paper, sufficient conditions for stochastic permanence (Theorem 2.12) and extinction (Theorem 3.3) are presented. Our results reveal that the stochastic permanence and extinction of the species have close relations with the stationary distribution of the Markov chain.

(c) The moment average in time (Theorem 4.1) and asymptotic pathwise properties (Lemma 5.1 and Theorem 5.2) are estimated.

(d) From our results we can see that the Markovian switching plays important roles in the model, it can switch the overall property of the system.

Some interesting topics deserve further investigation. In this paper, we present sufficient conditions for stochastic permanence and extinction, unfortunately, the critical value between them are not obtained and the gap between the value of permanence and extinction deserves further consideration. Moreover, one may consider a more general regime whose generator depends on $x(t)$, see [33, 34].

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8. Appendix A. Proof of Theorem 2.3

Proof. The method of this proof is classical, here we adopt the ideas of Luo and Mao [24]. Since the coefficients of the equation are locally Lipschitz continuous, for any initial data $\bar{x} \in \mathbb{R}^n_+, r(0) \in \mathbb{S}$ Eq.(4) has a unique maximal local solution $x(t)$ on $[0, \tau_e)$, where τ_e is the explosion time [24]. To show this solution is global, we only need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be so sufficiently large that every component of $\bar{x}$ lies in the interval $[1/k_0, k_0]$. For each integer $k > k_0$, define the sequence of stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : x_i(t) \notin (1/k, k) \ \ for \ \ some \ \ i = 1, 2, \cdots, n\}.$$

Clearly, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, then $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$, then $\tau_e = \infty$ a.s. In the sequel, we show that $\tau_\infty = \infty$. If this is not true, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Thus there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon \ \ for \ \ all \ \ k \geq k_1. \tag{31}$$

Define a C^2-function $V : \mathbb{R}^n_+ \to \mathbb{R}_+$ by

$$V(x) = \sum_{i=1}^{n} [\sqrt{x_i} - 1 - 0.5 \ln x_i], \ x \in \mathbb{R}^n_+.$$

Using the generalized Itô's formula [31] leads to

$$\begin{aligned} \mathrm{d}V(x(t)) \ = \ & \sum_{i=1}^{n} \Big\{0.5(x_i^{0.5} - 1)(b_i + \sum_{j=1}^{n} a_{ij}x_j) + (0.25 - 0.125x_i^{0.5})[\sum_{j=1}^{n} \sigma_{ij}x_j]^2\Big\}\mathrm{d}t \\ & + \sum_{i=1}^{n} \int_{\mathbb{Z}} \Big[x_i^{0.5}((1 + \gamma_i)^{0.5} - 1) - 0.5\ln(1 + \gamma_i)\Big]\lambda(\mathrm{d}z)\mathrm{d}t + \sum_{i=1}^{n} 0.5(x_i^{0.5} - 1) \sum_{j=1}^{n} \sigma_{ij}x_j \mathrm{d}B(t) \\ & + \sum_{i=1}^{n} \int_{\mathbb{Z}} \Big[x_i^{0.5}((1 + \gamma_i)^{0.5} - 1) - 0.5\ln(1 + \gamma_i)\Big]\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \end{aligned}$$

$$
\begin{aligned}
&= LV(x)\mathrm{d}t + \sum_{i=1}^{n} 0.5(x_i^{0.5}-1)\sum_{j=1}^{n}\sigma_{ij}x_j\mathrm{d}B(t) \\
&\quad + \sum_{i=1}^{n}\int_{\mathbb{Z}}\Big[x_i^{0.5}((1+\gamma_i)^{0.5}-1) - 0.5\ln(1+\gamma_i)\Big]\widetilde{N}(\mathrm{d}t,\mathrm{d}z), \qquad (32)
\end{aligned}
$$

where

$$
\begin{aligned}
LV(x) &= \sum_{i=1}^{n}\Big\{0.5(x_i^{0.5}-1)(b_i+\sum_{j=1}^{n}a_{ij}x_j) + (0.25-0.125x_i^{0.5})[\sum_{j=1}^{n}\sigma_{ij}x_j]^2\Big\}\mathrm{d}t \\
&\quad + \sum_{i=1}^{n}\int_{\mathbb{Z}}\Big[x_i^{0.5}((1+\gamma_i)^{0.5}-1) - 0.5\ln(1+\gamma_i)\Big]\lambda(\mathrm{d}z)\mathrm{d}t \qquad (33)
\end{aligned}
$$

Here, for simplicity, we omit t^- in $x(t^-)$ and $r(t)$ from $b_i(r(t))$, etc. Note that

$$
\begin{aligned}
\sum_{i=1}^{n}(x_i^{0.5}-1)(b_i+\sum_{j=1}^{n}a_{ij}x_j) &\le \sum_{i=1}^{n}|b_i|(x_i^{0.5}+1) + \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|x_j + \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|x_i^{0.5}x_j \qquad (34) \\
&\le \sum_{i=1}^{n}|b_i|(x_i^{0.5}+1) + \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}|x_j + \sum_{i=1}^{n}\sum_{j=1}^{n}0.5|a_{ij}|(x_i+x_j^2) \\
&= \sum_{i=1}^{n}\Big[|b_i|(x_i^{0.5}+1) + \sum_{j=1}^{n}(|a_{ji}|+0.5|a_{ij}|)x_i + 0.5\sum_{j=1}^{n}|a_{ji}|x_i^2\Big]
\end{aligned}
$$

and

$$
\sum_{i=1}^{n}\Big[\sum_{j=1}^{n}\sigma_{ij}x_j\Big]^2 \le \sum_{i=1}^{n}\Big[\sum_{j=1}^{n}\sigma_{ij}^2\sum_{j=1}^{n}x_j^2\Big] = |\sigma|^2\sum_{i=1}^{n}x_i^2. \qquad (35)
$$

In addition, by Assumption 2.1,

$$
\sum_{i=1}^{n}x_i^{0.5}\Big[\sum_{j=1}^{n}\sigma_{ij}x_j\Big]^2 \ge \sum_{i=1}^{n}\sigma_{ii}^2x_i^{2.5}. \qquad (36)
$$

Substituting (34), (35), (36) into (33), we obtain

$$
\begin{aligned}
LV(x) &\le \sum_{i=1}^{n}\Big[0.5|b_i|(x_i^{0.5}+1) + \sum_{j=1}^{n}(0.5|a_{ji}|+0.25|a_{ij}|)x_i + 0.25(\sum_{j=1}^{n}|a_{ji}|+|\sigma|^2)x_i^2 \\
&\quad -0.125\sigma_{ii}^2x_i^{2.5} + \int_{\mathbb{Z}}\Big[x_i^{0.5}((1+\gamma_i)^{0.5}-1) - 0.5\ln(1+\gamma_i)\Big]\lambda(\mathrm{d}z)\Big].
\end{aligned}
$$

Therefore $LV(x)$ is bounded in $\mathbb{R}_+^n\times\mathbb{S}$, namely, there exists a constant $K>0$ such that

$$
LV(x) \le K.
$$

By now, Eq.(32) changes into

$$
\begin{aligned}
\mathrm{d}V(x(t)) &\le Kdt + \sum_{i=1}^{n}0.5(x_i^{0.5}-1)\sum_{j=1}^{n}\sigma_{ij}x_j\mathrm{d}B(t) \\
&\quad + \sum_{i=1}^{n}\int_{\mathbb{Z}}\Big[x_i^{0.5}((1+\gamma_i)^{0.5}-1) - 0.5\ln(1+\gamma_i)\Big]\widetilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}
$$

Integrating the above inequality from 0 to $\tau_k \wedge T$, then taking expectations, we get

$$\mathbb{E}[V(x(\tau_k \wedge T))] \leq V(\bar{x}) + K\mathbb{E}(\tau_k \wedge T) \leq V(\bar{x}) + KT. \tag{37}$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$, by (31), $\mathbb{P}(\Omega_k) \geq \varepsilon$. From the definition of τ_k we have for every $\omega \in \Omega_k$, there is some i such that $x_i(\tau_k, \omega)$ equals either k or $1/k$, therefore $V(x(\tau_k, \omega))$ is no less than either

$$\sqrt{k} - 1 - 0.5 \ln k$$

or

$$\sqrt{1/k} - 1 - 0.5 \ln(1/k) = \sqrt{1/k} - 1 + 0.5 \ln k.$$

This results in

$$V(x(\tau_k, \omega)) \geq [\sqrt{k} - 1 - 0.5 \ln k] \wedge [0.5 \ln k - 1].$$

From (37) we have

$$\begin{aligned} V(\bar{x}) + KT &\geq \mathbb{E}[I_{\Omega_k}(\omega) V(x(\tau_k, \omega))] \\ &\geq \varepsilon([\sqrt{k} - 1 - 0.5 \ln k] \wedge [0.5 \ln k - 1]). \end{aligned}$$

Letting $k \to \infty$ leads to the contradiction

$$\infty > V(\bar{x}) + KT = \infty.$$

Therefore we conclude that $\tau_\infty = \infty$ a.s. This completes the proof. □

9. Appendix B. Proof of Lemma 2.11

Proof. Our proof is similar to [17]. Without loss of generality, we assume $u = m$ in Assumption 2.9, namely

$$q_{im} > 0, \ \ 1 \leq i \leq m - 1.$$

It is easy to see that

$$\begin{aligned} detA(\theta) &= \begin{vmatrix} \xi_1(\theta) & -q_{12} & \cdots & -q_{1m} \\ \xi_2(\theta) & \xi_2(\theta) - q_{22} & \cdots & -q_{2m} \\ \vdots & \vdots & \cdots & -q_{m-1,m} \\ \xi_m(\theta) & -q_{m2} & \cdots & \xi_m(\theta) - q_{mm} \end{vmatrix} \\ &= \sum_{k=1}^{m} \xi_k(\theta) M_k(\theta), \end{aligned}$$

where $M_k(\theta)$ is the corresponding minor of $\xi_k(\theta)$ in the first column, i.e.

$$M_1(\theta) = (-1)^{1+1} \begin{vmatrix} \xi_2(\theta) - q_{22} & \cdots & -q_{2m} \\ \vdots & \cdots & \vdots \\ -q_{m-1,2} & \cdots & -q_{m-1,m} \\ -q_{m,2} & \cdots & \xi_m(\theta) - q_{mm} \end{vmatrix},$$

$$\vdots$$

$$M_m(\theta) = (-1)^{m+1} \begin{vmatrix} -q_{12} & \cdots & -q_{1m} \\ \xi_2(\theta) - q_{22} & \cdots & -q_{2m} \\ \vdots & \cdots & \vdots \\ -q_{m-1,2} & \cdots & -q_{m-1,m} \end{vmatrix}.$$

Note that

$$\xi_k(0) = 0 \ \ and \ \ \frac{d}{d\theta}\xi_k(0) = \hat{\beta}(k),$$

so we have

$$\frac{d}{d\theta}detA(0) = \sum_{k=1}^{m} \hat{\beta}(k) M_k(0).$$

This means that

$$\frac{d}{d\theta}detA(0) = \begin{vmatrix} \hat{\beta}(1) & -q_{12} & \cdots & -q_{1m} \\ \hat{\beta}(2) & -q_{22} & \cdots & -q_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\beta}(m) & -q_{m2} & \cdots & -q_{mm} \end{vmatrix}. \tag{38}$$

According to Appendix A in [27], under Assumption 2.9, the condition $\sum_{k=1}^{m} \pi_k \hat{\beta}(k) > 0$ is equivalent to

$$\begin{vmatrix} \hat{\beta}(1) & -q_{12} & \cdots & -q_{1m} \\ \hat{\beta}(2) & -q_{22} & \cdots & -q_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\beta}(m) & -q_{m2} & \cdots & -q_{mm} \end{vmatrix} > 0.$$

Together with (38), we see that

$$\frac{d}{d\theta}detA(0) > 0.$$

By $detA(0) = 0$, we can find a sufficiently small $0 < \theta < 1$ such that $detA(\theta) > 0$ and

$$\xi_k(\theta) = \theta\Big[2\hat{b}(k) - \int_{\mathbb{Z}} \frac{2\hat{\gamma}(k,z)}{(1 + |\gamma(\check{k},z)|)^2}\lambda(\mathrm{d}z)\Big] > -q_{km}, \ \ 1 \le k \le m-1. \tag{39}$$

For every $1 \le k \le m-1$, we consider the leading principle sub-matrix

$$A_k(\theta) := \begin{vmatrix} \xi_1(\theta) - q_{11} & -q_{12} & \cdots & -q_{1k} \\ -q_{21} & \xi_2(\theta) - q_{22} & \cdots & -q_{2k} \\ \vdots & \cdots & \vdots & \\ -q_{k1} & -q_{k2} & \cdots & \xi_k(\theta) - q_{kk} \end{vmatrix}$$

of $A(\theta)$. Clearly, $A_k(\theta) \in Z^{k\times k} := \{A = (a_{ij})_{k\times k} : a_{ij} \le 0, i \ne j\}$. By (39) we follow that each row of this sun-matrix has the sum

$$\xi_k(\theta) - \sum_{j=1}^{k} q_{kj} \ge \xi_k(\theta) + q_{kN} > 0.$$

By Lemma 5.3 [29], we have $detA_k(\theta) > 0$. In other words, we reach that all the leading principle minors of $A(\theta)$ are positive. According to Theorem 2.10 [29], we obtain the desired assertion. □

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