On Certain Class of Meromorphically Multivalent Reciprocal Starlike Functions Associated with the Liu-Srivastava Operator Defined by Subordination

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Abstract. In the paper, we introduce the class of meromorphically p-valent reciprocal starlike functions associated with the Liu-Srivastava operator defined by subordination. Some sufficient conditions for functions belonging to this class are derived. The results presented here improve and generalize some known results.

1. Introduction and Preliminaries

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^{k-p+1} \quad (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1}$$

which are analytic and p-valent in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \backslash \{0\},$$

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we set $\Sigma_1 = \Sigma$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, if there exists a Schwartz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \ \text{ and } \ |\omega(z)| < 1 \ (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)) \ \ (z \in \mathbb{U}).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function g is univalent in $\mathbb{U}$, then the following equivalent relationship holds(see for details [4, 11]; see also [17]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \ \text{ and } \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f \in \Sigma_p$ is said to be in the class $S_p^*(\alpha)$ of meromorphically p-valent starlike of order α if it satisfies the inequality

$$\mathfrak{R}\left(\frac{zf'(z)}{pf(z)}\right) < -\alpha \quad (0 \le \alpha < 1; z \in \mathbb{U}). \tag{2}$$

As usual, we let $S_p^*(0) \equiv S_p^*$. Furthermore, a function $f \in S_p^*$ is said to be in the class $M_p(\alpha)$ of meromorphically p-valent starlike of reciprocal order α if and only if

$$\mathfrak{R}\left(\frac{pf(z)}{zf'(z)}\right) < -\alpha \quad (0 \le \alpha < 1; z \in \mathbb{U}). \tag{3}$$

In particular, we set $M_1(\alpha) \equiv M(\alpha)$.

Remark 1.1. *In view of the fact that*

$$\mathfrak{R}(p(z)) < 0 \Rightarrow \mathfrak{R}\left(\frac{1}{p(z)}\right) = \mathfrak{R}\left(\frac{\overline{p(z)}}{|p(z)|^2}\right) < 0,$$

it follows that a meromorphically p-valent starlike function of reciprocal order 0 is same as a meromorphically p-valent starlike function. When $0 < \alpha < 1$, the function $f \in \Sigma_p$ is meromorphically p-valent starlike of reciprocal order α if and only if

$$\left|\frac{zf'(z)}{pf(z)} + \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}). \tag{4}$$

For $p = 1$, this class $M(\alpha)$ was considered by Sun et al. [18].

For arbitrary fixed real numbers A and $B(-1 \le B < A \le 1)$, we denote by $P(A, B)$ the class of functions of the form

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

which is analytic in the unit disk $\mathbb{U}$ and satisfies the condition

$$q(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{5}$$

where the symbol $\prec$ stands for usual subordination. The class $P(A, B)$ was introduced and studied by Janowski [8].

We also observe from (5) (see, also [15]) that a function $q(z) \in P(A, B)$ if and only if

$$\left|q(z) - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2} \quad (B \ne -1; z \in \mathbb{U}) \tag{6}$$

and

$$\mathfrak{R}\{q(z)\} > \frac{1 - A}{2} \quad (B = -1; z \in \mathbb{U}). \tag{7}$$

For functions $f \in \Sigma_p$ given by (1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^{k-p+1} \quad (p \in \mathbb{N}), \tag{8}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) := z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^{k-p+1} = (g * f)(z). \tag{9}$$

The linear operator $L_p(a,c)$ is defined as follows(see[10])

$$L_p(a,c)f(z) := \phi_p(a,c;z) * f(z) \quad (f \in \Sigma_p), \tag{10}$$

and $\phi_p(a,c;z)$ is defined by

$$\phi_p(a,c;z) := z^{-p} + \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p+1} \quad (z \in \mathbb{U}^*; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \cdots), \tag{11}$$

where $(\lambda)_n$ is the Pochhammer symbol(or the shifted factorial) defined (in terms of the Gamma function) by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & n \in \mathbb{N}. \end{cases} \tag{12}$$

The Liu-Srivastava operator $L_p(a,c)$, analogous to the Carlson-Shafer operator, was considered by Liu and Srivastava [10] on the space of analytic and meromorphically p-valent functions. The carlson-Shafer operator $\mathcal{L}_p(a,c)$ was introduced earlier by Saitoh [14] on the space of analytic and p-valent functions in $\mathbb{U}$. In present, the Carlson-Shafer operator was generalized to the Dziok-Srivastava operator by Dziok and Srivastava [5, 6]. Recently, Aouf et al.[1] constructed a new operator by applying the Liu-Srivastava operator $L_p(a,c)$.

In [10], making use of the Liu-Srivastava operator $L_p(a,c)$, Liu and Srivastava discussed the subclass of Σ_p such that

$$\frac{z(L_p(a,c)f(z))'}{pL_p(a,c)f(z)} \prec -\frac{1 + Az}{1 + Bz}. \tag{13}$$

By using the Liu-Srivastava operator $L_p(a,c)$, we now introduce a new subclass $M_{a,c}(p; \beta; A, B)$ satisfying the following subordination condition for $f \in \Sigma_p$ given by (1),

$$\frac{p}{1 - p\beta} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} \prec -\frac{1 + Az}{1 + Bz}, \tag{14}$$

where $-1 \le B < A \le 1, a > 0, c > 0, 0 \le p\beta < 1, p \in \mathbb{N}$.

We note that (14) is equivalent to(by (6) and (7))

$$\left| \frac{p}{1 - p\beta} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} + \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (B \ne -1; z \in \mathbb{U}) \tag{15}$$

and

$$\Re \left\{ \frac{p}{1 - p\beta} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} \right\} < -\frac{1 - A}{2} \quad (B = -1; z \in \mathbb{U}). \tag{16}$$

We note (16) is equivalent to

$$\left| \frac{1 - p\beta}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) + \frac{1}{1 - A} \right| < \frac{1}{1 - A} \quad (B = -1, A \ne 1; z \in \mathbb{U}) \tag{17}$$

and

$$\left| \frac{p}{1 - p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) + 1 \right| < 1 \quad (B = -1, A = 1; z \in \mathbb{U}) \tag{18}$$

By specializing the parameters p, a, c, A, B and β, we obtain the following classes studied by different authors,

(1) $M_{1,1}(1; 0; 1 - 2\alpha, -1) = M(\alpha)(0 \le \alpha < 1)$ (see [18]);

(2) $M_{1,1}(1; 0; B, b(A - B) + B) = \Sigma^*[b; A, B](b < 0, -1 \le B < A \le 1)$ (see [3]);

(3) $M_{1,1}\left(p; 0; B, B + \dfrac{(A - B)(p - a)}{p}\right) = Q_k[p, a, A, B](0 \le a < p, -1 \le A < B \le 1, 0 \le B \le 1, A + B \ge 0)$ (see [16]).

In recent years, more and more researchers are interested in the reciprocal case of the starlike functions(see [19],[9],[13],[2]).

In the present investigation, we give some sufficient conditions for the function belonging to the class $M_{a,c}(p; \beta; A, B)$. In order to establish our main results, we need the following lemmas.

Lemma 1.2. *(Jack's lemma [7]) Let the (nonconstant) function $\omega(z)$ be analytic in $\mathbb{U}$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then $z_0\omega'(z_0) = \gamma\omega(z_0)$, where γ is a real number and $\gamma \ge 1$.*

Lemma 1.3. *[12] Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \le -\dfrac{1 + x^2}{2}$. If the function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$.*

Lemma 1.4. *[20] Let $\rho(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in $\mathbb{U}$ and η be analytic and starlike (with respect to the origin) univalent in $\mathbb{U}$ with $\eta(0) = 0$. If*

$$z\rho'(z) \prec \eta(z),$$

then

$$\rho(z) \prec 1 + \int_0^z \frac{\eta(t)}{t} dt.$$

2. Main Results

Unless otherwise mentioned we shall assume through this paper that $a > 0, c > 0, -1 \le B < A \le 1, 0 \le p\beta < 1, p \in \mathbb{N}$.

We begin by presenting the following coefficient sufficient condition for functions belonging to the class $M_{a,c}(p; \beta; A, B)$.

Theorem 2.1. *If $f \in \Sigma_p$ satisfies anyone of the following conditions:*

(i) For $B \ne -1$,

$$\sum_{k=0}^{\infty} \left(|k - p + 1| + \frac{|p(1 - B^2)[1 + (k - p + 1)\beta] + (1 - AB)(1 - p\beta)(k - p + 1)|}{(1 - p\beta)(A - B)}\right) \frac{(a)_k}{(c)_k} |a_k| < p(1 - |B|), \tag{19}$$

(ii) For $B = -1, A \ne 1$,

$$\sum_{k=0}^{\infty} \left(|1 + (k - p + 1)\beta| + \left|1 + (k - p + 1)\beta + \frac{(1 - A)(1 - p\beta)(k - p + 1)}{p}\right|\right) \frac{(a)_k}{(c)_k} |a_k| < (1 - p\beta)(1 - |A|), \tag{20}$$

(iii) For $B = -1, A = 1$,

$$\sum_{k=0}^{\infty} \left(|k - p + 1| + \frac{k + 1}{1 - p\beta}\right) \frac{(a)_k}{(c)_k} |a_k| < p, \tag{21}$$

then $f \in M_{a,c}(p; \beta; A, B)$.

Proof. (i) If $B \neq -1$, by the condition (15), we only need to show that

$$\left| \frac{p(1-B^2)}{(1-p\beta)(A-B)} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} + \frac{1-AB}{A-B} \right| < 1 \quad (z \in \mathbb{U}). \tag{22}$$

We first observe that

$$\left| \frac{p(1-B^2)}{(1-p\beta)(A-B)} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} + \frac{1-AB}{A-B} \right|$$

$$= \left| \frac{Bp + \sum_{k=0}^{\infty} \frac{p(1-B^2)[1+(k-p+1)\beta]+(1-AB)(1-p\beta)(k-p+1)}{(1-p\beta)(A-B)} \frac{(a)_k}{(c)_k} a_k z^{k+1}}{-p + \sum_{k=0}^{\infty} (k-p+1)\frac{(a)_k}{(c)_k} a_k z^{k+1}} \right|$$

$$\leq \frac{|B|p + \sum_{k=0}^{\infty} \frac{|p(1-B^2)[1+(k-p+1)\beta]+(1-AB)(1-p\beta)(k-p+1)|}{(1-p\beta)(A-B)} \frac{(a)_k}{(c)_k} |a_k||z|^{k+1}}{p - \sum_{k=0}^{\infty} |k-p+1|\frac{(a)_k}{(c)_k} |a_k||z|^{k+1}}$$

$$< \frac{|B|p + \sum_{k=0}^{\infty} \frac{|p(1-B^2)[1+(k-p+1)\beta]+(1-AB)(1-p\beta)(k-p+1)|}{(1-p\beta)(A-B)} \frac{(a)_k}{(c)_k} |a_k|}{p - \sum_{k=0}^{\infty} |k-p+1|\frac{(a)_k}{(c)_k} |a_k|} \tag{23}$$

Now, by using the inequality (19), we have

$$\frac{|B|p + \sum_{k=0}^{\infty} \frac{|p(1-B^2)[1+(k-p+1)\beta]+(1-AB)(1-p\beta)(k-p+1)|}{(1-p\beta)(A-B)} \frac{(a)_k}{(c)_k} |a_k|}{p - \sum_{k=0}^{\infty} |k-p+1|\frac{(a)_k}{(c)_k} |a_k|} < 1, \tag{24}$$

which, in conjunction with (23), completes the proof of (i) for Theorem 2.1.

(ii) If $B = -1, A \neq 1$, by virtue of the condition (17), we only need to show that

$$\left| \frac{(1-A)(1-p\beta)}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) + 1 \right| < 1 \quad (z \in \mathbb{U}). \tag{25}$$

We first observe that

$$\left| \frac{(1-A)(1-p\beta)}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) + 1 \right|$$

$$= \left| \frac{A(1-p\beta) + \sum_{k=0}^{\infty} \left[1 + (k-p+1)\beta + \frac{(1-A)(1-p\beta)(k-p+1)}{p} \right] \frac{(a)_k}{(c)_k} a_k z^{k+1}}{(1-p\beta) + \sum_{k=0}^{\infty} [1 + (k-p+1)\beta]\frac{(a)_k}{(c)_k} a_k z^{k+1}} \right|$$

$$\leq \frac{|A|(1-p\beta) + \sum_{k=0}^{\infty} \left| 1 + (k-p+1)\beta + \frac{(1-A)(1-p\beta)(k-p+1)}{p} \right| \frac{(a)_k}{(c)_k} |a_k||z|^{k+1}}{(1-p\beta) - \sum_{k=0}^{\infty} |1 + (k-p+1)\beta|\frac{(a)_k}{(c)_k} |a_k||z|^{k+1}}$$

$$< \frac{|A|(1-p\beta) + \sum_{k=0}^{\infty} \left| 1 + (k-p+1)\beta + \frac{(1-A)(1-p\beta)(k-p+1)}{p} \right| \frac{(a)_k}{(c)_k} |a_k|}{(1-p\beta) - \sum_{k=0}^{\infty} |1 + (k-p+1)\beta|\frac{(a)_k}{(c)_k} |a_k|} \tag{26}$$

Now, by using the inequality (20), we have

$$\frac{|A(1 - p\beta)| + \sum\limits_{k=0}^{\infty} \left|1 + (k - p + 1)\beta + \frac{(1-A)(1-p\beta)(k-p+1)}{p}\right| \frac{(a)_k}{(c)_k}|a_k|}{|1 - p\beta| - \sum\limits_{k=0}^{\infty} |1 + (k - p + 1)\beta| \frac{(a)_k}{(c)_k}|a_k|} < 1, \tag{27}$$

which, in conjunction with (26), completes the proof of (ii) for Theorem 2.1.

(iii) If $B = -1, A = 1$, by virtue of the condition (18), we only need to show that

$$\left|\frac{p}{1 - p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right) + 1\right| < 1 \quad (z \in \mathbb{U}). \tag{28}$$

We first observe that

$$\left|\frac{p}{1 - p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right) + 1\right| = \left|\frac{\sum\limits_{k=0}^{\infty}\left(\frac{k+1}{1-p\beta}\right)\frac{(a)_k}{(c)_k}a_k z^{k+1}}{-p + \sum\limits_{k=0}^{\infty}(k - p + 1)\frac{(a)_k}{(c)_k}a_k z^{k+1}}\right|$$

$$\leq \frac{\sum\limits_{k=0}^{\infty}\frac{k+1}{1-p\beta}\frac{(a)_k}{(c)_k}|a_k||z|^{k+1}}{p - \sum\limits_{k=0}^{\infty}|k - p + 1|\frac{(a)_k}{(c)_k}|a_k||z|^{k+1}}$$

$$< \frac{\sum\limits_{k=0}^{\infty}\frac{k+1}{1-p\beta}\frac{(a)_k}{(c)_k}|a_k|}{p - \sum\limits_{k=0}^{\infty}|k - p + 1|\frac{(a)_k}{(c)_k}|a_k|} \tag{29}$$

Now, by using the inequality (21), we have

$$\frac{\sum\limits_{k=0}^{\infty}\frac{k+1}{1-p\beta}\frac{(a)_k}{(c)_k}|a_k|}{p - \sum\limits_{k=0}^{\infty}|k - p + 1|\frac{(a)_k}{(c)_k}|a_k|} < 1, \tag{30}$$

which, in conjunction with (29), completes the proof of (iii) for Theorem 2.1. □

Taking $a = c = 1, \beta = 0$ in Theorem 2.1, we obtain the following Corollary.

Corollary 2.2. *If $f \in \Sigma_p$ satisfies anyone of the following conditions:*
(i) For $B \neq -1$,

$$\sum_{k=0}^{\infty}\left(|k - p + 1| + \frac{|p(1 - B^2) + (k - p + 1)(1 - AB)|}{(A - B)}\right)|a_k| < p(1 - |B|),$$

(ii) For $B = -1, A \neq 1$,

$$\sum_{k=0}^{\infty}\left(1 + \left|1 + \frac{(1 - A)(k - p + 1)}{p}\right|\right)|a_k| < (1 - |A|),$$

(iii) For $B = -1, A = 1$,

$$\sum_{k=0}^{\infty}(|k - p + 1| + k + 1)|a_k| < p,$$

then $f \in M_{1,1}(p; 0; A, B)$.

Taking $p = 1, A = 1 - 2\alpha, 0 < \alpha < 1, B = -1$ in Corollary 2.2, we obtain the following result.

Corollary 2.3. *[18] If $f \in \Sigma$ satisfies*

$$\sum_{k=0}^{\infty} (1 + k\alpha) |a_k| < \frac{1}{2}(1 - |1 - 2\alpha|),$$

then $f \in M(\alpha)$, for $0 < \alpha < 1$.

Theorem 2.4. *If $f \in \Sigma_p$ satisfies anyone of the following conditions:*
(i) For $B \neq -1$,

$$\left| 1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right| < \frac{(1 - p\beta)(A - B)}{(1 - p\beta)(A - B) + (1 + |B|)}, \tag{31}$$

(ii) For $B = -1, -1 < A \leq 0$,

$$\left| 1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right| < \frac{(1 - p\beta)(1 - A)(1 + A)}{2p\beta(1 + A) + 2(1 - A)}, \tag{32}$$

(iii) For $B = -1, A = 1$,

$$\left| 1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right| < \frac{1 - p\beta}{2 - p\beta}, \tag{33}$$

then $f(z) \in M_{a,c}(p; \beta; A, B)$.

Proof. (i) If $B \neq -1$, let

$$\omega(z) = \frac{1 + \frac{1+|B|}{1+|B|+A-B} \cdot \frac{p}{1-p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right)}{1 - \frac{1+|B|}{1+|B|+A-B}} - 1 \quad (z \in \mathbb{U}). \tag{34}$$

Then the function ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. We easily find from (34) that

$$\frac{pL_p(a,c)f(z)}{z(L_p(a,c)f(z))'} = \frac{(1 - p\beta)(A - B)\omega(z) - (1 + |B|)}{(1 + |B|)} \quad (z \in \mathbb{U}). \tag{35}$$

Differentiating both sides of (35) logarithmically, we obtain

$$1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} = -\frac{(1 - p\beta)(A - B)z\omega'(z)}{(1 - p\beta)(A - B)\omega(z) - (1 + |B|)}, \tag{36}$$

by virtue of (31) and (36), we find that

$$\left| 1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right| = (1 - p\beta)(A - B)\left| \frac{z\omega'(z)}{(1 - p\beta)(A - B)\omega(z) - (1 + |B|)} \right|$$

$$< \frac{(1 - p\beta)(A - B)}{(1 - p\beta)(A - B) + (1 + |B|)}.$$

Next, we claim that $|\omega(z)| < 1$. Indeed, if not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \quad (z_0 \in \mathbb{U}). \tag{37}$$

Applying Jack's Lemma to $\omega(z)$ at the point z_0, we have

$$z_0\omega'(z_0) = \gamma\omega(z_0) \quad (\gamma \geq 1).$$

Now, upon setting

$$\omega(z_0) = e^{i\theta}(0 \leq \theta < 2\pi).$$

If we put $z = z_0$ in (36), we get

$$\left|1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)}\right| = (1 - p\beta)(A - B)\left|\frac{z_0\omega'(z_0)}{(1 - p\beta)(A - B)\omega(z_0) - (1 + |B|)}\right|$$

$$= (1 - p\beta)(A - B)\left|\frac{\gamma}{(1 - p\beta)(A - B) - (1 + |B|)e^{-i\theta}}\right|$$

$$\geq (1 - p\beta)(A - B)\left|\frac{1}{(1 - p\beta)(A - B) - (1 + |B|)e^{-i\theta}}\right|.$$

This implies that

$$\left|1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)}\right|^2$$

$$\geq \frac{[(1 - p\beta)(A - B)]^2}{[(1 - p\beta)(A - B)]^2 + (1 + |B|)^2 - 2(1 - p\beta)(A - B)(1 + |B|)\cos\theta} \tag{38}$$

Since the right hand side of (38) takes it minimum value for $\cos\theta = -1$, we have that

$$\left|1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)}\right|^2 \geq \frac{[(1 - p\beta)(A - B)]^2}{[(1 - p\beta)(A - B) + (1 + |B|)]^2}$$

This contradicts our condition (31) of Theorem 2.4. Therefore, we conclude that $|\omega(z)| < 1$, which shows that

$$\left|\frac{1 + \frac{1+|B|}{1+|B|+A-B} \cdot \frac{p}{1-p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right)}{1 - \frac{1+|B|}{1+|B|+A-B}} - 1\right| < 1 \quad (B \neq -1; z \in \mathbb{U}).$$

This implies that

$$\left|\frac{p}{1 - p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} + 1\right| < \frac{A - B}{1 + |B|} \quad (B \neq -1; z \in \mathbb{U}), \tag{39}$$

then, we have

$$\left|\frac{p}{1 - p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} + \frac{1 - AB}{1 - B^2}\right| \leq \left|\frac{p}{1 - p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} + 1\right| + \left|\frac{1 - AB}{1 - B^2} - 1\right|$$

$$< \frac{A - B}{1 + |B|} + \frac{|B|(A - B)}{1 - B^2}$$

$$= \frac{A - B}{1 - B^2} \quad (B \neq -1; z \in \mathbb{U}).$$

Therefore, we conclude that $f(z) \in M_{a,c}(p; \beta; A, B)$ for $B \neq -1$.

(ii) If $B = -1, -1 < A \leq 0$, analogously to Theorem 2.2 in [18], we let

$$\omega(z) = \frac{1 + \frac{1-A}{2}\frac{1}{\frac{p}{1-p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right)}}{1 - \frac{1-A}{2}} - 1 \tag{40}$$

Then the function ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. We easily find from (40) that

$$\frac{z(L_p(a,c)f(z))'}{pL_p(a,c)f(z)} = \frac{-(1-A)+(1+A)\omega(z)}{(1-A)-p\beta(1+A)\omega(z)} \quad (z \in \mathbb{U}). \tag{41}$$

Differentiating both sides of (41) logarithmically, we obtain

$$1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} = \frac{(1-p\beta)(1-A)(1+A)z\omega'(z)}{(-(1-A)+(1+A)\omega(z))((1-A)-p\beta(1+A)\omega(z))}, \tag{42}$$

by virtue of (32) and (42), we find that

$$\left| 1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right|$$

$$= (1-p\beta)(1-A)(1+A) \left| \frac{z\omega'(z)}{(-(1-A)+(1+A)\omega(z))((1-A)-p\beta(1+A)\omega(z))} \right|$$

$$< \frac{(1-p\beta)(1-A)(1+A)}{2p\beta(1+A)+2(1-A)}.$$

Next, we claim that $|\omega(z)| < 1$. Indeed, if not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \le |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \quad (z_0 \in \mathbb{U}). \tag{43}$$

Applying Jack's Lemma to $\omega(z)$ at the point z_0, we have

$$z_0\omega'(z_0) = \gamma\omega(z_0) \quad (\gamma \ge 1). \tag{44}$$

Now, upon setting

$$\omega(z_0) = e^{i\theta}(0 \le \theta < 2\pi).$$

If we put $z = z_0$ in (42), we get

$$\left| 1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)} \right|$$

$$= (1-p\beta)(1-A)(1+A) \left| \frac{\gamma}{((1+A)-(1-A)e^{-i\theta})(-p\beta(1+A)+(1-A)e^{-i\theta})} \right|$$

$$\ge (1-p\beta)(1-A)(1+A) \left| \frac{1}{-p\beta(1+A)^2+(1+p\beta)(1-A)(1+A)e^{-i\theta}-(1-A)^2e^{-2i\theta}} \right|.$$

This implies that

$$\left| 1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)} \right|^2 \ge \frac{((1-p\beta)(1-A)(1+A))^2}{P\cos^2\theta+Q\cos\theta+R}, \tag{45}$$

where $P = 4p\beta(1+A)^2(1-A)^2, Q = -2(1+p\beta)(1-A)(1+A)[(1-A)^2+p\beta(1+A)^2], R = 2(1+A^2)[p^2\beta^2(1+A)^2+(1-A)^2]$.

Since the right hand side of (54) takes it minimum value for $\cos\theta = -1$, we have that

$$\left| 1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)} \right|^2 \ge \frac{[(1-p\beta)(1-A)(1+A)]^2}{[2p\beta(1+A)+2(1-A)]^2}. \tag{46}$$

This contradicts our condition (32) of Theorem 2.4. Therefore, we conclude that $|\omega(z)| < 1$, which shows that

$$\left| \frac{1 + \frac{1-A}{2} \frac{1}{\frac{p}{1-p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right)}}{1 - \frac{1-A}{2}} - 1 \right| < 1. \tag{47}$$

This implies that

$$\left| \frac{1 - p\beta}{p} \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} + 1 \right| < \frac{2}{1-A} - 1, \tag{48}$$

then, we have

$$\begin{aligned}
\left| \frac{1-p\beta}{p} \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} + \frac{1}{1-A} \right| &\leq \left| \frac{1-p\beta}{p} \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} + 1 \right| + \left| \frac{1}{1-A} - 1 \right| \\
&< \frac{2}{1-A} - 1 + 1 - \frac{1}{1-A} \\
&= \frac{1}{1-A} \quad (-1 < A \leq 0; z \in \mathbb{U}).
\end{aligned}$$

Therefore, we conclude that $f(z) \in M_{a,c}(p; \beta; A, B)$ for $B = -1, -1 < A \leq 0$.

(iii) If $B = -1, A = 1$, we let

$$\omega(z) = \frac{p}{1-p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) + 1. \tag{49}$$

Then the function ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. We easily find from (49) that

$$\frac{pL_p(a,c)f(z)}{z(L_p(a,c)f(z))'} = (1 - p\beta)\omega(z) - 1 \quad (z \in \mathbb{U}). \tag{50}$$

Differentiating both sides of (50) logarithmically, we obtain

$$1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} = -\frac{(1-p\beta)z\omega'(z)}{(1-p\beta)\omega(z) - 1}, \tag{51}$$

by virtue of (33) and (51), we find that

$$\left| 1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right| = (1-p\beta) \left| \frac{z\omega'(z)}{(1-p\beta)\omega(z) - 1} \right| < \frac{1-p\beta}{2-p\beta}.$$

Next, we claim that $|\omega(z)| < 1$. Indeed, if not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \quad (z_0 \in \mathbb{U}). \tag{52}$$

Applying Jack's Lemma to $\omega(z)$ at the point z_0, we have

$$z_0\omega'(z_0) = \gamma\omega(z_0) \quad (\gamma \geq 1). \tag{53}$$

Now, upon setting

$$\omega(z_0) = e^{i\theta}(0 \leq \theta < 2\pi).$$

If we put $z = z_0$ in (51), we get

$$\left| 1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)} \right| = (1-p\beta)\left| \frac{z_0\omega'(z_0)}{(1-p\beta)\omega(z_0)-1} \right|$$
$$= (1-p\beta)\left| \frac{\gamma}{(1-p\beta)-e^{-i\theta}} \right|$$
$$\geq (1-p\beta)\left| \frac{1}{(1-p\beta)-e^{-i\theta}} \right|.$$

This implies that

$$\left| 1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)} \right|^2 \geq \frac{(1-p\beta)^2}{1+(1-p\beta)^2-2(1-p\beta)\cos\theta}. \tag{54}$$

Since the right hand side of (54) takes it minimum value for $\cos\theta = -1$, we have that

$$\left| 1 + \frac{z_0(L_p(a,c)f(z_0))''}{(L_p(a,c)f(z_0))'} - \frac{z_0(L_p(a,c)f(z_0))'}{L_p(a,c)f(z_0)} \right|^2 \geq \frac{(1-p\beta)^2}{(2-p\beta)^2} \tag{55}$$

This contradicts our condition (33) of Theorem 2.4. Therefore, we conclude that $|\omega(z)| < 1$, which shows that

$$\left| \frac{p}{1-p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) + 1 \right| < 1. \tag{56}$$

This implies that

$$\frac{p}{1-p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) \prec -\frac{1+z}{1-z}. \tag{57}$$

Therefore, we conclude that $f(z) \in M_{a,c}(p;\beta;A,B)$ for $B = -1, A = 1$. $\quad\square$

Putting $p = 1, a = c = 1, A = 1-2\alpha, \frac{1}{2} \leq \alpha < 1, B = -1$ and $\beta = 0$ in Theorem 2.4, we obtain the following Corollary.

Corollary 2.5. *[18] If $f \in \Sigma$ satisfies*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < 1-\alpha,$$

then $f \in M(\alpha)$, for $\frac{1}{2} \leq \alpha < 1$.

Theorem 2.6. *If $f \in \Sigma_p$ satisfies*

$$\Re\left(1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} \right) < \begin{cases} \dfrac{(1-A)+p\beta(A-B)}{2(1-p\beta)(A-B)} & (B + \dfrac{1-B}{2(1-p\beta)} \leq A \leq 1) \\[3mm] \dfrac{(1-p\beta)(A-B)}{2[(1-A)+p\beta(A-B)]} & (B < A \leq B + \dfrac{1-B}{2(1-p\beta)}), \end{cases} \tag{58}$$

then $f(z) \in M_{a,c}(p;\beta;A,B)$.

Proof. Suppose that

$$g(z) := \frac{\frac{-p}{1-p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} - \frac{1-A}{1-B}}{1 - \frac{1-A}{1-B}} \quad (-1 \le B < A \le 1; z \in \mathbb{U}).$$ (59)

Then g is analytic in $\mathbb{U}$. It follows from (59) that

$$\frac{-pL_p(a,c)f(z)}{z(L_p(a,c)f(z))'} = \frac{(1-p\beta)(A-B)g(z) + (1-A) + p\beta(A-B)}{1-B}.$$ (60)

Differentiating (60) logarithmically, we obtain

$$-1 - \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} + \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} = \frac{(1-p\beta)(A-B)zg'(z)}{(1-p\beta)(A-B)g(z) + (1-A) + p\beta(A-B)} := \Phi(g(z), zg'(z); z),$$

where

$$\Phi(r,s;t) = \frac{(1-p\beta)(A-B)s}{(1-p\beta)(A-B)r + (1-A) + p\beta(A-B)}.$$

For all real x and y satisfying $y \le -\dfrac{1+x^2}{2}$, we have

$$
\begin{aligned}
\Re(\Phi(ix, y; z)) &= \Re \frac{(1-p\beta)(A-B)y}{i(1-p\beta)(A-B)x + (1-A) + p\beta(A-B)} \\
&= \frac{(1-p\beta)(A-B)[(1-A) + p\beta(A-B)]y}{[(1-A) + p\beta(A-B)]^2 + [(1-p\beta)(A-B)]^2 x^2} \\
&\le -\frac{1+x^2}{2} \cdot \frac{(1-p\beta)(A-B)[(1-A) + p\beta(A-B)]}{[(1-A) + p\beta(A-B)]^2 + [(1-p\beta)(A-B)]^2 x^2} \\
&\le \begin{cases} -\dfrac{(1-A) + p\beta(A-B)}{2(1-p\beta)(A-B)} & (B + \dfrac{1-B}{2(1-p\beta)} \le A \le 1), \\[2ex] -\dfrac{(1-p\beta)(A-B)}{2[(1-A) + p\beta(A-B)]} & (B < A \le B + \dfrac{1-B}{2(1-p\beta)}). \end{cases}
\end{aligned}
$$

We now put

$$\Omega = \left\{\xi : \Re(\xi) > \begin{cases} -\dfrac{(1-A) + p\beta(A-B)}{2(1-p\beta)(A-B)} & (B + \dfrac{1-B}{2(1-p\beta)} \le A \le 1), \\[2ex] -\dfrac{(1-p\beta)(A-B)}{2[(1-A) + p\beta(A-B)]} & (B < A \le B + \dfrac{1-B}{2(1-p\beta)}) \end{cases}\right\},$$

then $\Phi(ix; y; z) \notin \Omega$ for all real x, y such that $y \le -\frac{1+x^2}{2}$. Moreover, in view of (58), we know that $\Phi(g(z), zg'(z); z) \in \Omega$. Thus, by Lemma 1.3, we deduce that

$$\Re(g(z)) > 0 \quad (z \in \mathbb{U}),$$

which shows that the desired assertion of Theorem 2.6 holds. □

Putting $p = 1, a = c = 1, A = 1 - 2\alpha, 0 \le \alpha < 1, B = -1$ and $\beta = 0$ in Theorem 2.6, we obtain the following Corollary.

Corollary 2.7. *[18] If $f \in \Sigma$ satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \begin{cases} \dfrac{\alpha}{2(1-\alpha)} & (0 \le \alpha \le \dfrac{1}{2}), \\[2mm] \dfrac{1-\alpha}{2\alpha} & (\dfrac{1}{2} \le \alpha < 1), \end{cases}$$

then $f \in M(\alpha)$, for $0 \le \alpha < 1$.

Theorem 2.8. *If $f \in \Sigma_p$ satisfies*

$$\Re\left\{\frac{pL_p(a,c)f(z)}{z(L_p(a,c)f(z))'}\left(1 + \eta\frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'}\right)\right\} > -\frac{1}{2}(1-p\beta)(\frac{A-B}{1-B})\eta + p\eta - (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)}, \quad (61)$$

then $f \in M_{a,c}(p; \beta; A, B)$ for $\eta \ge 0$.

Proof. We define the function $h(z)$ by

$$h(z) := \frac{\frac{-p}{1-p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} - \frac{1-A}{1-B}}{1 - \frac{1-A}{1-B}} \quad (-1 \le B < A \le 1; z \in \mathbb{U}). \quad (62)$$

Then h is analytic in $\mathbb{U}$. It follows from (62) that

$$\frac{-pL_p(a,c)f(z)}{z(L_p(a,c)f(z))'} = \frac{(1-p\beta)(A-B)h(z) + (1-A) + p\beta(A-B)}{1-B} \quad (63)$$

and

$$1 + \eta\frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} = \frac{P + Qh(z) + Rzh'(z)}{(1-p\beta)(A-B)h(z) + (1-A) + p\beta(A-B)}, \quad (64)$$

where

$$P = -p\eta(1-B) + (1-\eta)[(1-A) + p\beta(A-B)]; Q = (1-p\beta)(A-B)(1-\eta); R = -(1-p\beta)(A-B)\eta.$$

Combining (63) and (64), we get

$$\frac{-pL_p(a,c)f(z)}{z(L_p(a,c)f(z))'}\left(1 + \eta\frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'}\right)$$

$$= -p\eta + (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)} + (1-p\beta)(\frac{A-B}{1-B})(1-\eta)h(z) - (1-p\beta)(\frac{A-B}{1-B})\eta zh'(z)$$

$$:= \Phi(h(z), zh'(z); z),$$

where

$$\Phi(r,s;t) = -(1-p\beta)(\frac{A-B}{1-B})\eta s + (1-p\beta)(\frac{A-B}{1-B})(1-\eta)r - p\eta + (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)}.$$

For all real x and y satisfying $y \le -\dfrac{1+x^2}{2}$, we have

$$\begin{aligned} \Re(\Phi(ix,y;z)) &= \Re\{-(1-p\beta)(\frac{A-B}{1-B})\eta y + (1-p\beta)(\frac{A-B}{1-B})(1-\eta)ix - p\eta + (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)}\} \\[2mm] &= -(1-p\beta)(\frac{A-B}{1-B})\eta y - p\eta + (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)} \\[2mm] &\ge \frac{(1+x^2)}{2}(1-p\beta)(\frac{A-B}{1-B})\eta - p\eta + (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)} \\[2mm] &\ge \frac{1}{2}(1-p\beta)(\frac{A-B}{1-B})\eta - p\eta + (1-\eta)\frac{[(1-A)+p\beta(A-B)]}{(1-B)} \end{aligned}$$

If we set

$$\Omega = \left\{ \xi : \Re(\xi) < \frac{1}{2}(1 - p\beta)(\frac{A - B}{1 - B})\eta - p\eta + (1 - \eta)\frac{[(1 - A) + p\beta(A - B)]}{(1 - B)} \right\},$$

then $\Phi(ix, y; z) \notin \Omega$ for all real x, y such that $y \le -\frac{1 + x^2}{2}$. Furthermore, by virtue of (61), we know that $\Phi(h(z), zh'(z); z) \in \Omega$. Thus, by Lemma 1.3, we conclude that

$$\Re(h(z)) > 0 \quad (z \in \mathbb{U}),$$

which implies that the assertion of Theorem 2.8 holds true. $\square$

Taking $p = 1, a = c = 1, A = 1 - 2\alpha, 0 \le \alpha < 1, B = -1$, and $\beta = 0$ in Theorem 2.8, we revise the result of Theorem 2.4 in [18] and obtain the following Corollary.

Corollary 2.9. *If $f \in \Sigma$ satisfies*

$$\Re\left(\frac{f(z)}{zf'(z)}\left(1 + \eta\frac{zf''(z)}{f'(z)}\right)\right) > \frac{1}{2}\eta(1 + 3\alpha) - \alpha,$$

then $f \in M(\alpha)$, for $0 \le \alpha < 1, \eta \ge 0$.

Theorem 2.10. *If $f \in \Sigma_p$ satisfies anyone of the following conditions:*
(i) For $B \ne -1$,

$$\left|\left\{\frac{p(1 - B^2)}{(1 - p\beta)(A - B)}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} + \frac{1 - AB}{A - B}\right\}'\right| \le \eta|z|^\tau, \tag{65}$$

(ii) For $B = -1, A \ne 1$,

$$\left|\left(1 + \frac{(1 - A)(1 - p\beta)}{p}\left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'}\right)\right)'\right| \le \eta|z|^\tau, \tag{66}$$

(iii) For $B = -1, A = 1$,

$$\left|\left\{\frac{p}{1 - p\beta}\left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right) + 1\right\}'\right| \le \eta|z|^\tau, \tag{67}$$

then $f \in M_{a,c}(p; \beta; A, B)$, for $0 < \eta \le \tau + 1$ and $\tau \ge 0$.

Proof. (i) For $f \in \Sigma_p$, if $B \ne -1$, we define the function $\Psi(z)$ by

$$\Psi(z) = z\left\{\frac{p(1 - B^2)}{(1 - p\beta)(A - B)}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} + \frac{1 - AB}{A - B}\right\} \quad (z \in \mathbb{U}).$$

Then $\Psi(z)$ is regular in $\mathbb{U}$ and $\Psi(0) = 0$. The condition of Theorem gives us that

$$\left|\left\{\frac{p(1 - B^2)}{(1 - p\beta)(A - B)}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta\right\} + \frac{1 - AB}{A - B}\right\}'\right| = \left|\left(\frac{\Psi(z)}{z}\right)'\right| \le \eta|z|^\tau.$$

It follows that

$$\left|\left(\frac{\Psi(z)}{z}\right)\right| = \left|\int_0^z \left(\frac{\Psi(t)}{t}\right)' dt\right| \le \int_0^{|z|} \eta|t|^\tau d|t| = \frac{\eta}{\tau + 1}|z|^{\tau+1}.$$

This implies that

$$\left|\left(\frac{\Psi(z)}{z}\right)\right| \le \frac{\eta}{\tau + 1}|z|^{\tau+1} < 1 \quad (0 < \eta \le \tau + 1, \tau \ge 0).$$

Therefore, by the definition of $\Psi(z)$, we conclude that

$$\left| \frac{p(1-B^2)}{(1-p\beta)(A-B)} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} + \frac{1-AB}{A-B} \right| < 1.$$

which is equivalent to

$$\left| \frac{p}{1-p\beta} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} + \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}.$$

Therefore, we conclude that $f(z) \in M_{a,c}(p;\beta;A,B)$ for $B \neq -1$.

(ii) For $f \in \Sigma_p$, if $B = -1, A \neq 1$, we define the function $\Psi(z)$ by

$$\Psi(z) = z \left(1 + \frac{(1-A)(1-p\beta)}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) \right) \quad (z \in \mathbb{U}).$$

Then $\Psi(z)$ is regular in $\mathbb{U}$ and $\Psi(0) = 0$. The condition of Theorem gives us that

$$\left| \left(1 + \frac{(1-A)(1-p\beta)}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) \right)' \right| = \left| \left(\frac{\Psi(z)}{z} \right)' \right| \leq \eta |z|^\tau.$$

It follows that

$$\left| \left(\frac{\Psi(z)}{z} \right) \right| = \left| \int_0^z \left(\frac{\Psi(t)}{t} \right)' dt \right| \leq \int_0^{|z|} \eta |t|^\tau d|t| = \frac{\eta}{\tau+1} |z|^{\tau+1}.$$

This implies that

$$\left| \left(\frac{\Psi(z)}{z} \right) \right| \leq \frac{\eta}{\tau+1} |z|^{\tau+1} < 1 \quad (0 < \eta \leq \tau + 1, \tau \geq 0).$$

Therefore, by the definition of $\Psi(z)$, we conclude that

$$\left| \frac{(1-A)(1-p\beta)}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) + 1 \right| < 1,$$

which is equivalent to

$$\left| \frac{(1-p\beta)}{p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) + \frac{1}{1-A} \right| < \frac{1}{1-A} \quad (A \neq 1; z \in \mathbb{U}).$$

Therefore, we conclude that $f(z) \in M_{a,c}(p;\beta;A,B)$ for $B = -1, A \neq 1$.

(iii) For $f \in \Sigma_p$, if $B = -1, A = 1$, we define the function $\Psi(z)$ by

$$\Psi(z) = z \left(\frac{p}{1-p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) + 1 \right) \quad (z \in \mathbb{U}).$$

Then $\Psi(z)$ is regular in $\mathbb{U}$ and $\Psi(0) = 0$. The condition (67) of Theorem gives us that

$$\left| \left(\frac{p}{1-p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) + 1 \right)' \right| = \left| \left(\frac{\Psi(z)}{z} \right)' \right| \leq \eta |z|^\tau.$$

It follows that

$$\left| \left(\frac{\Psi(z)}{z} \right) \right| = \left| \int_0^z \left(\frac{\Psi(t)}{t} \right)' dt \right| \leq \int_0^{|z|} \eta |t|^\tau d|t| = \frac{\eta}{\tau+1} |z|^{\tau+1}.$$

This implies that

$$\left| \left(\frac{\Psi(z)}{z} \right) \right| \leq \frac{\eta}{\tau+1} |z|^{\tau+1} < 1 \quad (0 < \eta \leq \tau + 1, \tau \geq 0).$$

Therefore, by the definition of $\Psi(z)$, we conclude that

$$\left| \frac{p}{1-p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) + 1 \right| < 1,$$

which is equivalent to

$$\frac{p}{1-p\beta} \left(\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right) \prec -\frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

Therefore, we conclude that $f(z) \in M_{a,c}(p;\beta;A,B)$ for $B = -1, A = 1$. $\quad\square$

Taking $p = 1, a = c = 1, A = 1 - 2\alpha, 0 < \alpha < 1, B = -1$ and $\beta = 0$ in Theorem 2.10, we obtain the following Corollary.

Corollary 2.11. *[18] If $f \in \Sigma$ satisfies*

$$\left| \left(1 + \frac{2\alpha z f'(z)}{f(z)} \right)' \right| \le \eta |z|^\tau,$$

then $f \in M(\alpha)$, for $0 < \alpha < 1, 0 < \eta \le \tau + 1$ and $\tau \ge 0$.

Theorem 2.12. *If $f \in \Sigma_p$ satisfies*

$$\left| \frac{\frac{1-p\beta}{p} z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} (1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))')'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'}) \right| < \frac{A-B}{1-A}, \quad (68)$$

then $f \in M_{a,c}(p;\beta;A,B)$, for $-1 \le B < A < \dfrac{1+B}{2}$.

Proof. Let

$$q(z) := \frac{-p}{1-p\beta} \left\{ \frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'} + \beta \right\} \quad (z \in \mathbb{U}). \tag{69}$$

Then the function $q(z)$ is analytic in $\mathbb{U}$. It follows from (69) that

$$z\left(\frac{1}{q(z)}\right)' = z\left\{ \frac{1-p\beta}{-p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) \right\}'$$

$$= \frac{1-p\beta}{-p} \left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right) \left(1 + \frac{z(L_p(a,c)f(z))''}{(L_p(a,c)f(z))'} - \frac{z(L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))')'}{L_p(a,c)f(z) + \beta z(L_p(a,c)f(z))'} \right)$$
$$\tag{70}$$

Combining (68) and (70), we find that

$$\left| z\left(\frac{1}{q(z)}\right)' \right| < \frac{A-B}{1-A} \quad (z \in \mathbb{U}),$$

that is

$$z\left(\frac{1}{q(z)}\right)' \prec \frac{A-B}{1-A} z \quad (z \in \mathbb{U}). \tag{71}$$

An application of Lemma 1.4 to (71) yields

$$q(z) \prec \frac{1-A}{1-A+(A-B)z} := F(z). \tag{72}$$

By noting that

$$\Re\left(1+\frac{zF''(z)}{F'(z)}\right) = \Re\left(\frac{1-A-(A-B)z}{1-A+(A-B)z}\right)$$

$$\geq \frac{1-A-(A-B)}{1-A+(A-B)}$$

$$= \frac{1-2A+B}{1-B}$$

$$> 0 \left(-1 \leq B < A < \frac{1+B}{2}; z \in \mathbb{U}\right),$$

which implies that the region $F(\mathbb{U})$ is symmetric with respect to the real axis and F is convex univalent in $\mathbb{U}$. Therefore, we have

$$\Re(F(z)) > F(1) = \frac{1-A}{1-B} \quad (z \in \mathbb{U}). \tag{73}$$

Combining (69), (72) and (73), we deduce that

$$\Re\left\{\frac{p}{1-p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'}+\beta\right\}\right\} < -\frac{1-A}{1-B} \quad \left(-1 \leq B < A < \frac{1+B}{2}; z \in \mathbb{U}\right),$$

which is equivalent to

$$\frac{p}{1-p\beta}\left\{\frac{L_p(a,c)f(z)}{z(L_p(a,c)f(z))'}+\beta\right\} < -\frac{1+Az}{1+Bz} \quad \left(-1 \leq B < A < \frac{1+B}{2}; z \in \mathbb{U}\right).$$

This evidently completes the proof of Theorem 2.12. □

Putting $p = 1, a = c = 1, A = 1-2\alpha, \frac{1}{2} < \alpha < 1, B = -1$ and $\beta = 0$ in Theorem 2.12, we obtain the following Corollary.

Corollary 2.13. *[18] If $f \in \Sigma$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{1}{\alpha}-1,$$

then $f \in M(\alpha)$, for $\frac{1}{2} < \alpha < 1$.

Acknowledgements

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