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Zero-Term Rank Inequalities and their Extreme Preservers

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Abstract. The zero-term rank of a matrix A over a semiring $\mathcal{S}$ is the least number of lines (rows or columns) needed to include all the zero entries in A. In this paper, we characterize linear operators that preserve the sets of matrix ordered pairs which satisfy extremal properties with respect to zero-term rank inequalities of matrices over nonbinary Boolean algebras.

1. Introduction

Linear preserver problems concern the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant([9]). In 1897, Frobenius characterized the linear operators that preserve determinant of matrices over real field, which was the first results on linear preserver problems. After his result, many researchers have studied the linear operators that preserve some matrix functions, say, rank and permanent of matrices and so on([9]).

Beasley and Guterman([1]) investigated rank inequalities of matrices over semirings. And they characterized the equality cases for some inequalities in [2]. These characterization problems are open even over fields(see [3]). The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a pair of matrices which belongs to it and to act on this pair by various linear operators that preserve this variety. The complete classification of linear operators that preserve equality cases in matrix rank inequalities over fields was obtained in [4]. For details on linear operators preserving matrix invariants one can see [9]. Almost all research on linear preserver problems over semirings have dealt with those semirings without zero-divisors to avoid the difficulties of multiplication arithmetic for the elements in those semirings([2]-[7]). But nonbinary Boolean algebra is not the case. That is, all elements except 0 and 1 in most nonbinary Boolean algebras are zero-divisors. So there are few results on the linear preserver problems for the matrices over nonbinary Boolean algebra([8], [10]). Kirkland and Pullman characterized the linear operators that preserve rank of matrices over nonbinary Boolean algebra in [8].

In this paper, we characterize the linear operators that preserve the sets of matrix pairs over nonbinary Boolean algebra which satisfy the extreme cases for certain zero-term rank inequalities. For this purpose, we also study the inequalities of zero-term rank for the sum or multiplication of matrices over nonbinary Boolean algebra. We also construct the sets of matrix pairs that satisfy the equalities for those zero-term rank inequalities.

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2. Preliminaries and Basic Results

A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ with two binary operations, addition and multiplication, such that:

· $\mathcal{S}$ is an Abelian monoid under addition (the identity is denoted by 0);

· $\mathcal{S}$ is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);

· multiplication is distributive over addition on both sides;

· $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

A semiring $\mathcal{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

A semiring $\mathcal{S}$ is called a *Boolean algebra* if $\mathcal{S}$ is equivalent to a set of subsets of a given set $\mathcal{Y}$, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set $\mathcal{Y}$.

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k and $\mathbb{B}_k$ be a Boolean algebra of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, which is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a Boolean algebra $\mathbb{B}_k$ is a commutative and antinegative semiring. If $\mathbb{B}_k$ consists of only the empty subset and S_k then it is called a *binary Boolean algebra*. If $\mathbb{B}_k$ is not binary Boolean algebra then it is called a *nonbinary Boolean algebra*. Let $\mathbb{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from the Boolean algebra $\mathbb{B}_k$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_k)$ instead of $\mathbb{M}_{n,n}(\mathbb{B}_k)$.

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes a nonbinary Boolean algebra, which contains at least 3 elements. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. The matrix $E_{i,j}$, which is called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the $(i, j)^{th}$ entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere.

Let $\mathbb{B}_k$ be a nonbinary Boolean algebra. An operator $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is called *linear* if it satisfies $T(X + Y) = T(X) + T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\alpha \in \mathbb{B}_k$.

A *line* of a matrix A is a row or a column of the matrix A.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *zero-term rank* k ($z(A) = k$) if the least number of lines needed to include all zero elements of A is equal to k.

The matrix $D \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *term rank* h ($t(D) = h$) if the least number of lines needed to include all nonzero elements of D is equal to h.

Let us denote by $c(D)$ the least number of columns needed to include all nonzero elements of A and by $r(D)$ the least number of rows needed to include all nonzero elements of A.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank* r if there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We denote $b(A) = r$.

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

Arithmetic properties of zero-term rank of Boolean matrices are restricted by the following list of inequalities established in [1]:

1. $z(A + B) \geq 0$;
2. $z(A + B) \leq \min\{z(A), z(B)\}$;
3. $z(AB) \geq 0$;
4. $z(AB) \leq z(A) + z(B)$.

Below, we use the following notations in order to denote sets of Boolean matrices that arise as extremal cases in the inequalities listed above:

$$\mathbf{Z}_{sn}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | z(X + Y) = \min\{z(X), z(Y)\}\};$$

$$\mathbf{Z}_{sz}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 | z(X + Y) = 0\};$$

$$\mathbf{Z}_{mz}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{n}(\mathbb{B}_k)^2 | z(XY) = 0\};$$

$$\mathbf{Z}_{ms}(\mathbb{B}_k) = \{(X, Y) \in \mathbb{M}_{n}(\mathbb{B}_k)^2 | z(XY) = z(X) + z(Y)\}.$$

We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs, provided that $(X, Y) \in \mathcal{P}$ implies $(T(X), T(Y)) \in \mathcal{P}$.

An operator T *strongly preserves* the set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs, provided that $(X, Y) \in \mathcal{P}$ if and only if $(T(X), T(Y)) \in \mathcal{P}$.

The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i, j) entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

An operator T is called a *(P, Q, B)-operator* if there exist permutation matrices P and Q, and a matrix B with no zero entries, such that

$$T(X) = P(X \circ B)Q \tag{2.1}$$

for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$, or, if $m = n$,

$$T(X) = P(X \circ B)^t Q \tag{2.2}$$

for all $X \in \mathbb{M}_{m,n}(\mathcal{S})$, where X^t denotes the transpose of X. Operator of the form (2.1) is called *non-transposing (P, Q, B)-operator*; operators of the form (2.2) is called *transposing (P, Q, B)-operator*. A (P, Q, B)-operator is called a *(P, Q)-operator* if $B = J$, the matrix of all ones.

It was shown in [4] that linear preservers for extremal cases of classical matrix rank inequalities over fields were characterized. On the other hand, linear preservers for various rank functions over semirings have been the object of much study during the last 30 years, see for example [2]-[9]. In particular zero-term rank was investigated in the last years, see for example [5–7]. The aim of the present paper is to classify linear operators that preserve pairs of matrices that attain extreme cases in the above zero-term rank inequalities $1 \sim 4$.

We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

We begin with some basic results.

Theorem 2.1. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then the following conditions are equivalent:*

(a) T is bijective;

(b) T is surjective;

(c) T is injective;

(d) there exists a permutation σ on $\{(i, j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proof. (a), (b) and (c) are equivalent since $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is a finite set.

(d)⇒(b) For any $D \in \mathbb{M}_n(\mathbb{B}_k)$, we may write

$$D = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{i,j} E_{i,j}.$$

Since σ is a permutation, there exist $\sigma^{-1}(i,j)$ and

$$D' = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)} E_{\sigma^{-1}(i,j)}$$

such that

$$T(D') = T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)} E_{\sigma^{-1}(i,j)}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma\sigma^{-1}(i,j)} E_{\sigma\sigma^{-1}(i,j)}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j} E_{i,j} = D.$$

(a)⇒(d) We assume that T is bijective. Suppose that $T(E_{i,j}) \neq E_{\sigma(i,j)}$ where σ be a permutation on $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$. Then there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}$ $(\alpha \neq 1)$ or some pairs $(i,j),(r,s)$ and (u,v) $((r,s) \neq (u,v))$ such that $T(E_{i,j}) = \alpha E_{r,s}+\beta E_{u,v}+Z$ $(\alpha \neq 0,\ \beta \neq 0,\ Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros.

Case 1) Suppose that there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}$ $(\alpha \neq 1)$. Since T is bijective, there exist $X_{r,s} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = E_{r,s}$. Then $\alpha T(X_{r,s}) = \alpha E_{r,s} = T(E_{i,j})$, and $T(\alpha X_{r,s}) = T(E_{i,j})$. Hence $\alpha X_{r,s} = E_{i,j}$, which contradicts the fact that $\alpha \neq 1$.

Case 2) Suppose that there exist some pairs $(i,j),(r,s)$ and (u,v) such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z$ $(\alpha \neq 0,\ \beta \neq 0,\ Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros. Since T is bijective, there exist $X_{r,s}$, $X_{u,v}$ and $Z' \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = \alpha E_{r,s}$, $T(X_{u,v}) = \beta E_{u,v}$, and $T(Z') = Z$. Thus $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z = T(X_{r,s}) + T(X_{u,v}) + T(Z') = T(X_{r,s} + X_{u,v} + Z')$. So $E_{i,j} = X_{r,s} + X_{u,v} + Z'$, a contradiction. □

One can easily verify that if $m = 1$ or $n = 1$, then all operators under consideration are (P,Q,B)-operators and if $m = n = 1$, then all operators under consideration are (P,P^T,B)-operators.

Henceforth we will always assume that $m,n \geq 2$.

Lemma 2.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator which maps lines to lines and let T be defined by the rule $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$. Then T is a (P,Q)-operator.*

Proof. Since no combination of p rows and q columns can dominate J for any nonzero p and q with $p+q = m$, we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and image of each column is a row. Thus there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$, $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^TQ$, $T(C_j) \leq P(C_j)^TQ$. Since each nonzero entry of a cell lies in the intersection of a row and a column and T maps cells to cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m = n$, $T(E_{i,j}) = P(E_{i,j})^TQ$. □

Example 2.3. *Consider the linear operator $T : \mathbb{M}_{3,3}(\mathbb{B}_3) \to \mathbb{M}_{3,3}(\mathbb{B}_3)$ defined by $T(X) = X \circ B$ for all $X \in \mathbb{M}_{3,3}(\mathbb{B}_3)$ with $\mathbb{B}_3 = \mathcal{P}(\{a,b,c\})$. For some B, such that $z(B) = 0$ and $b(B) = 1$, we show that T does not preserves the zero-term rank if $B \neq J$.*

For, let $X = \begin{bmatrix} \{a,b\} & \{a,b,c\} & \{a,b\} \\ \{a,c\} & \{a,c\} & \{a,b\} \\ \{a\} & \{b,c\} & \{a,b\} \end{bmatrix}$ *and* $B = \begin{bmatrix} \{a\} & \{b\} & \{c\} \\ \{a\} & \{b\} & \{c\} \\ \{a\} & \{b\} & \{c\} \end{bmatrix}$. *Then $t(X) = 3$, but*

$$T(X) = X \circ B = \begin{bmatrix} \{a\} & \{b\} & 0 \\ \{a\} & 0 & 0 \\ \{a\} & \{b\} & 0 \end{bmatrix}.$$

That is, $z(T(X)) = z(X \circ B) = 2 \neq 0 = z(X)$. Thus $z(X) = 0$, but T does not preserves the zero-term rank since there are nonzero entries of B which are zero-divisors. □

3. Extremes Preservers of Zero-Term Rank over Nonbinary Boolean Algebras

In this section, we characterize the linear operators that preserve the extreme set of matrix pairs, which are derived from the inequalities of the zero-term ranks of matrices over nonbinary Boolean algebra.

We begin with a Lemma.

Lemma 3.1. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a (P, Q)-operator. Then T preserves all zero-term ranks.*

Proof. Assume that T is a (P, Q)-operator. For any $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, we have

$$z(T(X)) = z(PXQ) = z(X)$$

or if m=n,

$$z(T(X)) = z(PX^tQ) = z(X^t) = z(X).$$

Hence any (P, Q)-operator preserves all zero-term ranks. □

Theorem 3.2. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear map. Then T preserves the set $\mathbf{Z}_{sn}(\mathbb{B}_k)$ if and only if T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.*

Proof. Since T is a linear surjective map, by Theorem 2.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}, E_{i,k}$ are the cells such that $T(E_{i,j}), T(E_{i,k})$ are not in the same line. Then one has that $z((J - E_{i,j} - E_{i,k}) + E_{i,k}) = 1 = z(J - E_{i,j} - E_{i,k})$, i.e. $(J - E_{i,j} - E_{i,k}, E_{i,k}) \in \mathbf{Z}_{sn}(\mathbb{B}_k)$, as far as $z(T(J - E_{i,j} - E_{i,k}) + T(E_{i,k})) = 1 < 2 = min\{z(T(J - E_{i,j} - E_{i,k})), z(T(E_{i,k})\}$, i.e. $(T(J - E_{i,j} - E_{i,k}), T(E_{i,k})) \notin \mathbf{Z}_{sn}(\mathbb{B}_k)$, a contradiction. Thus T maps lines to lines.

By Lemma 2.2 it follows that T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.

Conversely, assume that T is a (P, Q)-operator. Then T preserves all zero-term ranks by Lemma 3.1. Therefore for any $(X, Y) \in \mathbf{Z}_{sn}(\mathbb{B}_k)$, we have $z(X + Y) = \min\{z(X), z(Y)\}$. Thus $z(T(X) + T(Y)) = z(T(X + Y)) = z(X + Y) = \min\{z(X), z(Y)\} = \min\{z(T(X), z(T(Y)\}$. Hence (P, Q)-operator preserves the set $\mathcal{Z}_{sn}(\mathbb{B}_k)$. □

Theorem 3.3. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear map. Then T preserves the set $\mathbf{Z}_{sz}(\mathbb{B}_k)$ if and only if T is a permutation on the set of all cells.*

Proof. Assume that T is a permutation on the set of all cells. That is, $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$, where σ is a permutation on the set of pairs (i, j).

Consider $(A, B) \in \mathbf{Z}_{sz}(\mathbb{B}_k)$. Then $z(A + B) = 0$. From antinegativity it follows that sets of zero cells in A and B are disjoint. Thus the same holds for $T(A)$ and $T(B)$ since σ is a permutation. Hence in $(T(A) + T(B))$ there is no zero entry, and hence, $(T(A) + T(B)) \in \mathbf{Z}_{sz}(\mathbb{B}_k)$. Thus, such a linear operator T preserves the set $\mathbf{Z}_{sz}(\mathbb{B}_k)$.

Conversely, assume that T preserves the set $\mathbf{Z}_{sz}(\mathbb{B}_k)$. If T is not a permutation on the set of all cells, then there are two distinct cells $E_{i,j}, E_{h,k}$ such that $T(E_{i,j}) = T(E_{h,k}) = E_{p,q}$. Then $z(J) = 0$ but $z(T(J)) \geq 1$, and hence $(J, 0) \in \mathbf{Z}_{sz}(\mathbb{B}_k)$ but $(T(J), T(0)) \notin \mathbf{Z}_{sz}(\mathbb{B}_k)$, a contradiction. □

Theorem 3.4. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear surjective map. Then T preserves the set $\mathbf{Z}_{mz}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator, where P is a permutation matrix.*

Proof. By Lemma 3.1, nontransposing (P, P^t)-operators preserve all the zero-term ranks. Let $(X, Y) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$. Then $z(XY) = 0$ and hence XY has no zero entries. Since T is a nontransposing (P, P^t)-operator, one has $T(X)T(Y) = PXP^tPYP^t = PXYP^t$, which has no zero entries. Thus $(T(X), T(Y)) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$. Hence T preserves the set $\mathcal{Z}_{mz}(\mathbb{B}_k)$.

Conversely, assume that T preserves the set $\mathbf{Z}_{mz}(\mathbb{B}_k)$. Since T is a linear surjective map, by Theorem 2.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, where σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not, say $E_{i,j}, E_{i,k}$ are the cells such that $T^{-1}(E_{i,j}), T^{-1}(E_{i,k})$ are not in the same line. Let us consider $A = T^{-1}(J \setminus R_i)$. Thus there are no zero rows of A since T is a permutation on the set of cells and not all elements of i'th row lie in one row by the choice of i. Hence AJ does not have zero elements by the antinegativity and $z(AJ) = 0$. Thus $(A, J) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$ as far as $(T(A), T(J)) = (J \setminus R_i, T(J)) \notin \mathbf{Z}_{mz}(\mathbb{B}_k)$, a contradiction. Thus T^{-1} maps lines to lines. Hence T maps lines to lines.

By Lemma 2.2 it follows that T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.

In order to prove that transposition operator does not preserve $\mathbf{Z}_{mz}(\mathbb{B}_k)$ it suffices to take the pair (C_1, R_1). That is, $(C_1, R_1) \in \mathbf{Z}_{mz}(\mathbb{B}_k)$ but $(C_1^t, R_1^t) = (R_1, C_1) \notin \mathbf{Z}_{mz}(\mathbb{B}_k)$.

Now, let us show that $Q = P^t$. Assume in the contrary that $QP \ne I$. Thus there exists indexes i, j such that QP transforms i'th column into j'th column. In this case we take matrices $A = J \setminus (E_{1,1} + \ldots + E_{1,n}) + E_{1,i}$, $B = J \setminus E_{j,n}$. Thus AB has no zero elements, i.e., $z(AB) = 0$. However, the $(1, 1)$'th element of $QT(A)T(B)P$ is zero, i.e., $z(T(A)T(B)) \ne 0$. This contradiction concludes that $Q = P^t$. Thus T is a nontransposing (P, P^t)-operator. $\square$

Theorem 3.5. *Let $\mathbb{B}_k$ be a nonbinary Boolean algebra, and $T : \mathbb{M}_n(\mathbb{B}_k) \to \mathbb{M}_n(\mathbb{B}_k)$ be a linear surjective map. Then T preserves the set $\mathbf{Z}_{ms}(\mathbb{B}_k)$ if and only if T is a nontransposing (P, P^t)-operator, where P is a permutation matrix of order n.*

Proof. By Lemma 3.1, nontransposing (P, P^t)-operators preserve all the zero-term ranks. Let $(X, Y) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. Then $z(XY) = z(X+Y)$. Since T is a nontransposing (P, P^t)-operator, one has $T(X)T(Y) = PXP^tPYP^t = PXYP^t$, which has the same zero-term rank as $z(XY)$. And $z(T(X)+T(Y)) = z(T(X+Y)) = z(X+Y)$. Thus $(T(X), T(Y)) \in \mathbf{Z}_{ms}(\mathbb{B}_k)$. Hence T preserves the set $\mathcal{Z}_{ms}(\mathbb{B}_k)$.

Conversely, assume that T preserves the set $\mathbf{Z}_{ms}(\mathbb{B}_k)$. Since T is a linear surjective map, by Theorem 2.1 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, where σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are, say $E_{i,j}, E_{i,k}$ are the cells such that $T(E_{i,j}), T(E_{i,k})$ are not in the same line. Note that

$$z((J \setminus R_i)J) = z(J \setminus R_i) = 1 = 1 + 0 = z(J \setminus R_i) + z(J).$$

Thus $(J \setminus R_i, J) \in \mathcal{Z}_{ms}(\mathbb{B}_k)$. On the other hand, $T(J) = J$ and $T(J \setminus R_i)$ has at least two lines containing zero entries, so one has $z(T(J \setminus R_i)) + z(T(J)) \ge 2$. But $T(J \setminus R_i)$ has no rows containing only zero entries and $T(J) = J$, so one has $z(T(J \setminus R_i)T(J)) = z(J) = 0$. Hence $(T(J \setminus R_i), T(J)) \notin \mathcal{Z}_{ms}(\mathbb{B}_k)$. This contradiction shows that T maps lines to lines.

By Lemma 2.2 it follows that T is a (P, Q)-operator where P and Q are permutation matrices of appropriate sizes.

In order to prove that transposition operator does not preserve $\mathbf{Z}_{ms}(\mathbb{B}_k)$ it suffices to take the pair of matrices $X = J \setminus R_1$, $Y = J \setminus C_1$ since $(X, Y) \in \mathcal{Z}_{ms}(\mathbb{B}_k)$ but $(X^t, Y^t) \notin \mathcal{Z}_{ms}(\mathbb{B}_k)$.

Now, let us show that $Q = P^t$. Assume in the contrary that $QP \ne I$. Thus there exists indexes i, j such that QP transforms i'th column into j'th column. In this case we take matrices $A = J \setminus C_i$, $B = R_i$. Thus $AB = 0$ and hence $z(AB) = n$. And $z(A) + Z(B) = n$. Therefore $(A, B) \in \mathcal{Z}_{ms}(\mathbb{B}_k)$. However, $T(A)T(B) = PAQPBQ = P(J \setminus C_j)R_iQ = PJQ = J$ has zero-term rank 0 while $z(T(A)) + z(T(B)) = z(PAQ) + z(PBQ) = z(A) + z(B) = n$. Therefore $(T(A), T(B)) \notin \mathcal{Z}_{ms}(\mathbb{B}_k)$. This contradiction concludes that $Q = P^t$. Thus T is a nontransposing (P, P^t)-operator. $\square$

As a concluding remark, we have characterized the linear operators that preserve the extreme sets of matrix ordered pairs over nonbinary Boolean algebra which come from certain zero-term rank inequalities over nonbinary Boolean algebra.

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