



Radial Growth and Hardy-Littlewood-Type Theorems on Hyperbolic Harmonic Functions

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In celebration of Matti Vuorinen's 65-th birthday

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Abstract. In this paper, we first show that a result of Girela et al. on analytic functions can be extended to hyperbolic-harmonic functions, and then we establish Hardy-Littlewood-type theorems on hyperbolic harmonic functions.

1. Introduction and main results

For $n \geq 2$, let \mathbb{R}^n denote the usual real vector space of dimension n . Sometimes it is convenient to identify each point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with an $n \times 1$ column matrix so that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For $a = (a_1, \dots, a_n)$ and $x \in \mathbb{R}^n$, we define the Euclidean inner product $\langle \cdot, \cdot \rangle$ by

$$\langle x, a \rangle = x_1 a_1 + \dots + x_n a_n$$

so that the Euclidean length of x is defined by

$$|x| = \langle x, x \rangle^{1/2} = (|x_1|^2 + \dots + |x_n|^2)^{1/2}.$$

Denote a ball in \mathbb{R}^n with center x_0 and radius r by

$$\mathbb{B}^n(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}.$$

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In particular, \mathbb{B}^n denotes the unit ball $\mathbb{B}^n(0, 1)$. Set $\mathbb{D} = \mathbb{B}^2$, the open unit disk in the complex plane \mathbb{C} .

Let Ω be a proper subdomain of \mathbb{R}^n . A function $f \in C^2(\Omega)$ is called *hyperbolic harmonic* (briefly, *h-harmonic*, in the following) function in Ω if it satisfies the *hyperbolic Laplace’s equation*

$$\Delta_h u = (1 - |x|^2)^2 \Delta u + 2(n - 2)(1 - |x|^2) \langle \nabla u, x \rangle = 0,$$

where Δ denotes the ordinary *Laplacian operator* and ∇ denotes the gradient. Recall that hyperbolic harmonic functions are solutions of the *Laplace-Beltrami equation* with respect to the *Poincaré metric*

$$ds^2 = (1 - |x|^2)^2 \sum_{k=1}^n dx_k^2$$

in the unit ball \mathbb{B}^n .

Obviously, when $n = 2$, all h-harmonic functions are harmonic functions. We refer to [2, 3, 6, 14, 18, 28, 29] for more details of h-harmonic functions.

It turns out that if $\psi \in C(\partial\mathbb{B}^n)$, then the Dirichlet problem

$$\begin{cases} \Delta_h f = 0 & \text{in } \mathbb{B}^n \\ f = \psi & \text{on } \partial\mathbb{B}^n \end{cases}$$

has an unique solution in $C(\overline{\mathbb{B}^n})$ and can be represented by

$$f(x) = P_h[\psi](x) = \int_{\partial\mathbb{B}^n} P_h(x, \zeta) \psi(\zeta) d\sigma(\zeta), \tag{1}$$

where $d\sigma$ is the unique normalized surface measure on $\partial\mathbb{B}^n$ and $P_h(x, \zeta)$ is the *hyperbolic Poisson kernel* defined by

$$P_h(x, \zeta) = \left(\frac{1 - |x|^2}{|x - \zeta|^2} \right)^{n-1} \quad (x \in \mathbb{B}^n, \zeta \in \partial\mathbb{B}^n).$$

Throughout this paper, we use C to denote the various positive constants, whose value may change from one occurrence to the next.

A continuous increasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset Ω of \mathbb{R}^n , a function $f : \Omega \rightarrow \mathbb{R}^m$ ($m \geq 1$) is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that

$$|f(x) - f(y)| \leq C\omega(|x - y|) \text{ for all } x, y \in \Omega. \tag{2}$$

For $\delta_0 > 0$, let

$$\int_0^\delta \frac{\omega(t)}{t} dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0 \tag{3}$$

and

$$\delta \int_\delta^\infty \frac{\omega(t)}{t^2} dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0, \tag{4}$$

where ω is a majorant. A majorant ω is said to be *regular* if it satisfies the conditions (3) and (4) (see [12, 13, 26]).

Let Ω be a proper subdomain of \mathbb{R}^n . We use $d_\Omega(x)$ to denote the Euclidean distance from x to the boundary $\partial\Omega$ of Ω . In particular, we always use $d(x)$ to denote the Euclidean distance from x to the boundary of \mathbb{B}^n .

A proper subdomain G of \mathbb{R}^n is said to be Λ_ω -extension if $\Lambda_\omega(G) = \text{loc}\Lambda_\omega(G)$, where $\text{loc}\Lambda_\omega(G)$ denotes the set of all functions $f : G \rightarrow \mathbb{R}^m$ ($m \geq 1$) satisfying (2) with a fixed positive constant C , whenever $x \in G$ and $y \in G$ such that $|x - y| < \frac{1}{2}d_G(x)$. Obviously, \mathbb{B}^n is a Λ_ω -extension domain.

In [22], the author proved that G is a Λ_ω -extension domain if and only if each pair of points $x, y \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d_G(\tau))}{d_G(\tau)} ds(\tau) \leq C\omega(|x - y|) \tag{5}$$

with some fixed positive constant $C = C(G, \omega)$, where ds stands for the arc length measure on γ . Furthermore, the author also proved that Λ_ω -extension domains exist only for majorants ω satisfying (3). See [13, 15, 22] for more details on Λ_ω -extension domains.

For $p \in (0, \infty]$, the Hardy class $H^p(\mathbb{B}^n)$ consists of those functions $f : \mathbb{B}^n \rightarrow \mathbb{R}$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f), & \text{if } p \in (0, \infty), \\ \sup_{z \in \mathbb{B}^n} |f(z)|, & \text{if } p = \infty \end{cases} \quad \text{and } M_p(r, f) = \left(\int_{\partial \mathbb{B}^n} |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}.$$

A classical result of Hardy and Littlewood asserts that if $p \in (0, \infty]$, $\alpha \in (1, \infty)$ and f is an analytic function in \mathbb{D} , then

$$M_p(r, f') = O\left(\left(\frac{1}{1-r}\right)^\alpha\right) \text{ as } r \rightarrow 1$$

if and only if

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\alpha-1}\right) \text{ as } r \rightarrow 1,$$

Indeed the above result of Hardy and Littlewood provides a close relationship between the integral means of analytic functions and those of their derivatives [11, 19, 20]. In [16, Theorem 1(a)], Girela and Peláez refined the above result for the case $\alpha = 1$ as follows.

Theorem 1.1. ([16, Theorem 1(a)]) *Let $p \in (2, \infty)$. For $r \in (0, 1)$, if f is an analytic function in \mathbb{D} such that*

$$M_p(r, f') = O\left(\left(\frac{1}{1-r}\right)\right) \text{ as } r \rightarrow 1,$$

then for all $\beta > 1/2$,

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^\beta\right) \text{ as } r \rightarrow 1. \tag{6}$$

In [16, P₄₆₄, Equation (26)], Girela and Peláez asked whether β in (6) can be substituted by $1/2$. This problem was affirmatively settled by Girela, Pavlovic and Peláez in [17] (see [17, Theorem 1.1]). We show that Theorem 1.6 can be extended to h -harmonic functions in \mathbb{B}^n with $\beta = 1/2$. On the related topics, see [5, 7, 8, 10, 27].

Theorem 1.2. *Let $p \in [2, \infty)$ and ω be a majorant. For $r \in (0, 1)$, if f is h -harmonic from \mathbb{B}^n into \mathbb{R} such that*

$$M_p(r, \nabla f) \leq C\omega\left(\frac{1}{1-r}\right),$$

then

$$M_p(r, f) \leq \left[|f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2\omega(1)}{n-1} \int_0^1 \omega\left(\frac{1}{1-rt}\right) dt \right]^{\frac{1}{2}}.$$

Especially, if $n = 2$ and $\omega(t) = t$, then

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \text{ as } r \rightarrow 1 \tag{7}$$

and the estimate of (7) is sharp.

Krantz [21] proved a Hardy-Littlewood-type theorem for harmonic functions in \mathbb{B}^n with respect to the majorant $\omega(t) = \omega_\alpha(t) = t^\alpha$ ($0 < \alpha \leq 1$) as follows. For the extended discussion on this topic, see [4, 9].

Theorem 1.3. ([21, Theorem 15.8]) *Let u be a harmonic function in \mathbb{B}^n and $0 < \alpha \leq 1$. Then u satisfies*

$$|\nabla u(x)| \leq C \frac{\omega_\alpha(d(x))}{d(x)} \text{ for any } x \in \mathbb{B}^n$$

if and only if

$$|u(x) - u(y)| \leq C\omega_\alpha(|x - y|) \text{ for any } x, y \in \mathbb{B}^n.$$

We generalize Theorem 1.3 to the following form.

Theorem 1.4. *Let ω be a majorant satisfying (3), Ω be a Λ_ω -extension domain and f be a h -harmonic function from Ω into \mathbb{R} . Then $f \in \Lambda_\omega(\Omega)$ if and only if*

$$|\nabla f(x)| \leq C \frac{\omega(d_\Omega(x))}{d_\Omega(x)} \text{ for any } x \in \Omega.$$

Let ω be a majorant and D be a bounded set of \mathbb{R}^n . We use $\Lambda_\omega^B(D)$ to denote all the bounded continuous functions f in D with the norm

$$\|f\|_{\omega, D} = \sup_{x, y \in D, x \neq y} \left\{ \frac{|f(x) - f(y)|}{\omega(|x - y|)} \right\} < \infty.$$

Taking another majorant ω' , we define the operator norm

$$\|P_h\|_{\omega \rightarrow \omega'} = \sup_{f \in \Lambda_\omega^B(\partial\mathbb{B}^n), \|f\|_{\omega, \partial\mathbb{B}^n} \neq 0} \frac{\|P_h[f]\|_{\omega', \mathbb{B}^n}}{\|f\|_{\omega, \partial\mathbb{B}^n}}.$$

For each $a \in \partial\mathbb{B}^n$, we define

$$\eta_{a, \omega}(\zeta) = \omega(|\zeta - a|) \text{ for } \zeta \in \partial\mathbb{B}^n.$$

We refer to [1] for the similar definitions of harmonic functions.

Proposition 1.5. *Let ω be a majorant. Then $\eta_{a, \omega} \in \Lambda_\omega^B(\mathbb{B}^n)$.*

The following result is the classical Hardy-Littlewood Theorem.

Theorem 1.6. ([11, Theorem 5.1]) *Let f be an analytic function in \mathbb{D} and continuous in $\overline{\mathbb{D}}$. Then for some $0 < \alpha \leq 1$,*

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq C\omega_\alpha(|\theta_1 - \theta_2|) \text{ for any } 0 \leq \theta_1, \theta_2 < 2\pi$$

if and only if

$$|f'(z)| \leq C \frac{\omega_\alpha(d(z))}{d(z)} \text{ for any } z \in \mathbb{D}.$$

The following result is a Hardy-Littlewood-type theorem for hyperbolic functions. For the extensive discussions on this topics, see [1, 13, 23–25].

Theorem 1.7. *Let ω be a majorant. Then for each $a \in \partial\mathbb{B}^n$ and any $x \in \mathbb{B}^n$, there is a constant $C \geq 1$ such that $P_h[\eta_{a, \omega}](x) \leq C\omega(|x - a|)$, if and only if $\|P_h\|_{\omega \rightarrow \omega} < \infty$.*

We will prove Theorems 1.2, 1.4 and 1.7 in section 2.

2. Proofs of the main results

Lemma 2.1. ([28, Lemma 3.2]) *If $f \in C^2(\mathbb{B}^n)$, then*

- (a) $\frac{d}{dr} \int_{\partial\mathbb{B}^n} f(r\zeta) d\sigma(\zeta) = \frac{r^{1-n}(1-r^2)^{n-2}}{n} \int_{\mathbb{B}_R^n(r)} \Delta_h f(x) d\tau(x)$ and
- (b) $\int_{\partial\mathbb{B}^n} f(r\zeta) d\sigma(\zeta) = f(0) + \int_{\mathbb{B}_R^n(r)} g(|x|, r) \Delta_h f(x) d\tau(x)$, where

$$g(|x|, r) = \frac{1}{n} \int_{|x|}^r \frac{(1-s^2)^{n-2}}{s^{n-1}} ds,$$

$d\tau = \frac{dV_N}{(1-|x|^2)^n}$ and dV_N denotes the normalized Lebesgue volume measure on \mathbb{B}^n .

Proof of Theorem 1.2

Let f be h -harmonic in \mathbb{B}^n . By elementary calculations, we see that

$$\Delta_h(|f|^p) = p(p-1)(1-|x|^2)^2 |f|^{p-2} |\nabla f|^2. \tag{8}$$

For $r \in [0, 1)$, the Hölder’s inequality yields

$$\int_{\partial\mathbb{B}^n} |f(r\zeta)|^{p-2} |\nabla f(r\zeta)|^2 d\sigma(\zeta) \leq M_p^2(r, \nabla f) \cdot M_p^{p-2}(r, f). \tag{9}$$

By (8), (9) and Lemma 2.1, we obtain

$$\begin{aligned} M_p^p(r, f) &= |f(0)|^p + \int_{\mathbb{B}^n(0,r)} g(|x|, r) \Delta_h(|f(x)|^p) d\tau(x) \\ &= |f(0)|^p + p(p-1) \int_{\mathbb{B}^n(0,r)} |f(x)|^{p-2} |\nabla f(x)|^2 g(|x|, r) (1-|x|^2)^2 d\tau(x) \\ &= |f(0)|^p + np(p-1) \int_0^r \frac{\rho^{n-1} g(\rho, r)}{(1-\rho^2)^{n-2}} \int_{\partial\mathbb{B}^n} |f(\rho\zeta)|^{p-2} |\nabla f(\rho\zeta)|^2 d\sigma(\zeta) d\rho \\ &\leq |f(0)|^p + np(p-1) \int_0^r \frac{\rho^{n-1} g(\rho, r)}{(1-\rho^2)^{n-2}} M_p^2(\rho, \nabla f) M_p^{p-2}(\rho, f) d\rho. \end{aligned} \tag{10}$$

By computations, we obtain

$$\begin{aligned} g(\rho, r) &= \frac{1}{n} \int_{\rho}^r \frac{(1-s^2)^{n-2}}{s^{n-1}} ds \\ &\leq \frac{1}{n\rho^{n-1}} \int_{\rho}^r (1-s^2)^{n-2} ds \\ &\leq \frac{(1+r)^{n-2}}{n\rho^{n-1}} \int_{\rho}^r (1-s)^{n-2} ds \\ &\leq \frac{(1+r)^{n-2} (1-\rho)^{n-1}}{n(n-1) \rho^{n-1}}. \end{aligned} \tag{11}$$

Applying (10) and (11), we get

$$\begin{aligned}
 M_p^2(r, f) &\leq |f(0)|^2 + np(p-1) \int_0^r \frac{\rho^{n-1}g(\rho, r)}{(1-\rho^2)^{n-2}} M_p^2(\rho, \nabla f) d\rho \\
 &\leq |f(0)|^2 + \frac{p(p-1)(1+r)^{n-2}}{n-1} \int_0^r (1-\rho) M_p^2(\rho, \nabla f) d\rho \\
 &= |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}}{n-1} \int_0^1 (1-tr) M_p^2(tr, \nabla f) dt \\
 &\leq |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2}{n-1} \int_0^1 (1-tr)\omega^2\left(\frac{1}{1-tr}\right) dt \\
 &= |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2}{n-1} \int_0^1 \left[\omega\left(\frac{1}{1-tr}\right)(1-tr)\right] \omega\left(\frac{1}{1-tr}\right) dt \\
 &\leq |f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2\omega(1)}{n-1} \int_0^1 \omega\left(\frac{1}{1-tr}\right) dt,
 \end{aligned}$$

which gives

$$M_p(r, f) \leq \left[|f(0)|^2 + \frac{rp(p-1)(1+r)^{n-2}C^2\omega(1)}{n-1} \int_0^1 \omega\left(\frac{1}{1-tr}\right) dt \right]^{\frac{1}{2}}.$$

In particular, if $n = 2$ and $\omega(t) = t$, then the estimate of (7) is sharp. The proof of the sharpness part follows from [16, Theorem 1(b)]. The proof of this theorem is complete. \square

Lemma 2.2. *Let ω be a majorant. Then*

- (1) ω is subadditive, that is, if $t, s > 0$, then $\omega(s + t) \leq \omega(s) + \omega(t)$;
- (2) for $t > 0$, if $\lambda \geq 1$, then $\omega(\lambda t) \leq \lambda\omega(t)$.

Proof. We first prove (1). Since $\omega(t)/t$ is nonincreasing for $t > 0$, we see that for $s, t > 0$,

$$\begin{aligned}
 \omega(s) + \omega(t) - \omega(s + t) &= s \frac{\omega(s)}{s} + t \frac{\omega(t)}{t} - (s + t) \frac{\omega(s + t)}{s + t} \\
 &= s \left(\frac{\omega(s)}{s} - \frac{\omega(s + t)}{s + t} \right) + t \left(\frac{\omega(t)}{t} - \frac{\omega(s + t)}{s + t} \right) \\
 &\geq 0.
 \end{aligned}$$

(2) easily follows from the monotonicity of $\omega(t)/t$ for $t > 0$. The proof of this lemma is complete. \square

Proof of Theorem 1.4

We first prove the sufficiency. Since Ω is a Λ_ω -extension domain, we see that for any $x, y \in \Omega$, by using (5), there is a rectifiable curve $\gamma \subset \Omega$ joining x to y such that

$$\begin{aligned}
 |f(x) - f(y)| &\leq \int_\gamma |\nabla f(\zeta)| ds(\zeta) \\
 &\leq C \int_\gamma \frac{\omega(d_\Omega(\zeta))}{d_\Omega(\zeta)} ds(\zeta) \\
 &\leq C\omega(|x - y|).
 \end{aligned}$$

Now we come to prove the necessity. Let $x = (x_1, \dots, x_n) \in \Omega$ and $r = d_\Omega(x)/2$. Then by Lemma 2.2, for all $y = (y_1, \dots, y_n) \in \mathbb{B}^n(x, r)$,

$$|f(x) - f(y)| \leq C\omega(|x - y|) \leq 2C\omega(d_\Omega(x)).$$

For all $y \in \mathbb{B}^n(x, r)$, using (1), we get

$$f(y) = \int_{\partial\mathbb{B}^n} P_h(y, \zeta) f(r\zeta + x) d\sigma(\zeta),$$

where

$$P_h(y, \zeta) = \left(\frac{r^2 - |y - x|^2}{|y - x - r\zeta|^2} \right)^{n-1} \quad \text{and } \zeta = (\zeta_1, \dots, \zeta_n) \in \partial\mathbb{B}^n.$$

By elementary calculations, for each $k \in \{1, 2, \dots, n\}$, we have

$$\frac{\partial P_h(y, \zeta)}{\partial y_k} = -2(n-1) (P_h(y, \zeta))^{\frac{n-2}{n-1}} \frac{[(y_k - x_k)|y - x - r\zeta|^2 + (r^2 - |y - x|^2)(y_k - r\zeta_k - x_k)]}{|y - x - r\zeta|^4}.$$

Then for all $y \in \mathbb{B}^n(x, r/2)$,

$$\begin{aligned} & \left| \frac{\partial P_h(y, \zeta)}{\partial y_k} \right| \\ & \leq 2(n-1) \frac{(r^2 - |y - x|^2)^{n-2} [|y_k - x_k||y - x - r\zeta|^2 + (r^2 - |y - x|^2)|y_k - r\zeta_k - x_k|]}{|y - x - r\zeta|^{2n}} \\ & \leq 2(n-1) \frac{r^{2n-4}}{|y - x - r\zeta|^{2n}} \left(\frac{9r^3}{8} + \frac{3r^3}{2} \right) \\ & \leq \frac{21(n-1)}{4} \frac{r^{2n-1}}{\left(\frac{r}{2}\right)^{2n}} \\ & = \frac{2^{2(n-1)} \cdot 21(n-1)}{r}, \end{aligned}$$

which implies that

$$\begin{aligned} |\nabla f(y)| &= \left[\sum_{k=1}^n f_{y_k}^2(y) \right]^{\frac{1}{2}} \\ &= \left\{ \sum_{k=1}^n \left(\left| \int_{\partial\mathbb{B}^n} \frac{\partial}{\partial y_k} P_h(y, \zeta) (f(r\zeta + x) - f(x)) d\sigma(\zeta) \right|^2 \right)^{\frac{1}{2}} \right\} \\ &\leq \sum_{k=1}^n \left| \int_{\partial\mathbb{B}^n} \frac{\partial}{\partial y_k} P_h(y, \zeta) (f(r\zeta + x) - f(x)) d\sigma(\zeta) \right| \\ &\leq \sum_{k=1}^n \int_{\partial\mathbb{B}^n} \left| \frac{\partial}{\partial y_k} P_h(y, \zeta) \right| |f(r\zeta + x) - f(x)| d\sigma(\zeta) \\ &\leq \sqrt{n} \int_{\partial\mathbb{B}^n} |\nabla P_h(y, \zeta)| |f(r\zeta + x) - f(x)| d\sigma(\zeta) \\ &\leq \frac{2^{2(n-1)} \cdot 21n(n-1)}{r} \int_{\partial\mathbb{B}^n} |f(r\zeta + x) - f(x)| d\sigma(\zeta) \\ &\leq \frac{2^{2(n-1)} \cdot 21n(n-1)C\omega(r)}{r} \\ &\leq 2^{2n-1} \cdot 21n(n-1)C \frac{\omega(d_\Omega(x))}{d_\Omega(x)}. \end{aligned}$$

The proof of this theorem is complete. \square

Proof of Proposition 1.5

Let $\zeta_1, \zeta_2 \in \partial\mathbb{B}^n$. Without loss of generality, we may assume $|\zeta_1 - a| < |\zeta_2 - a|$. Then by Lemma 2.2, we have

$$\begin{aligned} |\eta_{a,\omega}(\zeta_2) - \eta_{a,\omega}(\zeta_1)| &= \omega(|\zeta_2 - a|) - \omega(|\zeta_1 - a|) \\ &= \omega(|\zeta_2 - a| - |\zeta_1 - a| + |\zeta_1 - a|) - \omega(|\zeta_1 - a|) \\ &\leq \omega(|\zeta_2 - a| - |\zeta_1 - a|) + \omega(|\zeta_1 - a|) - \omega(|\zeta_1 - a|) \\ &\leq \omega(|\zeta_2 - \zeta_1|), \end{aligned}$$

which gives

$$\frac{|\eta_{a,\omega}(\zeta_2) - \eta_{a,\omega}(\zeta_1)|}{\omega(|\zeta_2 - \zeta_1|)} \leq 1.$$

Hence $\|\eta_{a,\omega}\|_{\omega, \partial\mathbb{B}^n} \leq 1 < \infty$. On the other hand, for each $a \in \partial\mathbb{B}^n$,

$$\eta_{a,\omega}(\zeta) = \omega(|\zeta - a|) \leq \omega(2) < \infty \text{ for } \zeta \in \partial\mathbb{B}^n.$$

Then $\eta_{a,\omega}$ is bounded and $\eta_{a,\omega} \in \Lambda_\omega^B(\mathbb{B}^n)$. The proof of this Proposition is complete. \square

Proof of Theorem 1.7

We first prove the necessity. Let $f \in \Lambda_\omega^B(\partial\mathbb{B}^n)$. We only need to prove that for any $x, y \in \mathbb{B}^n$, there is a positive constant C such that

$$|\mathbb{P}_h[f](x) - \mathbb{P}_h[f](y)| \leq C\|f\|_{\omega, \partial\mathbb{B}^n} \omega(|x - y|).$$

Without loss of generality, we assume that

$$0 < d(y) \leq d(x). \tag{12}$$

Let $x_0, y_0 \in \partial\mathbb{B}^n$ such that $d(x) = |x - x_0|$ and $d(y) = |y - y_0|$, respectively. For $\zeta \in \partial\mathbb{B}^n$, let $F(\zeta) = f(\zeta) - f(x_0)$.

We divide the proof into two cases.

Case 1. $|x - y| \leq d(x)/2$. For $z \in \mathbb{B}^n(x, d(x)/2)$, using Lemma 2.2, we see that

$$|z - x_0| \leq |z - x| + |x - x_0| \leq \frac{d(x)}{2} + d(x) = \frac{3d(x)}{2},$$

which yields that

$$\begin{aligned} |\mathbb{P}_h[F](z)| &\leq \|f\|_{\omega, \partial\mathbb{B}^n} |\mathbb{P}_h[\eta_{x_0,\omega}](z)| \\ &\leq C\|f\|_{\omega, \partial\mathbb{B}^n} \omega(|x - y|) \\ &\leq C\|f\|_{\omega, \partial\mathbb{B}^n} \omega\left(\frac{3d(x)}{2}\right) \\ &\leq \frac{3C}{2}\|f\|_{\omega, \partial\mathbb{B}^n} \omega(d(x)). \end{aligned}$$

Using arguments similar to those in the necessity's proof of Theorem 1.4, for $z \in \mathbb{B}^n(x, d(x)/2)$, there is a positive constant C such that

$$|\nabla\mathbb{P}_h[f](z)| = |\nabla\mathbb{P}_h[F](z)| \leq C\|f\|_{\omega, \partial\mathbb{B}^n} \frac{\omega(d(x))}{d(x)},$$

which gives that

$$\begin{aligned} |P_h[f](x) - P_h[f](y)| &\leq \int_{[x,y]} |\nabla P_h[f](z)| dz \\ &\leq C \|f\|_{w, \partial B^n} \frac{\omega(d(x))}{d(x)} |x - y| \\ &\leq C \|f\|_{w, \partial B^n} \omega(|x - y|), \end{aligned}$$

where $[x, y]$ denotes the segment from x to y .

Case 2. $|x - y| > d(x)/2$. We use the similar approach as in the proof of [1, Theorem 1.1] to prove this case. By (12), we know that $|x - y| > d(x)/2 \geq d(y)/2$. By elementary calculations, we see that there is positive constant C such that

$$\begin{aligned} |P_h[f](x) - f(x_0)| &= |P_h[F](x)| \leq C \|f\|_{w, \partial B^n} |P_h[\eta_{x_0, \omega}](x)| \leq C \|f\|_{w, \partial B^n} \omega(d(x)) \\ &\leq C \|f\|_{w, \partial B^n} \omega(|x - y|). \end{aligned} \tag{13}$$

Similarly,

$$|P_h[f](y) - f(y_0)| \leq C \|f\|_{w, \partial B^n} \omega(|x - y|). \tag{14}$$

By using $|x_0 - y_0| \leq |x - y| + d(x) + d(y) < 5|x - y|$ and Lemma 2.2, we know that there is positive constant C such that

$$|f(x_0) - f(y_0)| \leq C \|f\|_{w, \partial B^n} \omega(|x_0 - y_0|) \leq 5C \|f\|_{w, \partial B^n} \omega(|x - y|). \tag{15}$$

By (13), (14) and (15), we conclude that

$$\begin{aligned} |P_h[f](x) - P_h[f](y)| &\leq |P_h[f](x) - f(x_0) - (P_h[f](y) - f(y_0)) + f(x_0) - f(y_0)| \\ &\leq |P_h[f](x) - f(x_0)| + |P_h[f](y) - f(y_0)| + |f(x_0) - f(y_0)| \\ &\leq 7C \|f\|_{w, \partial B^n} \omega(|x - y|). \end{aligned}$$

Now we come to prove the sufficiency. By Proposition 1.5, we have

$$\|P_h[\eta_{a, \omega}]\|_{\omega', B^n} \leq \|P_h\|_{\omega \rightarrow \omega} \|\eta_{a, \omega}\|_{\omega, \partial B^n} < \infty,$$

which implies that, for $x, y \in B^n$, there is a positive constant C such that

$$|P_h[\eta_{a, \omega}](x) - P_h[\eta_{a, \omega}](y)| \leq C\omega(|x - y|).$$

Let $x_0 \in \partial B^n$ such that $d(x) = |x_0 - x|$. By letting y tends to x_0 yields that

$$|P_h[\eta_{a, \omega}](x) - \eta_{a, \omega}(x_0)| \leq C\omega(|x_0 - x|),$$

which gives

$$P_h[\eta_{a, \omega}](x) \leq \eta_{a, \omega}(x_0) + C\omega(|x_0 - x|) \leq \omega(|x_0 - a|) + C\omega(|x - a|). \tag{16}$$

Applying (16) and the inequality $|x_0 - a| \leq |x_0 - x| + |x - a| \leq 2|x - a|$, we conclude that

$$P_h[\eta_{a, \omega}](x) \leq C\omega(|x - a|) + \omega(|x_0 - a|) \leq (C + 2)\omega(|x - a|).$$

The proof of this theorem is complete. \square

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