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A Note on Jensen Formula in $\mathbb{C}^n$

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Abstract. We provide some natural weighted generalizations of certain sharp assertions related with multidimensional Jensen formula which served as a base of recent sharp theorems on generators for spaces of entire functions for several complex variables in $\mathbb{C}^n$.

1. Introduction

The intention of this note to provide some natural weighted generalizations of certain assertions from [1], [2], [8] related with Jensen formula (in the disk for slice function, in the ball and in the polydisk) which serve in [1], [8] as base of certain sharp simple theorems on generators for spaces of entire functions for several complex variables in $\mathbb{C}^n$.

The mentioned assertions are closely related with multidimensional classical Jensen formula (in the disk for slice function, in the ball and in the polydisk) which has various applications in complex function theory not only of several but also one complex variables on the complex plane $\mathbb{C}$.

We refer the reader for those applications to [4], [5], [6], [7], for example, in one dimension.

We do not discuss applications of theorems we provided in this note to problems related with generators in weighted spaces of entire functions of several complex variables in $\mathbb{C}^n$ leaving this to our future collaborative work.

Note again, this short note has an intention to put a ground for further work and has roots and is based heavily on arguments from [1], [8].

Let us denote as usual the ball and sphere as follows

$$S(z,t) = \{w \in \mathbb{C}^n : |w - z| = t\},\ t > 0,$$

$$B(z,t) = \{w \in \mathbb{C}^n : |w - z| \le t\},\ t > 0.$$

We define polydisk and it is measure as

$$\Delta(z,r) = \{w \in \mathbb{C}^n :\ |w_k - z_k| \le r_k,\ k = 1,\dots,n\},\ r \in (0,\infty)^n,\ z \in \mathbb{C}^n,$$

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and

$$\nu(\Delta(z,r)) = \pi^n \left(\prod_{j=1}^{n} r_j^2 \right).$$

All q weights in this paper are positive and all q weights without integral conditions on them are assumed to be circular, that is $q(z) = q(|z|),\ z \in \mathbb{C}^n$ or $q(z) = q(|z_1|, \ldots, |z_n|),\ z \in \mathbb{C}^n$.

Let $N_f(z) = N_f(z, |z|),\ z \in \mathbb{C}^n,\ z \neq 0$ and

$$N_f(z,r) = \frac{1}{\sigma_n} \int_{S_1} d\sigma_\xi \int_0^r \cdot \frac{1}{t} n_{f,z}(t,\xi) dt,$$

where f is an entire function in $\mathbb{C}^n$, $f \neq 0$, $\xi \in S_1$, $S_1 = \{w \in \mathbb{C}^n : |w| = 1\}$ is the standard sphere in $\mathbb{C}^n$, $d\sigma_\xi$ is a Lebesgue measure in S_1, $n_{f,z}(t,\xi)$ is the number of zeroes of $f(z + \xi u)$ in $\{u \in \mathbb{C} : |u| \leq t\}$, $t \geq 0$, $\sigma_n = \frac{2\pi^n}{(n-1)!}$ is a volume of sphere and $N_f(0) = 0$ if $f(0) \neq 0$, $N_f(0) = \infty$ if $f(0) = 0$.

Let f be entire function in $\mathbb{C}^n$ and for fixed $z \in \mathbb{C}^n$ by f_z we define $f((1+w)z)$ a function of one complex variable w. Assume that $f(z) \neq 0$ and we define

$$\widetilde{\widetilde{N}}_f(z) = \int_0^1 \frac{n(f_z, t)}{t}\, dt$$

where $n(f_z, t)$ is the number of zeroes of f_z function according to their multiplicity in disk $\{w \in \mathbb{C} : |w| \leq t\}$.

If $f(z) = 0$ then we put $\widetilde{\widetilde{N}}_f(z) = +\infty$.

Let f be entire function in $\mathbb{C}^n$ and let

$$M_f(z,t) = \frac{1}{\sigma_n t^{2n-1}} \int_{S(z,t)} \ln |f(w)| d\sigma_w.$$

Let also

$$\widetilde{N}_f(z,t) = \frac{1}{w_n t^{2n}} \int_{B(z,t)} \ln |f(w)| dv_w,\ t > 0,$$

where $w_n = \frac{\pi^n}{n!}$ is a volume of the unit ball in $\mathbb{C}^n$ and v_w is a Lebesgue measure on $B(z,t)$.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

2. Main results

We formulate in this section our main results discussed in Introduction. The first result is based on Jensen formula in the ball, the second result on Jensen formula in the disk (but for slice function) and the third result on Jensen formula in the polydisk.

Theorem 2.1. *Let $p : [0,\infty) \to [0,\infty)$ be nondecreasing function, $p(z) := p(|z|)$, $p(0) = 0$, for $z \in \mathbb{C}^n$. Let f be entire function in $\mathbb{C}^n$, non constant, $\ln |f(z)| \leq p(z)$, $p(z) = \widetilde{p}(z)\widetilde{w}(z)$, $z \in \mathbb{C}^n$, $\widetilde{w}(z) : \mathbb{C}^n \to (0,\infty)$. We assume the same conditions are valid for all $\widetilde{p}$. If $\frac{1}{|z|^{2n-1}} \int_{S(z,|z|)} \widetilde{w}(r\tau)\, d\sigma_\tau \leq C_0$, $r \in (0,\infty)$, for all $z \in \mathbb{C}^n$ there is $r_0 \geq 0$ so that for all $|z| > r_0$ $f(z) \neq 0$, we have*

$$-C_1 p(4z) \leq N_f(z) + \ln |f(z)| \leq C_0 \widetilde{p}(2z),$$

for some constant C_1 and C_0.

Remark 2.2. *If $\widetilde{w} = const(\widetilde{w} = 1)$ integral condition is trivially satisfied and this result can be seen in [1], [2].*

Theorem 2.3. *Let $p_1 : [0,\infty)^n \to [0,\infty)$ is a function which is nondecreasing by each variable and $f : \mathbb{C}^n \to \mathbb{C}$ is entire function so that $|f(0)| = 1$, $\ln|f(z)| \le \widetilde{p}(z)$, $z \in \mathbb{C}^n$, $\widetilde{p}(z) = \prod_{j=1}^{2} p_j(z)$. Let p_2 be arbitrary positive weight in $\mathbb{C}^n$, $p_2(z) > 0$ and $p_1 = \widetilde{p}(|z_1|, \ldots, |z_n|)$.*

If $\frac{1}{2\pi}\int_0^{2\pi} p_2((1+e^{i\theta})z)d\theta \le C_1$, for all $z \in \mathbb{C}^n$ then, for some constant C_1 and C_2, we have the following estimate

$$-C_2\widetilde{p}(2z) \le \widetilde{\widetilde{N}}_f(z) + \ln|f(z)| \le C_1 p_1(2z),$$

for all $z \in \mathbb{C}^n$ with $f(z) \neq 0$.

Remark 2.4. *For $p_2 = 1$ this is known (see [1], [2], [8]).*

We finally formulate one more theorem similar to Theorem 2.1 and 2.3.

Let f be entire function in $\mathbb{C}^n$. For fixed $z \in \mathbb{C}^n$, $r \in (0,\infty)^n$, $\varphi \in [0,2\pi]^{n-1}$ we denote by $f_{z,r,\varphi}(w)$ an entire function

$$f(z_1 + \frac{r_1}{|r|}e^{i\varphi_1}w, \ldots, z_{n-1} + \frac{r_{n-1}}{|r|}e^{i\varphi_{n-1}}w, z_n + \frac{r_n}{|r|}w)$$

of one complex variable w.

For $s \in [0,\infty)$ let $n_{f,z,r}(\varphi, s)$ be the number of zeros of $f_{z,r,\varphi}(w)$ according to their multiplicity in $\{w \in C : |w| \leqslant s\}$. Let $f(z) \neq 0$ and we define

$$N_f(z,r,t) = \int_{[0,2\pi]^{n-1}\times[0,t]} \left[\frac{n_{f,z,r}(\varphi,s)}{s}\right] d\varphi ds,\ t \in [0,\infty).$$

We denote $N_f^\star(z) = N_f(z, r(z), |r(z)|)$ where $r(z) = (|z_1| + 1, \ldots, |z_n| + 1)$. If $f(z) = 0$ then $N_f^\star(z) = +\infty$.

Theorem 2.5. *Let $p : [0,\infty)^n \to [0,\infty)$ be nondecreasing by each variable function, $p(0) = 0$ and $f : \mathbb{C}^n \to \mathbb{C}$ be an entire function so that $|f(0)| = 1$, $\ln|f(z)| \le p(z)$, $p(z) = \widetilde{p}(z)w(z)$, $w : \mathbb{C}^n \to (0,\infty)$, $z \in \mathbb{C}^n$.*

If $\int_{\Delta(0,2r(z))} w(z)dv_z \le C_2 v(\Delta(0, 2r(z)))$, then, for some constant C_1 and C_2, we have the following estimate

$$-C_2 p(2r(z)) \le N_f^\star(z) + \ln|f(z)| \le C_1\widetilde{p}(2r(z)),$$

for all $z \in \mathbb{C}^n$, $f(z) \neq 0$, $\widetilde{p}(z) = p(|z_1|, \ldots, |z_n|)$, $r(z) = (|z_1| + 1, \ldots, |z_n| + 1)$.

Remark 2.6. *For $w = 1$ case this theorem can be seen in [1], [8].*

All theorems are based on same simple idea and we will show the proof of only one assertion. Note all assertions are based heavily on arguments from [1], [8] and we generalized them modifying a little to weighted case. Our main purpose is ahead, it is to apply these assertions for some problems related to generators in this more general "weighted" setting. We note that in less general case this procedure can be seen in [1], [2], [3].

The proof of Theorem 2.1.

By classical Jensen formula we have

$$N_f(z) + \ln|f(z)| = M_f(z, |z|)$$

for all $z \in \mathbb{C}^n \setminus \{0\}$, $f(z) \neq 0$. Then from estimate $\ln|f(z)| \le \widetilde{p}(z)\widetilde{w}(z)$, which we have in formulation of theorem, we have what we need immediately

$$N_f(z) + \ln|f(z)| \le C_0\widetilde{p}(2z).$$

Since

$$\frac{1}{\sigma_n|z|^{2n-1}}\int_{S(z,|z|)} \ln|f(w)|d\sigma_w \le C_0\widetilde{p}(2z).$$

We show the reverse. If $f \neq const$ then we have that there is $\xi \in \mathbb{C}^n$, $|f(\xi)| \geq 1$. But by plurisubharmonicity we have now that (see also [1], [2], [3])

$$\widetilde{N}_f(\xi, 3|z|) \geq \ln|f(\xi)| \geq 0,\ |z| > r_0 = |\xi|,\ z \in \mathbb{C}^n.$$

Hence we have the following estimate

$$\int_{B(\xi,3|z|)} \ln|f(w)|dv_w \geq 0.$$

Hence we have for $|z| > r_0$

$$\widetilde{N}_f(z, |z|) = \frac{1}{w_n|z|^{2n}} \int_{B(z,|z|)} \ln|f(w)|dv_w$$

$$\geq \frac{-1}{w_n|z|^{2n}} \int_{B(\xi,3|z|)} \ln|f(w)|dv_w \geq C_1(-3^{2n}) \max_{w\in B(\xi,3|z|)} \ln|f(w)|$$

$$\geq C_1(-3^{2n})p(3z + r_0) \geq C_1(-3^{2n})p(4z).$$

Since

$$M_f(z, |z|) \geq \widetilde{N}_f(z, |z|),\ z \neq 0,$$

we have finally

$$N_f(z) + \ln|f(z)| \geq (-3^{2n})p(4z),\ \forall\, z \in \mathbb{C}^n,\ |z| > r_0,\ f(z) \neq 0.$$

Theorem 2.1 is proved. □

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