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On John Domains in Banach Spaces

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Abstract. We study the stability of John domains in Banach spaces under removal of a countable set of points. In particular, we prove that the class of John domains is stable in the sense that removing a certain type of closed countable set from the domain yields a new domain which also is a John domain. We apply this result to prove the stability of the inner uniform domains. Finally, we consider a wider class of domains, so called ψ-John domains and prove a similar result for this class.

1. Introduction

The class of domains, nowadays known as John domains and originally introduced by John [12] in the study of elasticity theory, has been investigated during the past three decades by many people in connection with applications of classical analysis and geometric function theory. See for instance [3, 18, 19] and the references therein. Here we study the class of John domains and the wider class of ψ-John domains [9, 26] and the stability of these two classes of domains under the removal of a countable closed set of points. The motivation for this paper stems from the discussions in [10, 28], where the effect of the removal of a finite set of points was examined. See also the very recent paper [14].

Suppose that D is a domain in a real Banach space E with dimension at least 2 and let P_D denote a countable set in D such that the quasihyperbolic distance w.r.t. D between each pair of distinct points in P_D is at least b where $b > 0$ is a constant. We note that, in Banach spaces, the properties of the quasihyperbolic metric were first studied by Väisälä in a series of articles in 1990's [23–27]. The quasihyperbolic metric is a critical tool for studying quasiconformal mappings in the infinite dimensional Banach spaces because quasiconformality is defined in terms of it. The first main result of this paper shows that D is a c-John domain if and only if $D \setminus P_D$ is a c_1-John domain, where c and c_1 are two constants depending only on each other and on b. Applying this result, we show that D is inner uniform if and only if $D \setminus P_D$ is inner uniform. Our second main result shows that D is a ψ-John domain if and only if $D \setminus P_D$ is a ψ_1-John domain, where ψ and ψ_1 depend only on each other and on b.

The methods applied in the proofs rely on standard notions of metric space theory: curves, their lengths, and nearly length-minimizing curves. It should be noted that we employ several metric space structures

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on the domain D including hyperbolic type metrics. We use three metrics: the norm metric, the distance ratio metric and the quasihyperbolic metric on the domain D and, moreover, also on its subdomains.

2. Preliminaries and main results

Throughout the paper, we always assume that E denotes a real Banach space with dimension at least 2. The norm of a vector z in E is written as $|z|$, and for each pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. We always use $\mathbb{B}(x_0, r)$ to denote the open ball $\{x \in E : |x - x_0| < r\}$ centered at x_0 with radius $r > 0$. Similarly, for the closed balls and spheres, we use the usual notations $\overline{\mathbb{B}}(x_0, r)$ and $\mathbb{S}(x_0, r)$, respectively.

For each pair of points z_1, z_2 in D, the *distance ratio metric* $j_D(z_1, z_2)$ between z_1 and z_2 is defined by

$$j_D(z_1, z_2) = \log\Big(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big),$$

where $d_D(z)$ denotes the distance from z to the boundary ∂D of D.

The *quasihyperbolic length* of a rectifiable arc or a path γ in D is the number (cf. [1, 5, 6, 23])

$$\ell_k(\gamma) = \int_\gamma \frac{1}{d_D(z)}\,|dz|.$$

For each pair of points z_1, z_2 in D, the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D.

For all z_1, z_2 in D, we have (cf. [23])

$$k_D(z_1, z_2) \geq \inf\left\{\log\Big(1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}}\Big)\right\} \geq j_D(z_1, z_2) \tag{2.1}$$

$$\geq \left|\log \frac{d_D(z_2)}{d_D(z_1)}\right|,$$

where the infimum is taken over all rectifiable curves α in D connecting z_1 and z_2, $\ell(\alpha)$ denotes the length of α. Next, if $|z_1 - z_2| < d_D(z_1)$, then we have [23], [31, Lemma 2.11]

$$k_D(z_1, z_2) \leq \log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|}\Big) \leq \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|}, \tag{2.2}$$

where the last inequality follows from the following elementary inequality

$$\frac{r}{1 - r/2} \leq \log\frac{1}{1 - r} \leq \frac{r}{1 - r} \quad \text{for } 0 \leq r < 1\,.$$

Gehring and Palka [5] introduced the quasihyperbolic metric of a domain in R^n. Many of the basic properties of this metric may be found in [6, 11, 13, 15, 20, 21, 30]. Recall that an arc α from z_1 to z_2 is a *quasihyperbolic geodesic* if $\ell_k(\alpha) = k_D(z_1, z_2)$. Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in E exists if the dimension of E is finite, see [6, Lemma 1]. This is not true in infinite dimensional Banach spaces (cf. [23, Example 2.9]). In order to remedy this shortage, Väisälä introduced the following concepts [24].

Definition 2.3. *Let $D \neq E$ and $c \geq 1$. An arc $\alpha \subset D$ is a c-neargeodesic if $\ell_k(\alpha[x, y]) \leq c\, k_D(x, y)$ for all $x, y \in \alpha$.*

In [24], Väisälä proved the following property concerning the existence of neargeodesics in E.

Lemma 2.4. ([24, Theorem 3.3]) *Let z_1, $z_2 \in D \neq E$ and let $c > 1$. Then there is a c-neargeodesic joining z_1 and z_2 in D.*

Definition 2.5. *A domain D in E is called c-John domain in the norm metric provided there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc α in D such that for all $z \in \alpha$ the following holds:*

$$\min\{\ell(\alpha[z_1, z]),\ \ell(\alpha[z_2, z])\} \leq c\, d_D(z), \tag{2.6}$$

where $\alpha[z_j, z]$ denotes the part of α between z_j and z (cf. [3, 19]). The arc α is called to be a c-cone arc .

A domain D in E is said to be a *c-uniform domain* (cf. [17, 18, 22, 24, 27]) if there is a constant $c \geq 1$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc α satisfying (2.6) and

$$\ell(\alpha) \leq c\, |z_1 - z_2|. \tag{2.7}$$

We also say that α is a *c-uniform arc* (cf. [28]).

For z_1, $z_2 \in D$, the *inner length metric $\lambda_D(z_1, z_2)$* between these points is defined by

$$\lambda_D(z_1, z_2) = \inf\{\ell(\alpha) :\ \alpha \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2\}.$$

We say that a domain D in E is *an inner c-uniform domain* if there is a constant $c \geq 1$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc α satisfying (2.6) and

$$\ell(\alpha) \leq c\lambda_D(z_1, z_2). \tag{2.8}$$

Such an arc α is called to be an *inner c-uniform arc* (cf. [26]).

Obviously, uniform domains are inner uniform domains, but inner uniform does not imply uniform. See [2, 4, 6, 17, 18, 22, 24, 28] for more details on uniform domains and inner uniform domains.

Remarks. If we replace (2.6), (2.7) and (2.8) by

$$\min\{\operatorname{diam}(\alpha[z_1, z]),\ \operatorname{diam}(\alpha[z_2, z])\} \leq c\, d_D(z), \tag{2.9}$$

$$\operatorname{diam}(\alpha) \leq c\, |z_1 - z_2| \tag{2.10}$$

and

$$\operatorname{diam}(\alpha) \leq c\lambda_D(z_1, z_2), \tag{2.11}$$

then we get concepts which in the case $E = \mathbb{R}^n$ are n-quantitatively equivalent to c-John domain, c-uniform domain and inner c-uniform domain, respectively [19]. Note that in general Banach space, each of these three conditions leads to a wider class of domains. For example, the broken tube domain considered by Väisälä [23, 4.12] (see also [29]) is neither John nor quasiconvex (a metric space is *c-quasiconvex* if each pair of points x, y can be joined by an arc such that (2.7) holds). Nevertheless, one can join a given pair of points in this bounded domain by arcs satisfying (2.9), (2.10) and (2.11).

Various classes of domains have been studied in analysis (e.g. see [7]). For some classes, the removal of a finite number of points from a domain may yield a domain no longer in this class [7]. In [10], the authors proved that remove a finite number of points from a John domain still yields a John domain.

Proposition 2.12. ([10, Theorem 1.4]) *A domain $D \subset \mathbb{R}^n$ is a John domain if and only if $G = D \setminus P$ is also a John domain, where $P = \{p_1, p_2, \cdots, p_m\} \subset D$.*

In general, when P is an infinite closed set in D, $D \setminus P$ need not be a John domain ([10, Example 1.5]). In this paper, we continue the study of the removability properties of John domains and prove that if P satisfies a certain separation condition, then $D \setminus P$ is still a John domain if D is a John domain.

Let $b > 0$ be a constant. In what follows, for a domain D in E, and for a sequence $X = \{x_j : j = 1, 2, ...\}$ of points in D satisfying *the quasihyperbolic separation condition*

$$k_D(x_i, x_j) \geq b \;\text{ for }\; i \neq j,$$

we always write

$$P_D = X.$$

Further, we assume that the set P_D satisfying the quasihyperbolic separation condition contains at least two points, and in the following, without loss of generality, we may assume that $b = \frac{1}{2}$. Our main results are as follows.

Theorem 2.13. *A domain $D \subset E$ is a c-John domain if and only if $G = D \setminus P_D$ is a c_1-John domain, where $c \geq 1$ and $c_1 \geq 1$ depend only on each other.*

As an application, we get the following result concerning inner uniform domains.

Theorem 2.14. *A domain $D \subset E$ is an inner c-uniform domain if and only if $G = D \setminus P_D$ is an inner c_1-uniform domain, where $c \geq 1$ and $c_1 \geq 1$ depend only on each other.*

Moreover, as a generalization of John domains, we consider the ψ-John domains [9] whose definition will be given in Section 4.

Theorem 2.15. *A domain $D \subset E$ is a ψ-John domain if and only if $G = D \setminus P_D$ is a ψ_1-John domain, where ψ and ψ_1 are homeomorphisms depending only on each other.*

The proofs of Theorems 2.13 and 2.14 will be given in Sectin 3, and the proof of Theorem 2.15 will be given in section 4.

3. Proofs of Theorems 2.13 and 2.14

3.1. Some crucial lemmas

We first give some lemmas which are crucial to the proofs of our main results.

Let $D \subset E$ be a domain. Given $x \in D$ and $s \in (0, 1)$, for $z_1, z_2 \in \mathbb{B}(x, sd_D(x))$, we see from (2.2) that

$$k_D(z_1, z_2) \leq 2 \log(1/(1 - s)).$$

This fact, together with the definition of P_D, yields the following lemma.

Lemma 3.1. *For all $w \in D$, there exists at most one point x_i of P_D such that $x_i \in \mathbb{B}(w, \frac{1}{6}d_D(w))$.*

Lemma 3.2. ([28, Lemma 6.7]) *Suppose that G is a c-uniform domain and that $x_0 \in G$. Then $G_0 = G \setminus \{x_0\}$ is c_1-uniform with $c_1 = c_1(c)$ (This means that c_1 is a constant depending only on c). Moreover, from the proof of [28, Lemma 6.7] we see that $c_1 \leq 9c$.*

We note that each ball $\mathbb{B}(x, r)$ is 2-uniform and $\mathbb{B}(x, r) \setminus \{x\}$ is 10-uniform by the proof of [28, Theorem 6.5]. By Lemma 3.1 and 3.2, the following holds.

Lemma 3.3. *For $x_0 \in D$, $\mathbb{B}(x_0, \frac{1}{6}d_D(x_0)) \setminus P_D$ is c_2-uniform with $2 \leq c_2 \leq 18$.*

Lemma 3.4. *For $x, y \in D$, if there is a c_3-cone arc γ joining x, y in D, then for each $w \in \gamma$ the following holds:*

$$d_D(w) \geq \frac{1}{2c_3} \min\{d_D(x), d_D(y)\}.$$

Moreover, if $\ell(\gamma[x, w]) \leq \ell(\gamma[y, w])$, then

$$d_D(w) \geq \frac{1}{2c_3} d_D(x).$$

Otherwise,

$$d_D(w) \geq \frac{1}{2c_3} d_D(y).$$

Proof. Let $w_0 \in \gamma$ bisect the arclength of γ. Obviously, we only need to consider the case $w \in \gamma[x, w_0]$ since the discussion for the case $w \in \gamma[y, w_0]$ is similar.

If $\ell(\gamma[x, w]) \leq \frac{1}{2} d_D(x)$, then

$$d_D(w) \geq d_D(x) - \ell(\gamma[x, w]) \geq \frac{1}{2} d_D(x).$$

If $\ell(\gamma[x, w]) > \frac{1}{2} d_D(x)$, then we have

$$d_D(w) \geq \frac{1}{c_3} \ell(\gamma[x, w]) > \frac{1}{2c_3} d_D(x).$$

The proof is complete. □

Let us recall the following result from [16].

Lemma 3.5. ([16, Theorem 1.2]) *Suppose that D_1 and D_2 are convex domains in E, where D_1 is bounded and D_2 is c-uniform, and that there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $\mathbb{B}(z_0, r) \subset D_1 \cap D_2$. If there exist $R_1 > 0$ and a constant $c_0 > 0$ such that $R_1 \leq c_0 r$ and $D_1 \subset \overline{\mathbb{B}}(z_0, R_1)$, then $D_1 \cup D_2$ is a c'-uniform domain with $c' = \frac{1}{2}(c + 1)(6c_0 + 1) + c$.*

By Lemma 3.1, 2.4 and 3.5, we easily have the following lemma.

Lemma 3.6. *Let $D \subset E$ be a domain. For $y_1,\ w_1 \in D$, if*

$$\mathbb{B}(y_1, \frac{1}{32} d_D(y_1)) \cap \mathbb{B}(w_1, \frac{1}{32} d_D(w_1)) \neq \emptyset,$$

then $D_0 \setminus P_D$ is a $660c_2^2$-uniform domain, where $D_0 = \mathbb{B}(y_1, \frac{1}{16} d_D(y_1)) \cup \mathbb{B}(w_1, \frac{1}{32} d_D(w_1))$.

Lemma 3.7. *Let $D \subset E$ be a domain. For $z_1, z_2 \in G = D \setminus P_D$, let γ be a rectifiable arc joining z_1 and z_2 in D. Then there exists an arc $\alpha \subset G$ joining z_1 and z_2 such that $\ell(\alpha) \leq 660c_2^2 \ell(\gamma)$. Moreover, if γ is a c-cone arc in D, then α is a $(2^{18}cc_2^3 + 660c_2^2)$-cone arc in G, where $c > 1$ is a constant and c_2 is the constant from Lemma 3.3.*

Proof. For given z_1 and z_2 in G, let γ be a rectifiable arc joining z_1 and z_2 in D and let

$$U = \{u \in \gamma : d_D(u) > 64 d_G(u)\}.$$

If $U = \emptyset$, then let $\alpha_0 = \gamma$. Obviously, Lemma 3.7 holds.

In the following, we assume that $U \neq \emptyset$. We prove this lemma by considering three cases.

Case 3.8. *There exists some point $w_0 \in \gamma$ such that $\{z_1, z_2\} \subset \overline{\mathbb{B}}(w_0, \frac{1}{32} d_D(w_0))$.*

Then by Lemma 3.3, we know that there is a c_2-uniform arc α_1 joining z_1 and z_2 in G which is the desired one since

$$\ell(\alpha_1) \leq c_2 |z_1 - z_2| \leq c_2 \ell(\gamma).$$

Let $z_0 \in \gamma$ be a point such that $\ell(\gamma[z_1, z_0]) = \ell(\gamma[z_2, z_0])$.

Case 3.9. *For all $w \in \gamma$, $\{z_1, z_2\} \not\subseteq \overline{\mathbb{B}}(w, \frac{1}{32}d_D(w))$, but there is a point $w_1 \in \gamma[z_1, z_0]$ such that $z_2 \in \overline{\mathbb{B}}(w_1, \frac{1}{32}d_D(w_1))$ or a point $w_2 \in \gamma[z_2, z_0]$ such that $z_1 \in \overline{\mathbb{B}}(w_2, \frac{1}{32}d_D(w_2))$.*

Obviously, we only need to consider the former case since the discussion for the latter case is similar. Without loss of generality, we may assume that w_1 is the first point in $\gamma[z_1, z_0]$ along the direction from z_1 to z_0 such that $z_2 \in \overline{\mathbb{B}}(w_1, \frac{1}{32}d_D(w_1))$.

Subcase 3.10. $U \cap \gamma[z_1, w_1] = \emptyset$.

That is, for all $w \in \gamma[z_1, w_1]$, $d_D(w) \leq 64d_G(w)$. By Lemma 3.3, there exists a c_2-uniform arc η_1 joining w_1 and z_2 in G. Then we come to prove that $\alpha_2 = \gamma[z_1, w_1] \cup \eta_1$ is the desired arc. By the choice of η_1, we know that

$$\ell(\alpha_2) \leq c_2|w_1 - z_2| + \ell(\gamma[z_1, w_1]) \leq c_2\ell(\gamma).$$

Assume further that γ is a c-cone arc. Then we let $u_0 \in \eta_1$ be a point bisecting the arclength of η_1. If $w \in \gamma[z_1, w_1]$, then

$$\ell(\alpha_2[z_1, w]) = \ell(\gamma[z_1, w]) \leq cd_D(w) \leq 64cd_G(w).$$

If $w \in \eta_1[w_1, u_0]$, then Lemma 3.4 yields

$$\ell(\alpha_2[z_1, w]) = \ell(\gamma[z_1, w_1]) + \ell(\eta_1[w_1, w]) \leq 64cd_G(w_1) + c_2d_G(w) \leq (128c + 1)c_2d_G(w).$$

If $w \in \eta_1[u_0, z_2]$, then

$$\ell(\alpha[z_2, w]) = \ell(\eta_1[z_2, w]) \leq c_2d_G(w).$$

Hence α_2 is the desired arc.

Subcase 3.11. $U \cap \gamma[z_1, w_1] \neq \emptyset$.

If $z_1 \in U$, then let $y_1 = z_1$. Otherwise, let y_1 be the first point in $\gamma[z_1, w_1]$ along the direction from z_1 to w_1 such that

$$d_D(y_1) = 64d_G(y_1). \tag{3.12}$$

We first consider the case:

$$\mathbb{B}(y_1, \frac{1}{32}d_D(y_1)) \cap \mathbb{B}(w_1, \frac{1}{32}d_D(w_1)) \neq \emptyset.$$

By Lemma 3.6, we know that there is a $660c_2^2$-uniform arc η_2 joining y_1 and z_2 in G, then let $\alpha_3 = \gamma[z_1, y_1] \cup \eta_2$. Here and in the following, we assume that $\gamma[z_1, y_1] = \{z_1\}$ if $z_1 = y_1$.

If $y_1 = z_1$, then $\alpha_3 = \eta_2$, and obviously, it satisfies Lemma 3.7. If $y_1 \neq z_1$, then replacing c_2 by $660c_2^2$, similar arguments as in Subcase 3.10 show that α_3 is the desired arc.

In the following, we assume

$$\mathbb{B}(y_1, \frac{1}{32}d_D(y_1)) \cap \mathbb{B}(w_1, \frac{1}{32}d_D(w_1)) = \emptyset$$

and we come to construct an arc α_4 satisfying the lemma. We first show the following claim.

Claim 3.13. *There exists a sequence of points $\{y_i\}_{i=1}^{p_1}$ in γ, where $p_1 \geq 3$ is an odd number, satisfying the following conditions.*

1. *$y_1 = z_1$ or y_1 is first point in $\gamma[z_1, w_1]$ from z_1 to w_1 such that $d_D(y_1) = 64d_G(y_1)$;*
2. *For each even number $j \in \{1, 2, \ldots, p_1\}$, $d_G(y_j) \geq \frac{1}{66}d_D(y_j)$ and $d_D(y_{p_1}) \leq 128d_G(y_{p_1})$;*
3. *If $p_1 \geq 5$, then for each even number $j \in \{1, 2, \ldots, p_1 - 2\}$, y_{j+1} is the first point in $\gamma[y_j, w_1]$ from y_j to w_1 such that $d_D(y_{j+1}) = 128d_G(y_{j+1})$;*
4. *p_1 is the smallest integer with $y_{p_1} \in \mathbb{S}(w_1, \frac{1}{32}d_D(w_1))$ or $\mathbb{B}(y_{p_1}, \frac{1}{32}d_D(y_{p_1})) \cap \mathbb{B}(w_1, \frac{1}{32}d_D(w_1)) \neq \emptyset$(see Figures 1 and 2).*

For a proof, we let $y_2 \in \gamma[y_1, w_1] \cap \mathbb{S}(y_1, \frac{1}{32}d_D(y_1)$ be such that

$$\gamma(y_2, w_1] \cap \mathbb{S}(y_1, \frac{1}{32}d_D(y_1)) = \emptyset,$$

where $\gamma(y_2, w_1]$ denote the part γ from y_2 to w_1 such that $y_2 \notin \gamma[y_2, w_1]$. Then

$$d_D(y_2) \leq d_D(y_1) + |y_1 - y_2| = \frac{33}{32}d_D(y_1). \tag{3.14}$$

By Lemma 3.1 and (3.12), we know that there exists one and only one point, say x_s, in $\mathbb{B}(y_1, \frac{1}{6}d_D(y_1)) \cap P_D$, and so

$$d_G(y_2) = |y_2 - x_s| \geq |y_2 - y_1| - |y_1 - x_s| \geq \frac{1}{64}d_D(y_1),$$

which, together with (3.14), shows that

$$d_G(y_2) \geq \frac{1}{66}d_D(y_2). \tag{3.15}$$

By Lemma 3.3 we know that there exists an arc $\beta_1 \subset G$ joining y_1 and y_2 such that β_1 is c_2-uniform in G with $c_2 \leq 18$.

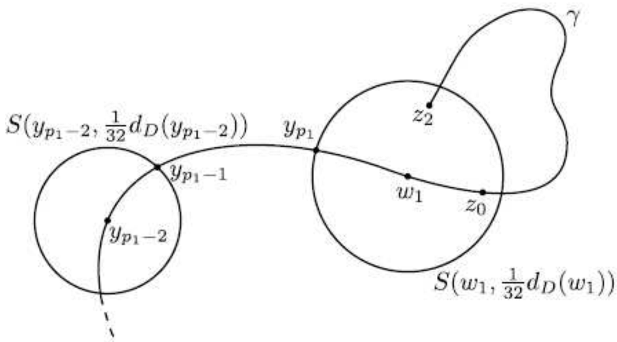

Figure 1: For all $w \in \gamma[y_{p_1-1}, w_1] \setminus \mathbb{B}(w_1, \frac{1}{32}d_D(w_1))$, $d_D(w) \leq 128 d_G(w)$

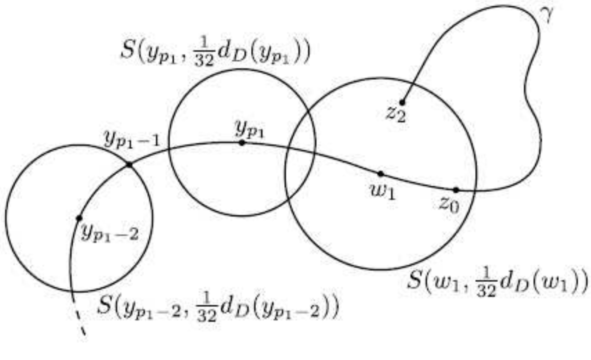

Figure 2: $\mathbb{B}(y_{p_1}, \frac{1}{32}d_D(y_{p_1})) \cap \mathbb{B}(w_1, \frac{1}{32}d_D(w_1)) \neq \emptyset$

If for all $w \in \gamma[y_2, w_1] \setminus \mathbb{B}(w_1, \frac{1}{32}d_D(w_1))$, $d_D(w) \leq 128 d_G(w)$, then the claim obviously holds by letting $y_3 \in \gamma[y_2, w_1] \cap \mathbb{S}(w_1, \frac{1}{32}d_D(w_1)$ and $p_1 = 3$.

If there is some $w_0 \in \gamma[y_2, w_1] \setminus \mathbb{B}(w_1, \frac{1}{32}d_D(w_1))$ such that $d_D(w_0) > 128d_G(w_0)$, then let y_3 be the first point in $\gamma[y_2, w_1]$ from y_2 to w_1 such that $d_D(y_3) = 128d_G(y_3)$. If $\mathbb{B}(y_3, \frac{1}{32}d_D(y_3)) \cap \mathbb{B}(w_1, \frac{1}{32}d_D(w_1)) \neq \emptyset$, then the claim holds and $p_1 = 3$. Otherwise, let $y_4 \in \gamma[y_3, w_1] \cap \mathbb{S}(y_3, \frac{1}{32}d_D(y_3))$ be such that

$$\gamma(y_4, w_1] \cap \mathbb{S}(y_3, \frac{1}{32}d_D(y_3)) = \emptyset.$$

Then by Lemma 3.1, and a similar argument as in the proof of (3.15), we have

$$d_G(y_4) \geq \frac{1}{66}d_D(y_4).$$

If for all $w \in \gamma[y_4, w_1] \setminus \mathbb{B}(w_1, \frac{1}{32}d_D(w_1))$, $d_D(w) \leq 128d_G(w)$, then we complete the proof of the claim by letting $y_5 \in \gamma[y_4, w_1] \cap \mathbb{S}(w_1, \frac{1}{32}d_D(w_1))$.

By repeating this process for finite steps, we get a sequence $\{y_i\}_{i=1}^{p_1} \in \gamma$ satisfying Claim 3.13, where $p_1 < \frac{M}{\log \frac{33}{32}}$, since for each $i \in \{1, 2, \ldots, \frac{p_1-1}{2}\}$,

$$\ell_{k_D}(\gamma[y_{2i-1}, y_{2i}]) \geq \log\Big(1 + \frac{|y_{2i-1} - y_{2i}|}{d_D(y_{2i-1})}\Big) = \log\frac{33}{32},$$

and $M = \ell_{k_D}(\gamma[z_1, z_2])$. Hence Claim 3.13 holds.

We continue the construction of α_4. Let $\gamma_1 = \gamma[z_1, y_1]$ and for each $j \in \{2, \ldots, \frac{p_1+1}{2}\}$, let $\gamma_j = \gamma[y_{2j-2}, y_{2j-1}]$. By Lemma 3.3, we know that for each $j \in \{1, 2, \ldots, \frac{p_1-1}{2}\}$, there exists a c_2-uniform arc $\beta_j \subset G$ joining y_{2j-1} and y_{2j}. By Lemmas 3.3 and 3.6, there exists a $660c_2^2$-uniform arc η_3 joining y_{p_1} and z_2. Take

$$\alpha_4 = \gamma_1 \cup \beta_1 \cup \gamma_2 \cup \ldots \cup \beta_{\frac{p_1-1}{2}} \cup \gamma_{\frac{p_1+1}{2}} \cup \eta_3.$$

Now, we are going to show that α_4 is the desired arc.

First observe that

$$\ell(\alpha_4) \leq 660c_2^2\ell(\gamma). \tag{3.16}$$

To prove that α_4 is a cone arc in G, it is enough to show that

$$\min\{\ell(\alpha_4[z_1, w]), \ell(\alpha_4[w, z_2])\} \leq (2^{18}cc_2^3 + 660c_2^2)d_G(w).$$

If $w \in \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_{\frac{p_1+1}{2}}$, then from the assumption, Claim 3.13 and (3.16), the above inequality obviously holds.

For the case where $w \in \beta_1 \cup \ldots \cup \beta_{\frac{p_1-1}{2}}$, we see that there exists some $i \in \{1, 2, \ldots, \frac{p_1-1}{2}\}$ such that $w \in \beta_i$. By Claim 3.13,

$$d_G(y_{2i}) \geq \frac{1}{66}d_D(y_{2i}) \geq \frac{1}{66}(d_D(y_{2i-1}) - |y_{2i-1} - y_{2i}|) \geq \frac{62}{33}d_G(y_{2i-1}),$$

whence Lemma 3.4 yields

$$d_G(w) \geq \frac{1}{2c_2}\min\{d_G(y_{2i-1}), d_G(y_{2i})\} = \frac{1}{2c_2}d_G(y_{2i-1}), \tag{3.17}$$

which, together with Claim 3.13, leads to

$$\begin{aligned} \ell(\alpha_4[z_1, w]) &\leq c_2\ell(\gamma[z_1, y_{2i-1}]) + c_2|y_{2i-1} - y_{2i}| \\ &\leq 4c_2(32c + 1)d_G(y_{2i-1}) \leq 8c_2^2(32c + 1)d_G(w). \end{aligned}$$

For the remaining case where $w \in \eta_3$, we let $u_0 \in \eta_3$ be a point which bisecting the arclength of η_3. If $w \in \eta_3[z_2, u_0]$, then, obviously,

$$\ell(\alpha_4[z_2, w]) \leq c_2 d_G(w).$$

If $w \in \eta_3[y_3, u_0]$, then by Lemma 3.4 and Claim 3.13, we have

$$\begin{aligned}\ell(\alpha_4[z_1, w]) &\leq c_2\ell(\gamma[z_1, y_{p_1}]) + \ell(\eta_3[y_{p_1}, w])\\ &\leq 128cc_2 d_G(y_{p_1}) + 660c_2^2 d_G(w) \leq (2^{18}cc_2^3 + 660c_2^2)d_G(w).\end{aligned}$$

Case 3.18. *$z_1 \notin \bigcup_{w\in\gamma[z_2,z_0]} \overline{\mathbb{B}}(w, \frac{1}{32}d_D(w))$ and $z_2 \notin \bigcup_{w\in\gamma[z_1,z_0]} \overline{\mathbb{B}}(w, \frac{1}{32}d_D(w))$.*

We may assume that $U \cap \gamma[z_1, z_0] \neq \emptyset$. Then by similar discussions as in the proof of Claim 3.13, we get the following Claim.

Claim 3.19. *There exists a sequence of points $\{u_i\}_{i=1}^{p_2}$ in γ, where $p_2 \geq 2$ is an integer, satisfying the following conditions.*

1. *$u_1 = z_1$ or u_1 is first point in $\gamma[z_1, z_0]$ from z_1 to z_0 such that $d_D(u_1) = 64d_G(u_1)$;*
2. *For each even number $j \in \{1, 2, \dots, p_2\}$, $d_D(u_j) \leq 66d_G(u_j)$ and if p_2 is an odd number, then $d_D(u_{p_2}) \leq 128d_G(u_{p_2})$;*
3. *If $p_2 \geq 4$, then for each even number $j \in \{1, 2, \dots, p_2 - 2\}$, u_{j+1} is the first point in $\gamma[u_j, z_2]$ from u_j to z_2 such that $d_D(u_{j+1}) = 128d_G(u_{j+1})$;*
4. *p_2 is the smallest integer such that $u_{p_2} = z_0$ or $z_0 \in \overline{\mathbb{B}}(u_{p_2-1}, \frac{1}{32}d_D(u_{p_2-1}))$.*

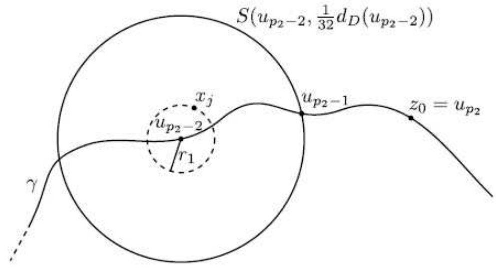

Figure 3: $r_1 = \frac{1}{128}d_D(u_{p_2-2})$ and $x_j \in P_D$

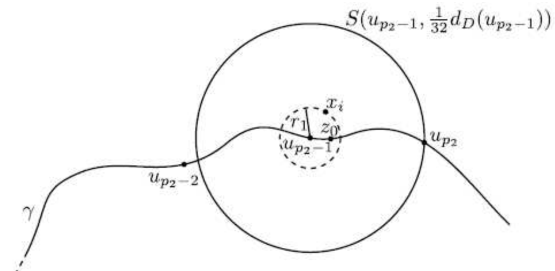

Figure 4: $r_1 = \frac{1}{128}d_D(u_{p_2-1})$ and $x_i \in P_D$

We note that there are two possibilities for u_{p_2} (see Figures 3 and 4) : One is $u_{p_2} = z_0$ and for all $w \in \gamma[u_{p_2-1}, u_{p_2}]$, $d_D(w) \leq 128d_G(w)$; and the other is $u_{p_2} \in \overline{\mathbb{B}}(u_{p_2-1}, \frac{1}{32}d_D(u_{p_2-1})) \cap \gamma[z_0, z_2]$. No matter in which case, the proof is similar. So, in the following, we assume that $u_{p_2} \in \overline{\mathbb{B}}(u_{p_2-1}, \frac{1}{32}d_D(u_{p_2-1})) \cap \gamma[z_0, z_2]$. Then p_2 is an even number, and by Claim 3.19, we note that

$$d_D(u_{p_2}) \leq 66d_G(u_{p_2}). \tag{3.20}$$

We assume that $U \cap \gamma[z_2, u_{p_2}] \neq \emptyset$. Then by similar discussions as in the proof of Claim 3.13, we also get the following claim.

Claim 3.21. *There exists a sequence of points $\{v_i\}_{i=1}^{p_3}$ in $\gamma[z_2, u_{p_2}]$, where $p_3 \geq 2$ is an integer, satisfying the following conditions.*

1. *$v_1 = z_2$ or v_1 is the first point in $\gamma[z_2, u_{p_2}]$ from z_2 to u_{p_2} such that $d_D(v_1) = 64d_G(v_1)$;*
2. *For each even number $j \in \{1, 2, \ldots, p_3\}$, $d_D(v_j) \leq 66d_G(v_j)$ and $d_D(u_{p_3}) \leq 66d_G(u_{p_3})$;*
3. *If $p_3 \geq 4$, then for each even number $j \in \{1, 2, \ldots, p_3 - 2\}$, v_{j+1} is the first point in $\gamma[v_j, u_{p_2}]$ from v_j to u_{p_2} such that $d_D(v_{j+1}) = 128d_G(v_{j+1})$;*
4. *p_3 is the smallest integer such that $u_{p_2} = v_{p_3}$ or $u_{p_2} \in \overline{\mathbb{B}}(v_{p_3-1}, \frac{1}{32}d_D(v_{p_3-1}))$.*

We note that what we consider here is the case when p_2 is an even number, then by using the similar method as in the discussion of Case 3.9, together with Lemma 3.3, Claims 3.19 and 3.21, we construct an arc $\alpha_5 = \alpha_5' \cup \alpha_5''$ in G such that

$$\alpha_5' = \gamma[z_1, u_1] \cup \beta_{1,1}[u_1, u_2] \cup \ldots \cup \beta_{1,i}[u_{2i-1}, u_{2i}] \cup \gamma[u_{2i}, u_{2i+1}] \cup \ldots \cup \beta_{1,\frac{p_2}{2}}[u_{p_2-1}, u_{p_2}]$$

and if p_3 is an odd number, then

$$\alpha_5'' = \gamma[z_2, v_1] \cup \beta_{2,1}[v_1, v_2] \cup \ldots \cup \beta_{2,i}[v_{2i-1}, v_{2i}] \cup \gamma[v_{2i}, v_{2i+1}] \cup \ldots \cup \gamma[v_{p_3-1}, u_{p_2}];$$

if p_3 is en even number, then

$$\alpha_5'' = \gamma[z_2, v_1] \cup \beta_{2,1}[v_1, v_2] \cup \ldots \cup \beta_{2,i}[v_{2i-1}, v_{2i}] \cup \gamma[v_{2i}, v_{2i+1}] \cup \ldots \cup \beta_{2,\frac{p_3}{2}}[v_{p_3-1}, u_{p_2}],$$

where $\beta_{1,i}[u_{2i-1}, u_{2i}]$ and $\beta_{2,i}[v_{2i-1}, v_{2i}]$ are c_2-uniform arcs in G.

Obviously,

$$\ell(\alpha_5) \leq c_2\ell(\gamma).$$

Moreover, if γ is a c-cone arc, we prove that for all $w \in \alpha_5$,

$$\min\{\ell(\alpha_5[z_1, w]), \ell(\alpha_5[w, z_2])\} \leq 8c_2^2(32c + 1)d_G(w).$$

To show this, we only need to consider the case $w \in \alpha_5'$.

If $w \in \gamma \cap \alpha_5'$, then

$$\ell(\alpha_5[z_1, w]) \leq c_2\ell(\gamma[z_1, w]) \leq 128cc_2d_G(w).$$

If $w \in \beta_{1,1}[u_1, u_2] \cup \ldots \cup \beta_{1,\frac{p_2}{2}}[u_{p_2-1}, u_{p_2}]$, then there exists some $i \in \{1, 2, \ldots, \frac{p_2}{2}\}$ such that $w \in \beta_{1,i}$. Hence by Lemma 3.4, Claim 3.19, (3.17) and (3.20), we have for all $w \in \beta_{1,i}[u_{2i-1}, u_{2i}]$,

$$\begin{aligned}
\ell(\alpha_5[z_1, w]) &\leq c_2\ell(\gamma[z_1, u_{2i-1}]) + \ell(\beta_{1,i}[u_{2i-1}, w]) \\
&\leq 128cc_2d_G(u_{2i-1}) + c_2|u_{2i-1} - u_{2i}| \\
&\leq 4c_2(32c + 1)d_G(u_{2i-1}) \leq 8c_2^2(32c + 1)d_G(w).
\end{aligned}$$

Hence, Lemma 3.7 is follows from Cases 3.8, 3.9 and 3.18. □

3.2. Proof of Theorem 2.13

Necessity: Let D be a c-John domain. For given $z_1, z_2 \in G$, there is a c-cone arc γ in D joining z_1 and z_2. By Lemma 3.7, we know that there exists a $(2^{18}cc_2^3 + 660cc_2^2)$-cone arc in G joining z_1 and z_2. Then G is a John domain.

Sufficiency: Let $c = \frac{65}{63}c_1$. For each $z_1, z_2 \in D$, we prove that there exists an arc $\beta \subset D$ joining z_1 and z_2 such that

$$\min\{\ell(\beta[z_1, w]), \ell(\beta[z_2, w])\} \le c d_D(w) \ \text{ for all } w \in \beta. \tag{3.22}$$

If $|z_1 - z_2| \le \frac{1}{4}\max\{d_D(z_1), d_D(z_2)\}$, then let

$$\beta = [z_1, z_2],$$

and obviously,

$$\min\{|z_1 - w|, |z_2 - w|\} \le d_D(w) \ \text{ for all } w \in [z_1, z_2],$$

which shows that (3.22) holds.

In the following, we assume that $|z_1 - z_2| > \frac{1}{4}\max\{d_D(z_1), d_D(z_2)\}$. If $z_1, z_2 \in G$, then let γ be a c_1-cone arc joining z_1 and z_2 in G, and take

$$\beta = \gamma.$$

Then β satisfies (3.22) since $G \subset D$.

If $z_1 \in P_D$ but $z_2 \notin P_D$, then let $x \in G$ be such that $|z_1 - x| = \frac{1}{64}d_D(z_1)$, and γ be a c_1-cone arc joining x and z_2 in G. Take

$$\beta = [z_1, x] \cup \gamma.$$

If $z_1 \notin P_D$ but $z_2 \in P_D$, then let $y \in G$ be such that $|z_2 - y| = \frac{1}{64}d_D(z_2)$, and γ be a c_1-cone arc joining y and z_1 in G. Take

$$\beta = [z_2, y] \cup \gamma.$$

If $z_1,\ z_2 \in P_D$, then let $x \in G$ such that $|z_1 - x| = \frac{1}{64}d_D(z_1)$ and $y \in G$ such that $|z_2 - y| = \frac{1}{64}d_D(z_2)$, and γ be a c_1-cone arc joining x and y in G. Take

$$\beta = [z_1, x] \cup \gamma \cup [y, z_2].$$

To prove that these three arcs β are cone arcs in D, it is enough to consider the third case where $z_1\ z_2 \in P_D$. In this case, $\beta = [z_1, x] \cup \gamma \cup [y, z_2]$.

Let $z_0 \in \gamma$ be a point which bisecting the arclength of γ. It suffices to prove that for all $w \in \beta[z_1, z_0]$,

$$\ell(\beta[z_1, w]) \le \frac{65}{63}c_1 d_D(w).$$

On one hand, if $w \in [z_1, x]$, then

$$\ell(\beta[z_1, w]) = |z_1 - w| \le \frac{1}{64}d_D(z_1) \le \frac{1}{63}d_D(w). \tag{3.23}$$

On the other hand, if $w \in \gamma[x, z_0]$, then Lemma 3.4 shows

$$d_D(w) \ge \frac{1}{2c_1}d_D(x),$$

which, together with (3.23), shows that

$$\ell(\beta[z_1, w]) = |z_1 - x| + \ell(\gamma[x, w]) \le \frac{1}{63}d_D(x) + c_1 d_D(w) \le \frac{65}{63}c_1 d_D(w).$$

Hence (3.22) holds, and so the proof of Theorem 2.13 is complete.

3.3. Proof of Theorem 2.14

We first prove the necessary part of the theorem, that is, if D is an inner c-uniform domain, we need to prove that each pair of points $z_1, z_2 \in G$ can be joined by an inner c_1-uniform arc in G, where $c_1 = 2^{18}c^2c_2^3 + 660c_2^2$, and c_2 $(2 \le c_2 \le 18)$ is a constant from Lemma 3.3.

For $z_1, z_2 \in G$, since D is an inner c-uniform domain, then there is an arc γ joining z_1 and z_2 in D such that for all $w \in \gamma$

$$\min\{\ell(\gamma[z_1, w]), \ell(\gamma[z_2, w])\} \le c d_D(w)$$

and

$$\ell(\gamma) \le c\lambda_D(z_1, z_2).$$

By Lemma 3.7, we know that there exists an arc $\alpha \subset G$ such that α is a $(2^{18}c^2c_2^3 + 660c_2^2)$-cone arc in G and $\ell(\alpha) \le 660c_2^2\ell(\gamma)$. Hence

$$\ell(\alpha) \le 660c_2^2\ell(\gamma) \le 660cc_2^2\lambda_D(z_1, z_2) \le 660cc_2^2\lambda_G(z_1, z_2),$$

which shows that α is the desired arc.

To prove the sufficient part of Theorem 2.14, we need to prove that for each $z_1, z_2 \in D$, there exists an arc β joining z_1 and z_2 in D such that

$$\min\{\ell(\beta[z_1, w]), \ell(\beta[z_2, w])\} \le (1485c_1c_2^2 + \frac{1}{8})d_D(w) \text{ for all } w \in \beta, \tag{3.24}$$

and

$$\ell(\beta) \le (1485c_1c_2^2 + \frac{1}{8})\lambda_D(z_1, z_2). \tag{3.25}$$

If $|z_1 - z_2| \le \frac{1}{4}\max\{d_D(z_1), d_D(z_2)\}$, then let

$$\beta = [z_1, z_2].$$

Obviously, β satisfies (3.24) and (3.25).

In the following, we assume that

$$|z_1 - z_2| > \frac{1}{4}\max\{d_D(z_1), d_D(z_2)\}. \tag{3.26}$$

We divide the proof of this case into two parts.

Case 3.27. $z_1, z_2 \in G$.

Since G is an inner c_1-uniform domain, then there is a c_1-cone arc γ joining z_1 and z_2 in G such that

$$\ell(\gamma) \le c_1\lambda_G(z_1, z_2). \tag{3.28}$$

Obviously, γ satisfies (3.24) since $G \subset D$. In order to prove γ satisfies (3.25), we let α be an arc joining z_1 and z_2 in D with

$$\ell(\alpha) \le 2\lambda_D(z_1, z_2). \tag{3.29}$$

By Lemma 3.7, we join z_1 and z_2 by an arc $\alpha_1 \subset G$ such that

$$\ell(\alpha_1) \le 660c_2^2\ell(\alpha),$$

which, together with (3.28) and (3.29), shows that

$$\ell(\gamma) \le c_1\lambda_G(z_1, z_2) \le c_1\ell(\alpha_1) \le 1320c_1c_2^2\lambda_D(z_1, z_2).$$

Now we take $\beta = \gamma$. Obviously, β satisfies (3.24) and (3.25).

Case 3.30. *$z_1 \notin G$ or $z_2 \notin G$.*

Without loss of generality, we may assume that $z_1 \notin G$ and $z_2 \notin G$, since the proof for the case $z_1 \in G$, $z_2 \notin G$ or $z_1 \notin G$, $z_2 \in G$ is similar. Let $x,\ y \in G$ be such that

$$|z_1 - x| = \frac{1}{64} d_D(z_1), \qquad |z_2 - y| = \frac{1}{64} d_D(z_2), \tag{3.31}$$

and let γ be an inner c_1-uniform arc joining x and y in G. Take

$$\beta = [z_1, x] \cup \gamma \cup [y, z_2].$$

By Theorem 2.13 and its proof, we know that β satisfies (3.24). It follows from Case 3.27 that

$$\ell(\gamma) \le 1320 c_1 c_2^2 c_2 \lambda_D(x, y),$$

which, together with (3.26) and (3.31), shows that

$$\begin{aligned} \ell(\beta[z_1, z_2]) &= |z_1 - x| + \ell(\gamma[x, y]) + |y - z_2| \\ &\le \frac{1}{8}|z_1 - z_2| + 1320 c_1 c_2^2 \lambda_D(x, y) \\ &\le (1485 c_1 c_2^2 + \frac{1}{8}) \lambda_D(z_1, z_2), \end{aligned}$$

from which we see that β satisfies (3.25). Hence the proof of Theorem 2.14 is complete.

4. Proof of Theorem 2.15

Definition 4.1. *([9]) A domain D is said to be a ψ-John domain if ψ is an increasing self-homeomorphism of $[0, \infty]$ and if for some fixed $x_0 \in D$ and for all $y \in D$, we have*

$$k_D(x_0, y) \le \psi\left(\frac{|x_0 - y|}{\min\{d_D(x_0), d_D(y)\}}\right).$$

The following lemma follows immediately from (2.1).

Lemma 4.2. *If $\psi : [0, \infty] \to [0, \infty]$ is a homeomorphism such that a domain is a ψ-John domain, then $\log(1+t) \le \psi(t)$ holds for all $t \ge 0$.*

By [28, Theorem 2.23], we have the following lemma which is useful for the discussion in the rest of this section.

Lemma 4.3. *Suppose that $D \subset E$ is a domain and that $D_1 \subset D$ is a c-uniform domain. Then for all $x, y \in D_1$,*

$$k_D(x, y) \le c_1 j_D(x, y)$$

with $c_1 = c_1(c) \le 7c^3$.

From Lemma 3.3 and Lemma 4.3, we easily get the following corollary.

Corollary 4.4. *Suppose that $D \subset E$ is a domain and $G = D \setminus P_D$. For $x, y \in D$, if $d_D(x) = 128 d_G(x)$ and $y \in \overline{\mathbb{B}}(x, \frac{1}{32} d_D(x))$, then $k_G(x, y) \le \mu j_G(x, y)$, where $\mu \le 7 \times 18^3$ is a constant.*

Meanwhile, [32, Lemma 3.7(2)] yields the following corollary.

Corollary 4.5. *Suppose that $D \subset E$ is a domain. For $x, y \in D$, if $|x - y| \le \frac{1}{2} \min\{d_D(x), d_D(y)\}$, then $k_D(x, y) \le 2 j_D(x, y)$.*

Before the statement of our main result in this section, we prove the following two lemmas.

Lemma 4.6. *Let D be a domain and $G = D \setminus P_D$. For each $x \in D$, there exists some point $w \in \mathbb{S}(x, \frac{1}{32}d_D(x))$ such that*

$$\frac{1}{48}d_D(x) < \frac{1}{33}d_D(w) \le d_G(w) \le \frac{33}{31}d_D(w).$$

Proof. Let $x \in D$. By Lemma 3.1, there exists at most one point in $P_D \cap \mathbb{B}(x, \frac{1}{6}d_D(x))$. On one hand, if $P_D \cap \mathbb{B}(x, \frac{1}{6}d_D(x)) = \emptyset$, let $w \in \mathbb{S}(x, \frac{1}{32}d_D(x))$. On the other hand, if $P_D \cap \mathbb{B}(x, \frac{1}{6}d_D(x)) \neq \emptyset$, then there exists one and only one point x_i in $P_D \cap \mathbb{B}(x, \frac{1}{6}d_D(x))$. Let l be a line determined by x and x_i, and take $w \in l \cap \mathbb{S}(x, \frac{1}{32}d_D(x))$ such that $d_G(w) \ge \frac{1}{32}d_D(x)$. Then

$$d_D(w) \le d_D(x) + |w - x| \le \frac{33}{32}d_D(x),$$

and so

$$d_G(w) \ge \frac{1}{32}d_D(x) \ge \frac{1}{33}d_D(w).$$

Hence

$$d_D(w) \ge d_D(x) - |x - w| = \frac{31}{32}d_D(x) \ge \frac{31}{33}d_G(w)$$

and

$$d_G(w) \ge \frac{1}{33}d_D(w) > \frac{1}{48}d_D(x).$$

The proof is complete. □

Lemma 4.7. *Let D be a domain and $G = D \setminus P_D$. For each $x \in D$ and $w \in \mathbb{S}(x, \frac{1}{32}d_D(x))$, if $d_D(x) \ge 128 d_G(x)$, then $d_G(w) \ge \frac{1}{44}d_D(w)$.*

Proof. Observe first that

$$d_D(w) \le d_D(x) + |w - x| \le \frac{33}{32}d_D(x).$$

Let $x \in D$. Since $d_D(x) \ge 128 d_G(x)$, then by Lemma 3.1, there exists one and only one point, namely x_i, in $P_D \cap \mathbb{B}(x, \frac{1}{6}d_D(x))$. Hence

$$d_G(w) = |w - x_i| \ge |x - w| - |x - x_i| \ge \frac{3}{128}d_D(x) \ge \frac{1}{44}d_D(w).$$

Thus the proof of the lemma is complete. □

Proof of Theorem 2.15. We first prove the necessary part of the theorem. For this, we assume that D is ψ-John domain with center x_0, where $x_0 \in D$. By Lemma 4.6, there exists some point w_0 in $\mathbb{S}(x_0, \frac{1}{32}d_D(x_0))$ such that

$$\frac{1}{48}d_D(x_0) < \frac{1}{33}d_D(w_0) \le d_G(w_0) \le \frac{33}{31}d_D(w_0) \tag{4.8}$$

and

$$d_D(w_0) \le d_D(x_0) + |x_0 - w_0| \le \frac{33}{32}d_D(x_0). \tag{4.9}$$

We come to prove that there exists some homeomorphism ψ' of $[0, \infty)$ such that G is a ψ'-John domain with center w_0. That is, we need to find a homeomorphism ψ' of $[0, \infty)$ such that for each $y \in G$,

$$k_G(w_0, y) \le \psi'\left(\frac{|w_0 - y|}{\min\{d_G(w_0), d_G(y)\}}\right). \tag{4.10}$$

For $y \in G$, if $|w_0 - y| \le \frac{1}{2}\max\{d_G(w_0), d_G(y)\}$, then Lemmas 4.2 and Corollary 4.5 show that

$$k_G(w_0, y) \le 2\log\left(1 + \frac{|w_0 - y|}{\min\{d_G(w_0), d_G(y)\}}\right) \le 2\psi\left(\frac{|w_0 - y|}{\min\{d_G(w_0), d_G(y)\}}\right),$$

which shows that (4.10) holds with $\psi_1(t) = 2\psi(t)$. Hence, in the following, we assume that

$$|w_0 - y| > \frac{1}{2}\max\{d_G(w_0), d_G(y)\}. \tag{4.11}$$

Let γ be a 2-neargeodesic joining w_0 and y in D. We leave the proof for a moment and prove the following claim.

Claim 4.12. *There exists a sequence of points $\{w_i\}_{i=0}^{p}$ in γ, where $p \ge 1$ is an integer, satisfying the following conditions.*

1. *For each even number $j \in \{0, \ldots, p-1\}$, $d_D(w_j) \le 44 d_G(w_j)$;*
2. *For each even number $j \in \{0, \ldots, p-1\}$, w_{j+1} is the first point in $\gamma[w_j, y]$ from w_j to y such that $d_D(w_{j+1}) = 128 d_G(w_{j+1})$;*
3. *If $p \ge 2$, then for each even number $j \in \{1, \ldots, p\}$, $w_j \in \overline{\mathbb{B}}(w_{j-1}, \frac{1}{32}d_D(w_{j-1}))$.*

Obviously, by (4.8), we have

$$d_D(w_0) \le 33 d_G(w_0) < 44 d_G(w_0).$$

If for all $w \in \gamma$, $d_D(w) < 128 d_G(w)$, then let $w_1 = y$. Then the claim obviously holds with $p = 1$. If there exists some point $v_0 \in \gamma$ such that $d_D(v_0) \ge 128 d_G(v_0)$, then by (4.8), we can choose a point $w_1 \in \gamma$ be the first point from w_0 to y such that

$$d_D(w_1) = 128 d_G(w_1).$$

If $y \in \overline{\mathbb{B}}(w_1, \frac{1}{32}d_D(w_1))$, then the claim holds by letting $w_2 = y$, and then $p = 2$. Otherwise, let $w_2 \in \gamma \cap \mathbb{S}(w_1, \frac{1}{32}d_D(w_1))$ such that

$$\gamma[w_2, y] \cap \mathbb{B}(w_1, \frac{1}{32}d_D(w_1)) = \emptyset.$$

Then by Lemma 4.7, we have

$$d_G(w_2) \ge \frac{1}{44} d_D(w_2).$$

If for all $w \in \gamma[w_2, y]$,

$$d_D(w) \le 128 d_G(w),$$

then the claim holds with $w_3 = y$, and then $p = 3$. Otherwise, let w_3 be the first point in $\gamma[w_2, y]$ from w_2 to y such that

$$d_G(w_3) = \frac{1}{128} d_D(w_3).$$

By repeating this process for finite steps, we get a sequence $\{w_i\}_{i=0}^{p} \in \gamma$ satisfying the claim, where $p < \frac{M}{\log \frac{33}{32}}$, since for each even number $i \in \{1, 2, \ldots, p\}$,

$$\ell_{k_D}(\gamma[w_{i-1}, w_i]) \ge \log\left(1 + \frac{|w_{i-1} - w_i|}{d_D(w_{i-1})}\right) = \log\frac{33}{32},$$

and $M = \ell_{k_D}(\gamma[w_0, y])$. Hence Claim 4.12 holds.

Now, we come back to the proof of the necessary part of the theorem. By Claim 4.12, we know that for each even number $j \in \{0, \ldots, p-1\}$ the following holds: for all $w \in \gamma[w_j, w_{j+1}]$,

$$d_D(w) \le 128 d_G(w).$$

Hence

$$k_G(w_j, w_{j+1}) \le \int_{\gamma[w_j,w_{j+1}]} \frac{|dw|}{d_G(w)} < 128\ell_{k_D}(\gamma[w_j, w_{j+1}]) \le 256k_D(w_j, w_{j+1}). \tag{4.13}$$

By Claim 4.12, we also know that if $p \ge 2$, then for each even number $j \in \{1, \dots, p\}$, $w_j \in \overline{\mathbb{B}}(w_{j-1}, \frac{1}{32}d_D(w_{j-1}))$. Hence by Corollary 4.4 and Claim 4.12, we have

$$k_G(w_{j-1}, w_j) \le \mu \log\left(1 + \frac{|w_{j-1} - w_j|}{\min\{d_G(w_{j-1}), d_G(w_j)\}}\right) \le 128\mu k_D(w_{j-1}, w_j), \tag{4.14}$$

where μ is the constant from Corollary 4.4.

Now we divided the rest part of proof into two cases.

Case 4.15. $d_G(y) \ge \frac{1}{128}d_D(y)$.

By (4.8) and (4.11), we have

$$|x_0 - w_0| = \frac{1}{32}d_D(x_0) < \frac{3}{2}d_G(w_0) \le 3|y - w_0| \tag{4.16}$$

and

$$|x_0 - y| \le |x_0 - w_0| + |y - w_0| \le 4|y - w_0|, \tag{4.17}$$

which, together with (4.8), (4.9), Claim 4.12, (4.13) and (4.14), shows that

$$\begin{aligned} k_G(w_0, y) &\le \sum_{i=0}^{p-1} k_G(w_i, w_{i+1}) \le 256\mu \sum_{i=0}^{p-1} k_D(w_i, w_{i+1}) \\ &\le 512\mu k_D(w_0, y) \le 512\mu(k_D(x_0, w_0) + k_D(x_0, y)) \\ &\le 512\mu\psi\left(\frac{|x_0 - w_0|}{\min\{d_D(x_0), d_D(w_0)\}}\right) \\ &+ 512\mu\psi\left(\frac{|x_0 - y|}{\min\{d_D(x_0), d_D(y)\}}\right) \\ &\le \psi_2\left(\frac{|y - w_0|}{\min\{d_G(y), d_G(w_0)\}}\right), \end{aligned} \tag{4.18}$$

where $\psi_2(t) = 1024\mu\psi(8t)$.

Case 4.19. $d_G(y) < \frac{1}{128}d_D(y)$.

In this case, by Claim 4.12, we see that p must be an even number and $p \ge 2$, and then $y \in \overline{\mathbb{B}}(w_{p-1}, \frac{1}{32}d_D(w_{p-1}))$. If $w_0 \in \overline{\mathbb{B}}(w_{p-1}, \frac{1}{32}d_D(w_{p-1}))$, then by Corollary 4.4 and Claim 4.12, we get

$$k_G(w_0, y) \le \mu \log\left(1 + \frac{|w_0 - y|}{\min\{d_G(w_0), d_G(y)\}}\right) \le \psi_3\left(\frac{|w_0 - y|}{\min\{d_G(w_0), d_G(y)\}}\right),$$

where $\psi_3(t) = \mu\psi(t)$.

If $w_0 \notin \overline{\mathbb{B}}(w_{p-1}, \frac{1}{32}d_D(w_{p-1}))$, then by (4.17),

$$d_G(y) < \frac{1}{128}d_D(y) \le \frac{1}{128}(d_D(w_{p-1}) + |w_{p-1} - y|) < \frac{1}{64}d_D(w_{p-1})$$

and

$$|w_{p-1} - x_0| \le |w_{p-1} - y| + |x_0 - y| < 5|y - w_0|,$$

which, together with Lemma 4.2, (4.8), (4.13), (4.14) and (4.16), shows that

$$\begin{aligned} k_G(w_0, y) &\le 256\mu \sum_{i=0}^{p-2} k_D(w_i, w_{i+1}) + k_G(w_{p-1}, y) \\ &\le 512\mu k_D(w_0, w_{p-1}) + \mu \log\left(1 + \frac{|w_{p-1} - y|}{\min\{d_G(w_{p-1}), d_G(y)\}}\right) \\ &\le 512\mu(k_D(x_0, w_0) + k_D(x_0, w_{p-1})) + \mu \log\left(1 + \frac{|w_{p-1} - y|}{\min\{d_G(w_{p-1}), d_G(y)\}}\right) \\ &\le 512\mu\left(\psi\left(\frac{|x_0 - w_0|}{\min\{d_D(x_0), d_D(w_0)\}}\right) + \psi\left(\frac{|x_0 - w_{p-1}|}{\min\{d_D(x_0), d_D(w_{p-1})\}}\right)\right) \\ &+ \mu \log\left(1 + \frac{|w_{p-1} - y|}{\min\{d_G(w_{p-1}), d_G(y)\}}\right) \\ &\le \psi_1\left(\frac{|w_0 - y|}{\min\{d_G(w_0), d_G(y)\}}\right), \end{aligned}$$

where $\psi_4(t) = 1025\mu\psi(8t)$. Hence (4.10) holds with $\psi'(t) = 1025\mu\psi(8t)$.

Now we are going to prove the sufficiency part of Theorem 2.15.

Assume that G is ψ_1-John domain with center z_0, where $z_0 \in G$. By Lemma 4.6, there exists some point y_0 in $\mathbb{S}(z_0, \frac{1}{32}d_D(z_0))$ such that

$$\frac{1}{48}d_D(z_0) < \frac{1}{33}d_D(y_0) \le d_G(y_0) \le \frac{33}{31}d_D(y_0). \tag{4.20}$$

We show that there exists a homeomorphism ψ of $[0, \infty)$ such that D is a ψ-John domain with center y_0. By the necessary part of the theorem, we know that $G_1 = G \setminus \{z_0\}$ is a ψ'-John domain with center y_0, where $\psi'(t) = 1025\mu\psi_1(8t)$.

For $y \in D$, if $|y_0 - y| \le \frac{1}{2}\max\{d_D(y_0), d_D(y)\}$, then Lemma 4.2 and Corollary 4.5 show that

$$k_D(y_0, y) \le 2\log\left(1 + \frac{|y_0 - y|}{\min\{d_D(y_0), d_D(y)\}}\right) \le \psi_1'\left(\frac{|y_0 - y|}{\min\{d_D(y_0), d_D(y)\}}\right),$$

where $\psi_1'(t) = 2\psi_1(t)$ and the constant 2 is from Corollary 4.5. In the following, we assume that

$$|y_0 - y| \ge \frac{1}{2}\max\{d_D(y), d_D(y_0)\}. \tag{4.21}$$

If $d_D(y) \le 62d_G(y)$, then by (4.20),

$$|y - z_0| \ge |y - y_0| - |y_0 - z_0| \ge \frac{1}{128}d_D(z_0).$$

Now we claim that

$$d_G(y) \le 129d_{G_1}(y). \tag{4.22}$$

In fact, if $d_G(y) = d_{G_1}(y)$, then the above inequality is obvious. If $d_G(y) > d_{G_1}(y)$, then $d_{G_1}(y) = |y - z_0|$. Hence

$$d_G(y) \le d_D(z_0) + |z_0 - y| \le 129d_{G_1}(y),$$

which shows (4.22).

Similarly, we have

$$d_G(y_0) \le 129 d_{G_1}(y_0). \tag{4.23}$$

Hence (4.20) and (4.22) yield

$$k_D(y_0, y) \le k_{G_1}(y_0, y) \le \psi'\left(\frac{|y_0 - y|}{\min\{d_{G_1}(y_0), d_{G_1}(y)\}}\right) \le \psi'_2\left(\frac{|y_0 - y|}{\min\{d_D(y_0), d_D(y)\}}\right),$$

where $\psi'_2(t) = 1025\mu\psi_1(2^{15}t)$.

If $d_D(y) \ge 62 d_G(y)$, then for $y_1 \in \mathbb{S}(y, \frac{1}{16}d_D(y))$, Lemma 3.1 implies

$$d_D(y_1) \le d_D(y) + |y_1 - y| \le 32 d_G(y_1). \tag{4.24}$$

Hence, a similar proof as to (4.22) leads to

$$d_G(y_1) \le 129 d_{G_1}(y_1),$$

which, together with Corollary 4.5, (4.20), (4.21), (4.23) and (4.24), shows that

$$\begin{aligned} k_D(y_0, y) &\le k_{G_1}(y_0, y_1) + k_D(y_1, y) \\ &\le \psi'\left(\frac{|y_0 - y_1|}{\min\{d_{G_1}(y_0), d_{G_1}(y_1)\}}\right) + 2\log\left(1 + \frac{|y_1 - y|}{\min\{d_D(y_1), d_D(y)\}}\right) \\ &\le \psi'_3\left(\frac{|y_0 - y|}{\min\{d_D(y_0), d_D(y)\}}\right), \end{aligned}$$

where $\psi'_3(t) = 1025(\mu + 2)\psi_1(2^{15}t)$, and μ is the constant from Corollary 4.4. By letting $\psi(t) = 1025(\mu + 2)\psi_1(2^{15}t)$, we get the sufficient part of the theorem. Hence the proof of the theorem is complete. □

Remark 4.25. *Let $\psi : [0, \infty] \to [0, \infty]$ be a homeomorphism and c, λ_1, λ_2 be positive constants. We define the following class:*

$$\Psi_{\lambda_1,\lambda_2} = \{\psi : \lambda_1 \le \frac{\psi(ct)}{\psi(t)} \le \lambda_2\}.$$

The proof of Theorem 2.15 yields the following quantitative statement: $\psi_1(t) = b_1\psi(b_2 t)$ and $\psi_2(t) = b_3\psi(b_4 t)$ for some positive constants b_j. Thus we see that if D is a ψ-John domain with $\psi \in \Psi_{\lambda_1,\lambda_2}$, then $D \setminus P_D$ is a ψ_1-John domain with $\psi_1 \in \Psi_{\lambda_1,\lambda_2}$. The converse implication also holds.

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