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On Modified Srivastava-Gupta Operators

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Abstract. Very recently the modified form of Srivastava-Gupta operators was studied in order to preserve the linear functions. Here, we estimate the rate of approximation for functions having bounded derivatives of the modified form.

1. Introduction

Srivastava and Gupta [17] introduced a general family of the summation-integral type operators. As the special cases of such an important operators, one obtains the Phillips operators [15] (see also [5], [3]), the Baskakov-Durrmmeyer type operators [8] and Bernstein-Durrmeyer operators [9]. Some other classes of linear positive operators were considered in [4] and [11]. These operators were termed as Srivastava-Gupta operators in [12], later some other approximation properties of these operators have been discussed in [18]. In the recent years several researchers have worked on different operators in this direction and they have obtained various approximation properties, we mention some of important papers and recent book as [1], [2], [14], [13], [16] , [20] and [6]. Very recently [19] considered the slight modification of Srivastava-Gupta operators and she obtained some direct results. The modified form is given as

$$G_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c)f\left(\frac{(n-c)t}{n}\right)dt + p_{n,0}(x,c)f(0), \tag{1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0 \\ (1+cx)^{-n/c}, & c \in \mathbb{N} := \{1,2,3,....\} \\ (1-x)^n, & c = -1. \end{cases}$$

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Here we discuss the cases $c \in \{0, 1, 2, 3, \cdots\}$ for $x \in [0, \infty)$. In case $c = -1$ the interval reduces to $[0, 1]$. These operators reproduce the constant as well as linear functions and are different from those considered in [10] and [7] etc.
We consider:

$$\Phi_{DB} = \{f : f(x) - f(0) = \int_0^x \phi(t)dt;\ f(t) = O(t^r),\ t \to \infty\},$$

where ϕ is bounded on every finite subinterval of the interval $[0, \infty)$. For fixed $x \in [0, \infty)$, $\lambda \geq 0$ and $f \in \Phi_{DB}$, we define the metric as

$$\Omega(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda] \cap [0,\infty)} |f(t) - f(x)|.$$

Lemma 1.1. *For $x \in [0, \infty)$, $\psi_x(t) = t - x$ and $c \in \mathbb{N} \cup \{-1, 0\}$, we immediately have by simple computation*

$$G_{n,c}(1, x) = 1,\ \ G_{n,c}(\psi_x, x) = 0,$$

$$G_{n,c}\left(\psi_x^2, x\right) = \frac{x^2 c(2n - c) + 2(n - c)x}{n(n - 2c)}.$$

For $r = 0, 1, 2, \ldots$, we have

$$G_{n,c}(\psi_x^r, x) = O\left(n^{-[(r+1)/2]}\right).$$

Application of Schwarz inequality, lead us to

$$G_{n,c}(|\psi_x^r|, x) \leq \sqrt{G_{n,c}(\psi_x^{2r}, x)} = O\left(n^{-r/2}\right).$$

For n sufficiently large, we can write

$$G_{n,c}(|\psi_x|, x) \leq \sqrt{\frac{2x(1 + cx)}{n}}.$$

The operators (1) has the form

$$G_{n,c}(f, x) = \int_0^\infty K_{n,c}(x, t)f(t)dt,$$

where the kernel $K_{n,c}(x, t)$ is given by

$$K_{n,c}(x, t) = \sum_{k=1}^{\infty} p_{n,k}(x, c)p_{n+c,k-1}(t, c) + p_{n,0}(x, c)\delta(t),$$

by $\delta(t)$ we mean the Dirac delta function.

Lemma 1.2. *For fixed $x \in (0, \infty)$ and n sufficiently large, one has*

$$\lambda_{n,c}(x, y) = \int_0^y K_{n,c}(x, t)dt \leq \frac{2x(1 + cx)}{n(x - y)^2}, 0 \leq y < x,$$

$$1 - \lambda_{n,c}(x, z) = \int_z^\infty K_{n,c}(x, t)dt \leq \frac{2x(1 + cx)}{n(z - x)^2}, x < z < \infty.$$

The proof of the above lemma is obvious, we just have to use Lemma 1.1.

In the present article we estimate the rate of approximation for functions belonging to the class Φ_{DB}.

2. Rate of Approximation

Theorem 2.1. *Let $f \in \Phi_{DB}$, $x \in (0,\infty)$ be fixed. Then for n sufficiently large, we have*

$$\begin{aligned}
&\left| G_{n,c}(f,x) - f(x) - \frac{\phi(x+)-\phi(x-)}{2}\sqrt{\frac{2x(cx+1)}{n}} \right| \\
\leq\ & \frac{2((3c+1)x+3)}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left(\phi_x, \frac{x}{k}\right) + O(n^{-r}),
\end{aligned}$$

where

$$\phi_x(t) = \begin{cases} \phi(t) - \phi(x-), & 0 \leq t < x \\ 0, & t = x \\ \phi(t) - \phi(x+), & x < t < \infty \end{cases}.$$

Proof. By simple calculation, we have

$$\begin{aligned}
G_{n,c}(f,x) - f(x) &\leq \frac{\phi(x+)-\phi(x-)}{2} G_{n,c}(|t-x|,x) + \frac{\phi(x+)+\phi(x-)}{2} G_{n,c}(t-x,x) \\
&\quad + \left(-\int_0^x + \int_x^{2x} + \int_{2x}^{\infty}\right)\left(\int_x^t \phi_x(u)du\right) d_t(\lambda_{n,c}(x,t)) \\
&=: \frac{\phi(x+)-\phi(x-)}{2}\sqrt{\frac{2x(cx+1)}{n}} - S_1 + S_2 + S_3
\end{aligned} \tag{2}$$

First we integrate by parts to have

$$\begin{aligned}
S_1 &= \int_t^x \phi_x(u)du\lambda_{n,c}(x,t)\Big|_0^x + \int_0^x \lambda_{n,c}(x,t)\phi_x(t)dt \\
&= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x\right)\lambda_{n,c}(x,t)\phi_x(t)dt.
\end{aligned}$$

As $\lambda_{n,c}(x,t) \leq 1$, the monotonicity of $\Omega_x(\phi_x,\lambda)$ and the definition of $\phi_x(t)$, implies that

$$\begin{aligned}
\left|\int_{x-x/\sqrt{n}}^x \lambda_{n,c}(x,t)\phi_x(t)dt\right| &\leq \frac{x}{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{\sqrt{n}}\right) \\
&\leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x\left(\phi_x, \frac{x}{k}\right).
\end{aligned}$$

Setting $t = \frac{x}{x-u}$ and applying Lemma 1.2, we obtain

$$\begin{aligned}
\left|\int_0^{x-x/\sqrt{n}} \lambda_{n,c}(x,t)\phi_x(t)dt\right| &\leq \frac{2x(1+cx)}{n}\int_0^{x-x/\sqrt{n}} \frac{\Omega_x\left(\phi_x, x-t\right)}{(x-t)^2}dt \\
&\leq \frac{2(cx+1)}{n}\int_1^{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{u}\right)du \\
&\leq \frac{2(cx+1)}{n}\sum_{k=1}^{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{k}\right).
\end{aligned}$$

Combining the above estimates, we have

$$|S_1| \leq \frac{2((c+1)x+1)}{n}\sum_{k=1}^{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{k}\right). \tag{3}$$

Next,

$$\begin{aligned} S_2 &= \int_x^{2x}\left(\int_x^t \phi_x(u)du\right) d_t\left(\lambda_{n,c}(x,t)\right) \\ &= -\int_x^{2x}\left(\int_x^t \phi_x(u)du\right) d_t\left(1-\lambda_{n,c}(x,t)\right) \\ &= -\int_x^{2x} \phi_x(u)du\left(1-\lambda_{n,c}(x,2x)\right) + \int_x^{2x} \phi_x(t)\left(1-\lambda_{n,c}(x,t)\right)dt. \end{aligned}$$

Using Lemma 1.2, we have

$$\left|-\int_x^{2x} \phi_x(u)du\left(1-\lambda_{n,c}(x,2x)\right)\right| \leq x\Omega_x(\phi_x,x)\frac{2x(cx+1)}{nx^2} = \frac{2(1+cx)}{n}\Omega_x(\phi_x,x).$$

Also, we have

$$\left|\int_x^{2x} \phi_x(t)\left(1-\lambda_{n,c}(x,t)\right)dt\right| \leq \frac{2(cx+1)}{n}\sum_{k=1}^{\sqrt{n}} \Omega_x\left(\phi_x,\frac{x}{k}\right).$$

Thus, we get

$$|S_2| \leq \frac{2(1+cx)}{n}\Omega_x(\phi_x,x) + \frac{2(cx+1)}{n}\sum_{k=1}^{\sqrt{n}} \Omega_x\left(\phi_x,\frac{x}{k}\right). \tag{4}$$

By the assumption $f(t)=O(t^{2r})$ as $t\to\infty$, for a certain constant $M>0$ depending only on f,x,r, we get

$$|S_3| = C\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_{2x}^{\infty} p_{n+c,k-1}(t,c)t^{2r}dt.$$

Making use of Lemma 1.1 and the inequality $t\leq 2(t-x)$ for $t\geq 2x$, with $C'=2^{2r}C$, we immediately have

$$|S_3| \leq C'\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c)(t-x)^{2r}dt = O(n^{-r}). \tag{5}$$

Combining the estimates in (2), (3), (4) and (5) the proof follows. □

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