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Properties For an Integral Operator on the Class of Close-to-Convex Functions

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Abstract. The purpose of this paper is to prove that the functions generated by the integral operator $I(f,g)(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} dt$ are in the class of close-to-convex functions, considering the analytical functions f_i and g_i from the classes of starlike and close-to-starlike functions.

1. Introduction and Definitions

Let $\mathcal{U} = \{z : |z| < 1\}$ be the open unit disk. By $\mathcal{A}$ we denote the class of all analytical functions in the open unit disk $\mathcal{U}$ and by $\mathcal{S}$ the class of univalent functions that contains all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition:

$$f(0) = f'(0) - 1 = 0.$$

To prove our main results we will recall here some known results about some subclasses of analytical functions. First we will recall the classes of starlike and convex functins of order α denoted by $S^*(\alpha)$ and $K(\alpha)$ and defined by:

$$S^*(\alpha) = \{f \in \mathcal{A} : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in \mathcal{U}\}$$

$$K(\alpha) = \{f \in \mathcal{A} : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in \mathcal{U}\}$$

for $0 \le \alpha < 1$.
Alexander studied for the first time the class of starlike functions in [1] and the class of convex functions was introduced in [9], by E. Study.

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A function $f \in \mathcal{A}$ is in the class $S^*(a, A)$ if it satisfy the condition:

$$\left|\frac{zf'(z)}{f(z)} - a\right| < A, \ |a-1| < A \le a, z \in \mathcal{U}. \tag{2}$$

We have that $a > \frac{1}{2}$ and $S^*(a, A) \subset S^*(a - A) \subset S^*(0) \equiv S^*$. This class was introduced in [5] by Jakubowski.

The class $K(a, A)$ contains all the functions $f \in \mathcal{A}$ such that:

$$\left|1 + \frac{zf''(z)}{f'(z)} - a\right| < A, \ |a-1| < A \le a, z \in \mathcal{U}. \tag{3}$$

Also for this class, $a > \frac{1}{2}$ and $K(a, A) \subset K(a - A) \subset K(0) \equiv K$. The relations (2) and (3) are equivalently with

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > a - A, \ z \in \mathcal{U}, |a-1| < A \le a,$$

respectively

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > a - A, \ z \in \mathcal{U}, |a-1| < A \le a.$$

The class of close-to-convex functions contains all the functions that satisfy the condition:

$$\int_{\theta_1}^{\theta_2} \text{Re}\left(1 + \frac{zf''(z)}{f'(z)} d\theta\right) > -\pi$$

where $0 \le \theta_1 < \theta_2 \le 2\pi, z = re^{i\theta}$ and $r < 1$ and is denoted by C_c.

This class was studied for certain analytic functions by Owa et al. in [6].

A function belongs to C_{s^*}, i.e. the class of close-to-starlike functions iff:

$$\int_{\theta_1}^{\theta_2} \text{Re}\frac{zf'(z)}{f(z)} d\theta > -\pi,$$

where $0 \le \theta_1 < \theta_2 \le 2\pi, z = re^{i\theta}$ and $r < 1$.

Shukla and Kumar introduced in [7] some subclasses of C_c and C_{s^*} and proved some important results for these.

The class $C_c(\beta, \rho)$ of close-to-convex functions of order β and type ρ contains all the functions that for a function $g \in S^*(\rho)$ satisfies the inequality:

$$\left|arg\left(\frac{zf'(z)}{g(z)}\right)\right| < \frac{\beta\pi}{2}, z \in \mathcal{U}, \beta \in [0, 1].$$

A function f is in the class of close-to-starlike functions of order β and type ρ, denoted by $C_{s^*}(\beta, \rho)$ if for some function $g \in S^*(\rho)$ we have the following inequality:

$$\left|arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\beta\pi}{2}, z \in \mathcal{U},$$

for $\beta \in [0, 1]$.

Is very clear that $C_c(0, \rho) = K(\rho)$ and $C_{s^*}(0, \rho) = S^*(\rho)$.

We consider the results proved by Shukla and Kumar in [7] about these two subclasses defined before.

Lemma 1.1. *[7] If $f \in S^*(\rho)$, then*

$$\rho(\theta_2 - \theta_1) \leq \int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{zf'(z)}{f(z)}d\theta \leq 2\pi(1-\rho) + \rho(\theta_2 - \theta_1),$$

where $z = re^{i\theta}$ and $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$.

Lemma 1.2. *[7] If $f \in C_{s^*}(\beta, \rho)$ then*

$$-\beta\pi + \rho(\theta_2 - \theta_1) \leq \int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{zf'(z)}{f(z)}d\theta \leq \beta\pi + 2\pi(1-\rho) + \rho(\theta_2 - \theta_1),$$

where $z = re^{i\theta}$ and $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$.

For the analytical functions f_i, g_i and the positive real numbers γ_i, for $i = \overline{1,n}$, we consider the integral operator:

$$I(f,g)(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} dt, \tag{4}$$

that was introduced by Ularu and Breaz in [10]. Integral operators make the subject of several articles, the authors studying some properties for them, for example the univalence (see for example [2], [8], [4], [11] and [3])

2. Main Results

Theorem 2.1. *Let the analytical functions f_i from the class $S^*(\eta_i)$, g_i from the class $S^*(\delta_i)$, and the positive real numbers γ_i, for $i = \overline{1,n}$. If $\sum_{i=1}^n \gamma_i \leq 1$, then $I(f,g)$ is in the class of close-to-convex functions C_c.*

Proof. From the definitions of $I(f,g)$ given in (4) by logarithmic differentions we obtain that

$$\frac{zI''(f,g)(z)}{I'(f,g)(z)} = \sum_{i=1}^n \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - \frac{zg_i'(z)}{g_i(z)}\right),$$

for $i = \overline{1,n}$ and $z \in \mathcal{U}$.

Using the definition of close-to-convex functions results:

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zI''(f,g)(z)}{I'(f,g)(z)}\right)d\theta = \int_{\theta_1}^{\theta_2} \mathrm{Re}\left[\sum_{i=1}^n \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - \frac{zg_i'(z)}{g_i(z)}\right)d\theta + 1\right]$$

$$= \int_{\theta_1}^{\theta_2} \sum_{i=1}^n \gamma_i \mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)d\theta - \int_{\theta_1}^{\theta_2} \sum_{i=1}^n \gamma_i \mathrm{Re}\left(\frac{zg_i'(z)}{g_i(z)}\right)d\theta + \int_{\theta_1}^{\theta_2} d\theta.$$

We use the hypothesis that $f_i \in S^*(\eta_i)$ and $g_i \in S^*(\delta_i)$ and according to Lemma 1.1 it follows that:

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zI''(f,g)(z)}{I'(f,g)(z)}\right)d\theta \geq \sum_{i=1}^n \gamma_i\eta_i(\theta_2 - \theta_1) - \sum_{i=1}^n \gamma_i\delta_i(\theta_2 - \theta_1) + (\theta_2 - \theta_1)$$

$$\geq \left(\sum_{i=1}^n \gamma_i(\eta_i - \delta_i) + 1\right)(\theta_2 - \theta_1),$$

for $z \in \mathcal{U}$ and $i = \overline{1,n}$. Because $\sum\limits_{i=1}^{n} \gamma_i(\eta_i - \delta_i) + 1 > 0$, and minimun is for $\theta_1 = \theta_2$, results that

$$\int\limits_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zI''(f,g)(z)}{I'(f,g)(z)}\right) > -\pi.$$

So, from the above inequality we obtain that $I(f,g) \in C_c$. □

If we consider $\eta_1 = \eta_2 = \cdots = \eta_n = \eta$ and $\delta_1 = \delta_2 = \cdots = \delta_n = \delta$ in Theorem 2.1 it follows:

Corollary 2.2. *Let $f_i, g_i \in \mathcal{A}$ and the positive real numbers γ_i, for $i = \overline{1,n}$. If $f_i \in S^*(\eta), g_i \in S^*(\delta)$ and $\sum\limits_{i=1}^{n} \gamma_i \leq 1$, then $I(f,g)$ is in the class of close-to-convex functions C_c.*

Theorem 2.3. *Let the analytical function $f_i \in C_{s^*}$, $g_i \in S^*(\delta_i)$ and γ_i positive real numbers, for $i = \overline{1,n}$. If $\sum\limits_{i=1}^{n} \gamma_i \leq 1$, then the functions generated by the operator $I(f,g)$ are in the class C_c.*

Proof. The proof follows the same idea as the proof of Theorem 2.1. Results that

$$\int\limits_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zI''(f,g)(z)}{I'(f,g)(z)}\right) d\theta = \int\limits_{\theta_1}^{\theta_2} \sum_{\gamma_i} \mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) d\theta - \int\limits_{\theta_1}^{\theta_2} \sum_{i=1}^{n} \gamma_i \mathrm{Re}\left(\frac{zg_i'(z)}{g_i(z)}\right) + \int\limits_{\theta_1}^{\theta_2} d\theta,$$

for all $z \in \mathcal{U}$ and $i = \overline{1,n}$.
We use that $f_i \in C_{s^*}, g_i \in S^*(\delta_i)$ and from Lemma 1.1 and Lemma 1.2 it follows that:

$$\begin{aligned}\int\limits_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zI''(f,g)(z)}{I'(f,g)(z)}\right) d\theta &\geq -\pi \sum_{i=1}^{n} \gamma_i - \sum_{i=1}^{n} \gamma_i \delta_i (\theta_2 - \theta_1) + (\theta_2 - \theta_1) \\ &\geq -\pi \sum_{i=1}^{n} \gamma_i - (\theta_2 \theta_1)(\sum_{i=1}^{n} \delta_i \gamma_i + 1),\end{aligned}$$

for all $z \in \mathcal{U}$ and $i = \overline{1,n}$.
Because $1 - \sum\limits_{i=1}^{n} \gamma_i \delta_i > 0$, minimum is for $\theta_1 = \theta_2$ we obtain that $I(f,g) \in C_c$. □

Theorem 2.4. *Let the analitical functions f_i, g_i and the positive real numbers γ_i, for all $i = \overline{1,n}$. If $f_i \in C_{s^*}(\beta_i, \rho_i), g_i \in C_{s^*}(\alpha_i, \eta_i)$ and $\sum\limits_{i=1}^{n} \gamma_i \beta_i \leq 1, \sum\limits_{i=1}^{n} \gamma_i \alpha_i \leq 1$, then the integral operator $I(f,g)$ is in the class C_c.*

Proof. We follow the same steps as in the proofs of the above theorems, but we use that the functions f_i are from the class $C_{s^*}(\beta_i, \rho_i)$ and the functions g_i are from $C_{s^*}(\alpha_i, \eta_i)$. Using these and applying Lemma 1.2 it follows that:

$$\begin{aligned}\int\limits_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zI''(f,g)(z)}{I'(f,g)(z)}\right) d\theta &\geq \sum_{i=1}^{n} \gamma_i [(-\beta_i \pi + \rho_i(\theta_2 - \theta_1)) - (-\alpha_i \pi + \eta_i(\theta_2 - \theta_1))] + (\theta_2 - \theta_1) \\ &\geq (\theta_2 - \theta_1)\left[\sum_{i=1}^{n} \gamma_i(\rho_i - \eta_i) + 1\right] - \sum_{i=1}^{n} \gamma_i \beta_i \pi + \sum_{i=1}^{n} \gamma_i \alpha_i \pi.\end{aligned}$$

Since $\sum_{i=1}^{n}\gamma_i(\rho_i-\eta_i)+1>0$, minimum is for $\theta_1=\theta_2$ and results

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right)d\theta>-\pi.$$

We obtain that $I(f,g)\in C_c$. □

If we consider $\beta_1=\beta_2=\cdots=\beta_n=\beta$ and $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$r in Theorem 2.4 we obtain:

Corollary 2.5. *Let the analitical functions f_i, g_i and the positive real numbers γ_i, for all $i=\overline{1,n}$. If $f_i\in C_{s^*}(\beta,\rho_i), g_i\in C_{s^*}(\alpha,\eta_i)$ and $\beta\sum_{i=1}^{n}\gamma_i\le 1$, respectivelly $\alpha\sum_{i=1}^{n}\gamma_i\le 1$, then the integral operator $I(f,g)$ is in the class C_c.*

Remark 2.6. *If we consider $\beta_i=0$ and $\alpha_i=0$, for $i=\overline{1,n}$ in Theorem 2.4 we obtain the results from Theorem 2.1.*

Theorem 2.7. *Let $f_i\in S^*(\alpha_i,\beta_i)$, for $|\alpha_i-1|<\beta_i\le\alpha_i$ and $g_i\in S^*(\xi_i,\eta_i)$, for $|\xi_i-1|<\eta_i\le\xi_i$, $\gamma_i>0$ for all $i=\overline{1,n}$ and $z\in\mathcal{U}$. Then the functions generated by the integral operator $I(f,g)$ are in the class $K(a_i,b_i)$, where $a_i=1+\sum_{i=1}^{n}\gamma_i(\alpha_i-\beta_i), b_i=\sum_{i=1}^{n}\gamma_i(\xi_i-\eta_i)$ and $\sum_{i=1}^{n}\gamma_i(\xi_i-\eta_i-\alpha_i+\beta_i)\le 1$, for all $i=\overline{1,n}$ and $z\in\mathcal{U}$.*

Proof. Using that $f_i\in S^*(\alpha_i,\beta_i)$ and $g_i\in S^*(\xi_i,\eta_i)$ results:

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right)&=\mathrm{Re}\left(1+\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i'(z)}{f_i(z)}-\frac{zg_i'(z)}{g_i(z)}\right)\right)\\&=1+\sum_{i=1}^{n}\gamma_i\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)-\sum_{i=1}^{n}\gamma_i\mathrm{Re}\left(\frac{zg_i'(z)}{g_i(z)}\right)\\&>1+\sum_{i=1}^{n}\gamma_i(\alpha_i-\beta_i)-\sum_{i=1}^{n}\gamma_i(\xi_i-\eta_i).\end{aligned}$$

From the above inequalitie and the definition of $K(a_i,b_i)$ we obtain that $I(f,g)(z)\in K(a_i,b_i)$, where a_i and b_i are defined as in the theorem hypothesis. □

For $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$ and $\xi_1=\xi_2=\cdots=\xi_n=\xi$ in Theorem 2.7 we obtain:

Corollary 2.8. *Let $f_i\in S^*(\alpha,\beta_i)$, for $|\alpha-1|<\beta_i\le\alpha$ and $g_i\in S^*(\xi,\eta_i)$, for $|\xi-1|<\eta_i\le\xi$, $\gamma_i>0$ for all $i=\overline{1,n}$ and $z\in\mathcal{U}$. Then the functions generated by the integral operator $I(f,g)$ are in the class $K(a_i,b_i)$, where $a_i=1+\sum_{i=1}^{n}\gamma_i(\alpha-\beta_i), b_i=\sum_{i=1}^{n}\gamma_i(\xi-\eta_i)$ and $\sum_{i=1}^{n}\gamma_i(\xi-\eta_i-\alpha+\beta_i)\le 1$, for all $i=\overline{1,n}$ and $z\in\mathcal{U}$.*

If in Theorem 2.7 we consider $\gamma_1=\gamma_2=\cdots=\gamma_n=\gamma$, then we obtain

Corollary 2.9. *Let $f_i\in S^*(\alpha_i,\beta_i)$, for $|\alpha_i-1|<\beta_i\le\alpha_i$ and $g_i\in S^*(\xi_i,\eta_i)$, for $|\xi_i-1|<\eta_i\le\xi_i$, $\gamma>0$ for all $i=\overline{1,n}$ and $z\in\mathcal{U}$. Then the functions generated by the integral operator $I(f,g)$ are in the class $K(a_i,b_i)$, where $a_i=1+\gamma\sum_{i=1}^{n}(\alpha_i-\beta_i), b_i=\gamma\sum_{i=1}^{n}(\xi_i-\eta_i)$ and $\gamma\sum_{i=1}^{n}(\xi_i-\eta_i-\alpha_i+\beta_i)\le 1$, for all $i=\overline{1,n}$ and $z\in\mathcal{U}$.*

Theorem 2.10. *Let $f_i\in S^*(\alpha_i)$ and $g_i\in S^*(\beta_i)$, for all $i=\overline{1,n}$. Then the integral operator $I(f,g)\in K(a_i,b_i)$, where $a_i=1+\sum_{i=1}^{n}\gamma_i\alpha_i, b_i=\sum_{i=1}^{n}\gamma_i\beta_i$ and $\sum_{i=1}^{n}\gamma_i(\beta_i-\alpha_i)\le 1$, for all $i=\overline{1,n}$ and $z\in\mathcal{U}$.*

Proof. The proof is similar to Theorem 2.7. □

If we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ and $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$ in Theorem 2.10 results:

Corollary 2.11. *Let $f_i \in S^*(\alpha)$ and $g_i \in S^*(\beta)$, for all $i = \overline{1,n}$. Then the integral operator $I(f,g) \in K(a_i, b_i)$, where $a_i = 1 + \alpha \sum_{i=1}^{n} \gamma_i, b_i = \beta \sum_{i=1}^{n} \gamma_i$ and $(\beta - \alpha) \sum_{i=1}^{n} \gamma_i \leq 1$, for all $i = \overline{1,n}$ and $z \in \mathcal{U}$.*

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